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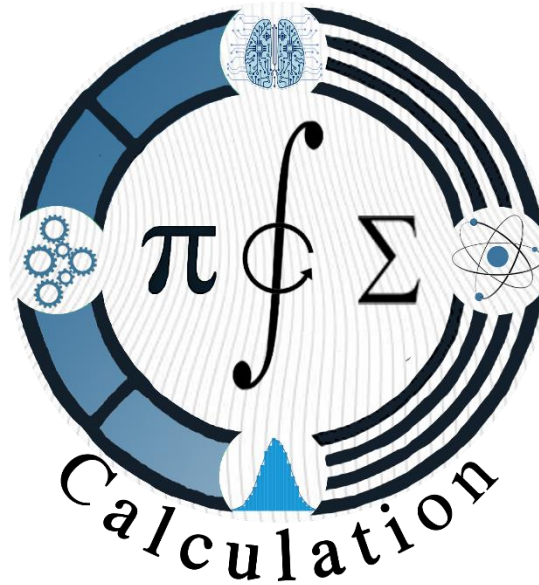
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This issue of Calculation Journal is dedicated to Prof. Dr. Rifat Güneş, who retired from İnönü University

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# The Journal CALCULATION



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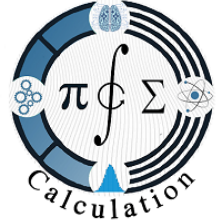
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### EDITORIAL

BAYRAM ŞAHİN  \*

Prof. Dr. Rifat GÜNEŞ is a Turkish mathematician whose research has focused primarily on differential geometry, semi-Riemannian geometry, lightlike submanifolds, CR-submanifolds, and related topics. He was born in Pütürge (Malatya, Türkiye) in 1959 and completed his primary and secondary education in Malatya. He completed his undergraduate, master's, and doctoral studies in mathematics at İnönü University, Malatya, Türkiye and later served as a faculty member in the Department of Mathematics, eventually becoming Professor of Mathematics.

Prof. Dr. Rifat GÜNEŞ has played a significant role in my academic development and scientific career. I had the opportunity to work with him during my graduate studies in the Department of Mathematics at İnönü University. I completed my master's and doctoral theses under his supervision. I remain grateful for his guidance, friendship, and the many years of productive collaboration we have shared.



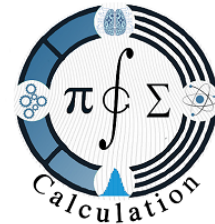
FIGURE 1. Prof. Dr. Rifat Güneş

Throughout his academic career, Prof. Dr. GÜNEŞ has made valuable contributions to mathematics through both his research and the many students he has trained. I consider myself fortunate to have been one of his students and collaborators. His mentorship, encouragement, and commitment to scientific excellence have left a lasting impact on my academic life.

We wish Prof. Dr. Rifat GÜNEŞ, who recently retired from İnönü University, a healthy and happy retirement with his family.

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\* Editor-in-Chief



**TYPE-1 INTERPOLATING SESQUI- $f$ -HARMONIC MAPS BETWEEN RIEMANNIAN MANIFOLDS**

SELÇEN YÜKSEL PERKTAŞ , FEYZA ESRA ERDOĞAN , ŞERİFE NUR BOZDAĞ ,  
AND BILAL EFTAL ACET  \*

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**Abstract.** In this paper, first, we introduce and study a new map called a type-1 interpolating sesqui- $f$ -harmonic map. Then, we provide necessary and sufficient conditions for a differentiable curve in a Riemannian space form to be a type-1 interpolating sesqui- $f$ -harmonic. These conditions are presented in a main theorem and investigated in several subcases. Moreover, we analyze type-1 interpolating sesqui- $f$ -harmonic curves on  $S^n(1)$  and  $H^n(-1)$ .

**Keywords:**  $f$ -Harmonic maps, bi- $f$ -harmonic maps, Riemannian manifolds

**2020 Mathematics Subject Classification:** 58E20, 53C43.

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1. INTRODUCTION

There are some physical applications of harmonic and biharmonic maps in differential geometry. In materials science, harmonic maps are used to study the deformation of materials under different stresses and strains [2]. Moreover, Branding [3] defined a new functional for maps between Riemannian manifolds by interpolating harmonic and biharmonic maps with the motivation of the motion functional for externally curvature bosonic strings used in physics. This functional is called an interpolating sesqui-harmonic map between Riemannian manifolds.

Interpolating sesqui-harmonic maps defined between Riemannian manifolds are similar to harmonic maps, but they also satisfy an additional condition that includes the sesquilinear form. Sesqui-harmonic maps arise in various physical contexts, such as the diffusion of heat and electric currents in conductive materials, the dynamics of fluid flows [6]. For instance, these maps are instrumental in analyzing heat conduction in metallic structures or modeling the stress-strain relationship in elastic media [10]. Furthermore, sesqui-harmonic maps find applications in cosmology, where they contribute to understanding the large-scale distribution

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of matter and the universe's expansion [7]. Finally, this functional, which appears in various places in the physics literature, is called a bosonic string with external curvature in string theory [13, 15].

After defining the interpolating sesqui-harmonic maps between Riemannian manifolds, Branding [4] investigated the analytical results of these maps, especially those with spherical image sets, and focused on the analytical aspects of interpolated sesqui-harmonic maps, which were built on the regularity theory developed for biharmonic maps. He introduced a conservation law for such maps and used it to demonstrate the differentiability of weak solutions.

After these studies, Karaca et al. [12] studied interpolating sesqui-harmonic Legendre curves in Sasakian space forms. Following this, Karaca [11] carried this work to generalized Sasakian space forms. The last study on interpolating sesqui-harmonic maps in the relevant context is by Iqbal et al. [9], where interpolating sesqui-harmonic slant curves are studied in generalized Sasakian space forms, and a definition of an interpolating sesqui-harmonic minimal curve is introduced, which is another interesting new concept.

The  $f$ -biharmonic and bi- $f$ -harmonic maps in differential geometry have physical applications similar to those of the harmonic and biharmonic maps. However, adding an  $f$ -functional to these maps provides a more precise and accurate representation of surface features, resulting in more accurate results in these applications. An  $f$ -functional is used to map local properties of a surface to a target space and to minimize the distortion of angles and lengths during the mapping of the surface to the target space.

In light of all the studies mentioned above, in this paper, we introduce type-1 interpolating sesqui- $f$ -harmonic maps, which can be considered as an intermediate value for  $f$ -harmonic and bi- $f$ -harmonic maps between Riemann manifolds. We provide a functional, called a type-1 interpolating sesqui- $f$ -energy integral, which enables the definition of type-1 interpolating sesqui- $f$ -harmonic maps, the corresponding Euler-Lagrange equations, and type-1 interpolating sesqui- $f$ -tension field definitions. Then, in the third section, we investigate the type-1 interpolating sesqui- $f$ -harmonicity conditions for curves on Riemannian manifolds using the type-1 interpolating sesqui- $f$ -harmonicity equations. We analyze these conditions for the curves in terms of the special cases of the elements of the Frenet frame and obtain some absence theorems and results from the data herein.

## 2. PRELIMINARIES

In this section, we present  $f$ -harmonic, bi- $f$ -harmonic, and sesqui-harmonic map equations between Riemannian manifolds.

**Definition 2.1.** [8] *Let  $(M, g)$  and  $(N, h)$  be Riemannian manifolds. Then, a harmonic map  $\varpi : (M, g) \rightarrow (N, h)$  is defined as the critical point of the energy functional*

$$E(\varpi) = \frac{1}{2} \int_M |d\varpi|^2 dv_g$$

where  $v_g$  is the volume element of  $(M, g)$ . Then, by using Euler-Lagrange equation  $\tau(\varpi)$  of the energy functional  $E(\varpi)$ , where it is the tension field of the map  $\varpi$ , the map is called as

harmonic if

$$\tau(\varpi) = \text{tr} \nabla d\varpi = 0$$

**Definition 2.2.** [8] A map  $\varpi : (M, g) \rightarrow (N, h)$  is defined as a biharmonic map if it is a critical point of the bienergy functional

$$E_2(\varpi) = \frac{1}{2} \int_M |\tau(\varpi)|^2 dv_g$$

for all variations. Then, for the bienergy functional  $E_2(\varpi)$ , the Euler-Lagrange equation  $\tau_2(\varpi)$  equals to

$$\tau_2(\varpi) = \text{tr}(\nabla^\varpi \nabla^\varpi - \nabla_{\frac{\varpi}{\varpi}}^\varpi) \tau(\varpi) - \text{tr}(R^N(d\varpi, \tau(\varpi))d\varpi) = 0$$

where  $\tau_2(\varpi)$  is the bitension field of the map  $\varpi$ , if  $\varpi$  is a biharmonic map where  $R^N$  is the curvature tensor field of  $N$  defined as

$$R^N(X, Y)Z = \nabla_X^N \nabla_Y^N Z - \nabla_Y^N \nabla_X^N Z - \nabla_{[X, Y]}^N Z$$

for all  $X, Y, Z \in \Gamma(TN)$ . Here,  $\nabla^\varpi$  is the pull-back connection.

**Definition 2.3.** [1, 5] A map  $\varpi : (M, g) \rightarrow (N, h)$  is said to be an  $f$ -harmonic if it is a critical point of the  $f$ -energy functional

$$E_f(\varpi) = \frac{1}{2} \int_M f |d\varpi|^2 dv_g$$

where  $f \in C^\infty(M, \mathbb{R})$  is a positive smooth function. Then, the  $f$ -harmonic map equation obtained by using Euler-Lagrange equation as follows:

$$\tau_f(\varpi) = f\tau(\varpi) + d\varpi(\text{grad}f) = 0$$

where  $\tau_f(\varpi)$  is the  $f$ -tension field of the map  $\varpi$ .

**Definition 2.4.** [16] A map  $\varpi : (M, g) \rightarrow (N, h)$  is said to be a bi- $f$ -harmonic if it is a critical point of the bi- $f$ -energy functional

$$E_{2,f}(\varpi) = \frac{1}{2} \int_M |\tau_f(\varpi)|^2 dv_g$$

The Euler-Lagrange equation for the bi- $f$ -harmonic map is defined by

$$\tau_{2,f}(\varpi) = \text{tr}(\nabla^\varpi f(\nabla^\varpi \tau_f(\varpi))) - f \nabla_{\frac{\varpi}{\varpi}}^\varpi \tau_f(\varpi) + f R^N(\tau_f(\varpi), d\varpi)d\varpi = 0$$

where  $\tau_{2,f}(\varpi)$  is the bi- $f$ -tension field of the map  $\varpi$ .

Furthermore, Branding [3] defined an interpolating sesqui-harmonic map between Riemannian manifolds. He introduced an action functional that interpolates between the actions for harmonic and biharmonic maps.

**Definition 2.5.** [3, 12] A map  $\varpi$  is called as interpolating sesqui-harmonic if it is a critical point of the following action functional that interpolates between the actions for harmonic and biharmonic maps:

$$E_{\delta_1, \delta_2}(\varpi) = \delta_1 \int_M |d\varpi|^2 dv_g + \delta_2 \int_M |\tau(\varpi)|^2 dv_g$$

where  $\delta_1, \delta_2 \in \mathbb{R}$ . Then, the interpolating sesqui-harmonic equation is provided as follows:  
 For  $\delta_1, \delta_2 \in \mathbb{R}$ ,

$$\tau_{\delta_1, \delta_2}(\varpi) = \delta_2 \tau_2(\varpi) - \delta_1 \tau(\varpi) = 0.$$

For more information about interpolating sesqui-harmonic maps, see [3, 4, 12].

### 3. INTERPOLATING SESQUI- $f$ -HARMONIC CURVES ON RIEMANNIAN MANIFOLDS

Branding [3] has obtained a functional by associating the critical points of the energy integral, which is crucial for harmonic maps, with the bi-energy integral that provides rise to biharmonic maps, enabling the definition of interpolated sesqui-harmonic maps between Riemann manifolds. Biharmonic maps are maps that roughly solve the Laplace equation in a space. Moreover,  $f$ -biharmonic maps emerge as maps satisfying the multivariable  $f$ -Laplace equation. The difference between these two maps arises from how these equations are solved and which properties are emphasized. In this study, we extend Branding’s work [3] to a broader class of maps, including  $f$ -harmonic and bi- $f$ -harmonic maps. We introduce and thoroughly study the type-1 interpolating sesqui- $f$ -harmonic map, an intermediate concept between  $f$ -harmonic and bi- $f$ -harmonic maps on Riemannian manifolds.

**3.1. Type-1 Interpolating Sesqui- $f$ -Harmonic Maps on Riemannian Manifolds.** In this section, first, we define the following functional

$$E_{\rho_1, \rho_2, f}(\varpi) = \rho_1 \int_M f |d\varpi|^2 dv_g + \rho_2 \int_M |\tau_f(\varpi)|^2 dv_g$$

where  $\rho_1, \rho_2 \in \mathbb{R} \setminus \{0\}$ . Let

$$\begin{aligned} \pi : M \times (-\epsilon, \epsilon) &\rightarrow N \\ (x, t) &\rightarrow \pi(x, t) = \varpi_t(x) \end{aligned}$$

be a smooth variation of  $\varpi$  with variation vector fields  $v$ , where  $(M, g)$  and  $(N, h)$  are Riemannian manifolds. Denote the pull-back connection on the vector bundle  $\pi^{-1} : TN \rightarrow M \times (-\epsilon, \epsilon)$  by  $\nabla^\pi$ . Since any vector field  $X \in \Gamma(TM)$  can be considered as a vector field on  $M \times (-\epsilon, \epsilon)$ , then  $[\frac{\partial}{\partial t}, X] = 0$ . As a first step, we calculate the following equality:

$$\frac{d}{dt} E_{\rho_1, \rho_2, f}(\varpi_t; M)|_{t=0} = \rho_1 \left( \frac{1}{2} \frac{d}{dt} \int_M f |d\varpi_t|^2 dv_g|_{t=0} \right) + \rho_2 \left( \frac{1}{2} \frac{d}{dt} \int_M |\tau_f(\varpi_t)|^2 dv_g|_{t=0} \right)$$

It is well known from [14] that

$$\frac{1}{2} \frac{d}{dt} \int_M f |d\varpi_t|^2 dv_g|_{t=0} = - \int_M h(\tau_f(\varpi), v) dv_g \tag{3.1}$$

and

$$\frac{1}{2} \frac{d}{dt} \int_M |\tau_f(\varpi_t)|^2 dv_g|_{t=0} = - \int_M h(\tau_{2, f}(\varpi), v) dv_g \tag{3.2}$$

where

$$\tau_f(\varpi) = f\tau(\varpi) + d\varpi(\text{grad}f)$$

and

$$\tau_{2, f}(\varpi) = \text{tr}(\nabla^\varpi f(\nabla^\varpi \tau_f(\varpi))) - f \nabla_{\nabla^\varpi}^\varpi \tau_f(\varpi) + f R^N(\tau_f(\varpi), d\varpi) d\varpi).$$

By using (3.1) and (3.2),

$$\frac{d}{dt}E_{\rho_1, \rho_2, f}(\varpi_t; M)|_{t=0} = -\rho_1 \int_M h(\tau_f(\varpi), v) dv_g - \rho_2 \int_M h(\tau_{2,f}(\varpi), v) dv_g$$

which implies

$$\frac{d}{dt}E_{\rho_1, \rho_2, f}(\varpi_t; M)|_{t=0} = - \int_M \langle \rho_1 \tau_f(\varpi) + \rho_2 \tau_{2,f}(\varpi), v \rangle dv_g. \quad (3.3)$$

**Definition 3.1.** A map  $\varpi : (M, g) \rightarrow (N, h)$  between Riemannian manifolds is called a type-1 interpolating sesqui- $f$ -harmonic map if it is a critical point of the functional

$$E_{\rho_1, \rho_2, f} = \rho_1 \int_M f |d\varpi|^2 dv_g + \rho_2 \int_M |\tau_f(\varpi)|^2 dv_g$$

The Euler-Lagrange equation associated with (3.3) is

$$\tau_{\rho_1, \rho_2, f}(\varpi) = \rho_1 \tau_f(\varpi) + \rho_2 \tau_{2,f}(\varpi)$$

where  $\tau_f(\varpi)$  and  $\tau_{2,f}(\varpi)$  are the  $f$ -tension and bi- $f$ -tension field of  $\varpi$ , respectively, and  $\rho_1, \rho_2 \in \mathbb{R} \setminus \{0\}$ .

**Theorem 3.1.** A map  $\varpi : (M, g) \rightarrow (N, h)$  is a type-1 interpolating sesqui- $f$ -harmonic map if and only if  $\rho_1 \tau_f(\varpi) + \rho_2 \tau_{2,f}(\varpi) = 0$ .

**Remark 3.1.** From the relevant definitions, the following hold:

If  $f$  is a constant, then  $f$ -harmonicity induces to harmonicity. In this case, interpolating sesqui- $f$ -harmonic maps overlap with interpolating sesqui-harmonic maps defined by Branding [3] such that  $\delta_1 = -\rho_1 f$  and  $\delta_2 = -\rho_2 f$ .

It is well known from [14] that any  $f$ -harmonic map is bi- $f$ -harmonic. Thus,  $f$ -harmonic maps are always interpolating sesqui- $f$ -harmonic maps, for any  $\rho_1, \rho_2 \in \mathbb{R}$ .

Let  $\alpha : I \rightarrow (N, h)$  be a differentiable curve on an  $n$ -dimensional Riemannian manifold  $(N, h)$  parametrized by the arclength  $s$ , where  $I$  is an open interval and  $\alpha' = T$ . Considering the  $f$ -tension field and bi- $f$ -tension field equations in [16],

$$\tau_f(\varpi) = f \nabla_T T + f' T$$

and

$$\tau_{2,f}(\varpi) = (ff''' + f'f'')T + (3ff'' + 2(f')^2)\nabla_T T + 4ff'\nabla_T^2 T + f^2\nabla_T^3 T + f^2 R^N(\nabla_T T, T)T$$

which imply

$$\begin{aligned} \tau_{\rho_1, \rho_2, f}(\varpi) &= (\rho_1 f' + \rho_2 (ff''' + f'f''))T + (\rho_1 f + \rho_2 (3ff'' + 2(f')^2))\nabla_T T + 4\rho_2 f f' \nabla_T^2 T \\ &\quad + \rho_2 f^2 \nabla_T^3 T + \rho_2 f^2 R^N(\nabla_T T, T)T. \end{aligned}$$

Thus, the following proposition can be obtained:

**Proposition 3.1.** *Let  $\alpha : I \rightarrow (N, h)$  be a differentiable curve parametrized by its arclength. Then,  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if*

$$0 = (\rho_1 f' + \rho_2 (f f''' + f' f''))T + (\rho_1 f + \rho_2 (3ff'' + 2(f')^2))\nabla_T T + 4\rho_2 f f' \nabla_T^2 T + \rho_2 f^2 \nabla_T^3 T + \rho_2 f^2 R^N(\nabla_T T, T)T. \tag{3.4}$$

Let  $\{E_1 = T, E_2, \dots, E_n\}$  be a Frenet frame along  $\alpha$  defined an  $n$ -dimensional Riemannian manifold  $(N, h)$ , where  $E_2$  is the unit normal vector field along  $\alpha$  and for every  $j \in \{3, 4, 5, \dots, n\}$ ,  $E_j$  is a unit vector field such that

$$\begin{cases} \nabla_T T = k_1 E_2 \\ \nabla_T E_2 = -k_1 T + k_2 E_3 \\ \nabla_T E_r = -k_{r-1} E_{r-1} + k_r E_{r+1}, \quad r \in \{3, 4, \dots, n-1\} \\ \nabla_T E_n = -k_{n-1} E_{n-1} \end{cases}.$$

Here,  $k_1 = \|\nabla_T T\|$  and  $k_2, \dots, k_{n-1}$  are nonnegative real-valued functions. Following the equations calculated in [16],

$$\nabla_T^2 T = -k_1^2 T + k_1' E_2 + k_1 k_2 E_3 \tag{3.5}$$

$$\nabla_T^3 T = -3k_1 k_1' T + (k_1'' - k_1^3 - k_1 k_2^2) E_2 + (2k_1' k_2 + k_1 k_2') E_3 + k_1 k_2 k_3 E_4 \tag{3.6}$$

and

$$R^N(\nabla_T T, T)T = k_1 R^N(E_2, T)T. \tag{3.7}$$

In view of (3.5)-(3.7) in (3.4), the following theorem can be obtained:

**Theorem 3.2.** *A differentiable curve  $\alpha : I \rightarrow (N, h)$  parametrized by its arclength is a type-1 interpolating sesqui- $f$ -harmonic map if and only if*

$$0 = (\rho_1 f' + \rho_2 (f f''' + f' f'')) - 4\rho_2 f f' k_1^2 - 3\rho_2 f^2 k_1 k_1' T + (\rho_1 f k_1 + \rho_2 k_1 (3ff'' + 2(f')^2)) + 4\rho_2 f f' k_1' + \rho_2 f^2 (k_1'' - k_1^3 - k_1 k_2^2) E_2 + (4\rho_2 f f' k_1 k_2 + \rho_2 f^2 (2k_1' k_2 + k_1 k_2')) E_3 + (\rho_2 f^2 k_1 k_2 k_3) E_4 + \rho_2 f^2 k_1 R^N(E_2, T)T.$$

If  $(N, h)$  is a Riemannian manifold of constant sectional curvature, then the following theorem can be obtained:

**Theorem 3.3.** *Let  $\alpha : I \rightarrow (N(c), h)$  be a differentiable curve on a Riemannian space form  $N(c)$ , parametrized by its arclength. Then,  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if*

$$\begin{cases} \rho_1 f' + \rho_2 (f f''' + f' f'') - 4\rho_2 f f' k_1^2 - 3\rho_2 f^2 k_1 k_1' = 0 \\ \rho_1 f k_1 + \rho_2 k_1 (3ff'' + 2(f')^2) + 4\rho_2 f f' k_1' + \rho_2 f^2 (k_1'' - k_1^3 - k_1 k_2^2) + c\rho_2 f^2 k_1 = 0 \\ 4\rho_2 f f' k_1 k_2 + \rho_2 f^2 (2k_1' k_2 + k_1 k_2') = 0 \\ \rho_2 f^2 k_1 k_2 k_3 = 0 \end{cases}. \tag{3.8}$$

From the first equation in (3.8), we consider the case in which  $\alpha$  is a geodesic as follows:

**Theorem 3.4.** *A geodesic curve on a Riemannian manifold is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if*

$$\frac{\rho_1}{\rho_2} = -\frac{(ff'')'}{f'}.$$

Assume that  $\alpha : I \rightarrow E^n$  is a differentiable curve on the  $n$ -dimensional Euclidean space parametrized by its arclength, where  $I \subseteq \mathbb{R}$  is an open interval. We investigate the following special cases:

**Case I.** Let  $k_1 \in \mathbb{R} \setminus \{0\}$  and  $k_2 = 0$ . Then, (3.8) reduces to

$$\begin{cases} \rho_1 f' + \rho_2 (ff''' + f' f'') - 4\rho_2 f f' k_1^2 = 0 \\ \rho_1 f k_1 + \rho_2 k_1 (3ff'' + 2(f')^2) - \rho_2 f^2 k_1^3 = 0 \end{cases}$$

which implies

$$\begin{cases} \rho_1 f' = \rho_2 (4ff' k_1^2 - (ff'')') \\ \rho_1 f = \rho_2 (-3ff'' - 2(f')^2 + f^2 k_1^2) \end{cases}. \quad (3.9)$$

From the second equation in (3.9),

$$ff'' = \frac{\rho_2 f^2 k_1^2 - \rho_1 f - 2\rho_2 (f')^2}{3\rho_2}.$$

By putting the last equation into the first equation in (3.9),

$$f' \left( 5\rho_2 f k_1^2 + 2\rho_2 f'' - \rho_1 \right) = 0.$$

Thus, the following theorem can be obtained:

**Theorem 3.5.** *Let  $\alpha : I \rightarrow E^n$  be a differentiable circle in the  $n$ -dimensional Euclidean space parametrized by its arclength with  $k_1 \in \mathbb{R} \setminus \{0\}$  and  $k_2 = 0$ . Then,  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if*

$$f(s) = c_1 \cos \left( \sqrt{\frac{5}{2}} k_1 s \right) + c_2 \sin \left( \sqrt{\frac{5}{2}} k_1 s \right) + \frac{\rho_1}{5k_1^2 \rho_2}.$$

**Case II.** Let  $k_1, k_2 \in \mathbb{R} \setminus \{0\}$ . In this case, (3.8) reduces to

$$\begin{cases} \rho_1 f' + \rho_2 (ff''' + f' f'') - 4\rho_2 f f' k_1^2 = 0 \\ \rho_1 f k_1 + \rho_2 k_1 (3ff'' + 2(f')^2) + \rho_2 f^2 (-k_1^3 - k_1 k_2^2) = 0 \\ 4\rho_2 f f' k_1 k_2 = 0 \\ \rho_2 f^2 k_1 k_2 k_3 = 0 \end{cases}$$

which implies

$$\begin{cases} \rho_1 + \rho_2 f (-k_1^2 - k_2^2) = 0 \\ f' = 0 \\ k_3 = 0 \end{cases}.$$

Since  $f$  is a positive real-valued and nonconstant function, we obtain the following theorem:

**Theorem 3.6.** *There does not exist a type-1 interpolating sesqui- $f$ -harmonic helix in the  $n$ -dimensional Euclidean space.*

**Case III.** Let  $k_1 \in \mathbb{R} \setminus \{0\}$  and  $k_2$  be a nonconstant and positive real-valued function. Then, by using (3.8),

$$\begin{cases} \rho_1 f' + \rho_2 (f f''' + f' f'') - 4\rho_2 f f' k_1^2 = 0 \\ \rho_1 f k_1 + \rho_2 k_1 (3f f'' + 2(f')^2) + \rho_2 f^2 (-k_1^3 - k_1 k_2^2) = 0 \\ 4\rho_2 f f' k_1 k_2 + \rho_2 f^2 k_1 k_2' = 0 \\ \rho_2 f^2 k_1 k_2 k_3 = 0 \end{cases}$$

which implies

$$\begin{cases} \rho_1 f' + \rho_2 (f f''' + f' f'' - 4f f' k_1^2) = 0 \\ \rho_1 f + \rho_2 (3f f'' + 2(f')^2 - f^2 (k_1^2 + k_2^2)) = 0 \\ \rho_2 (4f' k_1 k_2 + f k_1 k_2') = 0 \\ k_2 k_3 = 0 \end{cases}$$

Therefore, the following theorem can be obtained:

**Theorem 3.7.** Let  $\alpha : I \rightarrow E^n$  be a differentiable curve in the  $n$ -dimensional Euclidean space parametrized by its arclength with  $k_1 \in \mathbb{R} \setminus \{0\}$  and the nonconstant and positive real-valued function  $k_2$ . Then,  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if  $f(s) = c_1 k_2(s)^{-\frac{1}{4}}$  such that  $c_1$  is a positive integral constant,  $k_3 = 0$ , and

$$33(k_2')^3 - 52k_2 k_2' k_2'' + 16k_2^2 k_2''' - 48k_1^2 k_2^2 k_2' + 16k_2' k_2^4 = 0.$$

**Case IV.** Let  $k_1$  be a nonconstant and nonnegative real-valued function and  $k_2 = 0$ . Then, the following theorem can be obtained:

**Theorem 3.8.** Let  $\alpha : I \rightarrow E^n$  be a differentiable curve in the  $n$ -dimensional Euclidean space parametrized by its arclength with the nonconstant and nonnegative real-valued function  $k_1$  and  $k_2 = 0$ . Then,  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if

$$\rho_1 f' + \rho_2 (f f''' + f' f'') - 4\rho_2 f f' k_1^2 - 3\rho_2 f^2 k_1 k_1' = 0$$

and

$$\rho_1 f k_1 + \rho_2 k_1 (3f f'' + 2(f')^2) + 4\rho_2 f f' k_1' + \rho_2 f^2 (k_1'' - k_1^3) = 0$$

**Case V.** Let  $k_1$  be a nonconstant and nonnegative real-valued function and  $k_2 \in \mathbb{R} \setminus \{0\}$ . By using (3.8),

$$\begin{cases} \rho_1 f' + \rho_2 (f f''' + f' f'') - 4\rho_2 f f' k_1^2 - 3\rho_2 f^2 k_1 k_1' = 0 \\ \rho_1 f k_1 + \rho_2 k_1 (3f f'' + 2(f')^2) + 4\rho_2 f f' k_1' + \rho_2 f^2 (k_1'' - k_1^3 - k_1 k_2^2) = 0 \\ 4\rho_2 f f' k_1 k_2 + 2\rho_2 f^2 k_1' k_2 = 0 \\ k_1 k_3 = 0 \end{cases}$$

which implies

$$\begin{cases} \rho_1 f' + \rho_2 (f f''' + f' f'') - 4\rho_2 f f' k_1^2 - 3\rho_2 f^2 k_1 k_1' = 0 \\ \rho_1 f k_1 + \rho_2 k_1 (3f f'' + 2(f')^2) + 4\rho_2 f f' k_1' + \rho_2 f^2 (k_1'' - k_1^3 - k_1 k_2^2) = 0 \\ 2f' k_1 + f k_1' = 0 \\ k_3 = 0 \end{cases}$$

Thus, the following theorem can be obtained:

**Theorem 3.9.** *Let  $\alpha : I \rightarrow E^n$  be a differentiable curve in the  $n$ -dimensional Euclidean space parametrized by its arclength with the nonconstant and nonnegative real-valued function  $k_1$  and  $k_2 \in \mathbb{R} \setminus \{0\}$ . Then,  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if  $f(s) = c_2 k_1(s)^{-\frac{1}{2}}$  such that  $c_2$  is a positive integral constant,  $k_3 = 0$ , and*

$$15(k_1')^3 - 18k_1 k_1' k_1'' + 4k_1^2 k_1''' + 12k_1^4 k_1' + 4k_1^2 k_2^2 k_1' = 0$$

**CASE VI:** Let  $k_1$  and  $k_2$  be nonconstant and nonnegative real-valued functions. From the third equation in (3.8),

$$\begin{cases} 4f' k_1 k_2 + f(2k_1' k_2 + k_1 k_2') = 0 \\ \frac{f'}{f} = -\frac{1}{2} \frac{k_1'}{k_1} - \frac{1}{4} \frac{k_2'}{k_2} \end{cases}$$

which implies

$$f(s) = c_3 k_1(s)^{-\frac{1}{2}} k_2(s)^{-\frac{1}{4}}$$

where  $c_3$  is a positive integral constant. Therefore, the following theorem can be obtained:

**Theorem 3.10.** *Let  $\alpha : I \rightarrow E^n$  be a differentiable curve in the  $n$ -dimensional Euclidean space parametrized by its arclength with nonconstant and nonnegative real-valued functions  $k_1$  and  $k_2$ . Then,  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if  $f(s) = c_3 k_1(s)^{-\frac{1}{2}} k_2(s)^{-\frac{1}{4}}$  such that  $c_3$  is a positive constant,  $k_3 = 0$ ,*

$$\rho_1 f' + \rho_2 (f f''' + f' f'') - 4\rho_2 f f' k_1^2 - 3\rho_2 f^2 k_1 k_1' = 0$$

and

$$\rho_1 f k_1 + \rho_2 k_1 (3f f'' + 2(f')^2) + 4\rho_2 f f' k_1' + \rho_2 f^2 (k_1'' - k_1^3 - k_1 k_2^2) = 0$$

**3.2. Type-1 Interpolating Sesqui- $f$ -Harmonic Curves in  $S^n(1)$  and  $H^n(-1)$ .** This section presents type-1 interpolating sesqui- $f$ -harmonic curves in  $S^n(1)$  and  $H^n(-1)$ . In view of (3.8), we have the following theorem:

**Theorem 3.11.** *Let  $\alpha : I \rightarrow S^n(1)$  be a differentiable curve parametrized by its arclength. Then,  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if*

$$\begin{aligned} \rho_1 f' + \rho_2 (f f''' + f' f'') - 4\rho_2 f f' k_1^2 - 3\rho_2 f^2 k_1 k_1' &= 0 \\ \rho_1 f k_1 + \rho_2 k_1 (3f f'' + 2(f')^2) + 4\rho_2 f f' k_1' + \rho_2 f^2 (k_1'' - k_1^3 - k_1 k_2^2) + \rho_2 f^2 k_1 &= 0 \\ 4\rho_2 f f' k_1 k_2 + \rho_2 f^2 (2k_1' k_2 + k_1 k_2') &= 0 \end{aligned}$$

and

$$\rho_2 f^2 k_1 k_2 k_3 = 0$$

**Theorem 3.12.** Let  $\alpha : I \rightarrow S^n(1)$  be a differentiable curve parametrized by its arclength. If  $k_1 = 0$ , then  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if  $\frac{\rho_1}{\rho_2} = -\frac{(ff'')'}{f'}$ . If  $k_1 \in \mathbb{R} \setminus \{0\}$  and  $k_2 = 0$ , then  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if

$$f(s) = c_1 \cos\left(\sqrt{\frac{5k_1^2 + 1}{2}}s\right) + c_2 \sin\left(\sqrt{\frac{5k_1^2 + 1}{2}}s\right) + \frac{\rho_1}{(5k_1^2 + 1)\rho_2}.$$

If  $k_1, k_2 \in \mathbb{R} \setminus \{0\}$ , then  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if

$$f(s) = \frac{\rho_1}{4\rho_2} s^2 + c_1 s + c_2$$

where  $-5k_1^2 + k_2^2 = 1$ , or

$$f(s) = c_1 \cos\left(\sqrt{\frac{-5k_1^2 + k_2^2 - 1}{2}}s\right) + c_2 \sin\left(\sqrt{\frac{-5k_1^2 + k_2^2 - 1}{2}}s\right) + \frac{\rho_1}{(5k_1^2 - k_2^2 + 1)\rho_2}$$

where  $-5k_1^2 + k_2^2 > 1$ , or

$$f(s) = c_1 e^{\sqrt{\frac{5k_1^2 - k_2^2 + 1}{2}}s} + c_2 e^{-\sqrt{\frac{5k_1^2 - k_2^2 + 1}{2}}s} + \frac{\rho_1}{(5k_1^2 - k_2^2 + 1)\rho_2}$$

where  $-5k_1^2 + k_2^2 < 1$ .

In view of (3.8), the following theorem can be obtained:

**Theorem 3.13.** Let  $\alpha : I \rightarrow H^n(-1)$  be a differentiable curve parametrized by its arclength. Then,  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if

$$\begin{aligned} \rho_1 f' + \rho_2 (ff''' + f'f'') - 4\rho_2 f f' k_1^2 - 3\rho_2 f^2 k_1 k_1' &= 0 \\ \rho_1 f k_1 + \rho_2 k_1 (3ff'' + 2(f')^2) + 4\rho_2 f f' k_1' + \rho_2 f^2 (k_1'' - k_1^3 - k_1 k_2^2) - \rho_2 f^2 k_1 &= 0 \\ 4\rho_2 f f' k_1 k_2 + \rho_2 f^2 (2k_1' k_2 + k_1 k_2') &= 0 \end{aligned}$$

and

$$\rho_2 f^2 k_1 k_2 k_3 = 0$$

**Theorem 3.14.** Let  $\alpha : I \rightarrow H^n(-1)$  be a differentiable curve parametrized by its arclength. If  $k_1 = 0$ , then  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if  $\frac{\rho_1}{\rho_2} = -\frac{(ff'')'}{f'}$ . If  $k_1 \in \mathbb{R} \setminus \{0\}$  and  $k_2 = 0$ , then  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if

$$f(s) = \frac{\rho_1}{4\rho_2} s^2 + c_1 s + c_2$$

where  $k_1 = \pm \frac{1}{\sqrt{5}}$ , or

$$f(s) = c_1 \cos\left(\sqrt{\frac{5k_1^2 - 1}{2}}s\right) + c_2 \sin\left(\sqrt{\frac{5k_1^2 - 1}{2}}s\right) + \frac{\rho_1}{(5k_1^2 - 1)\rho_2}$$

where  $k_1 \in \left(-\infty, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$ , or

$$f(s) = c_1 e^{\sqrt{\frac{1-5k_1^2}{2}}s} + c_2 e^{-\sqrt{\frac{1-5k_1^2}{2}}s} + \frac{\rho_1}{(1-5k_1^2)\rho_2}$$

where  $k_1 \in \left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ .

If  $k_1, k_2 \in \mathbb{R} \setminus \{0\}$ , then  $\alpha$  is a type-1 interpolating sesqui- $f$ -harmonic curve if and only if

$$f(s) = \frac{\rho_1}{4\rho_2} s^2 + c_1 s + c_2$$

where  $5k_1^2 - k_2^2 = 1$ , or

$$f(s) = c_1 \cos\left(\sqrt{\frac{5k_1^2 - k_2^2 - 1}{2}}s\right) + c_2 \sin\left(\sqrt{\frac{5k_1^2 - k_2^2 - 1}{2}}s\right) + \frac{\rho_1}{(5k_1^2 - k_2^2 - 1)\rho_2}$$

where  $5k_1^2 - k_2^2 > 1$ , or

$$f(s) = c_1 e^{\sqrt{\frac{-5k_1^2 + k_2^2 + 1}{2}}s} + c_2 e^{-\sqrt{\frac{-5k_1^2 + k_2^2 + 1}{2}}s} + \frac{\rho_1}{(5k_1^2 - k_2^2 - 1)\rho_2}$$

where  $5k_1^2 - k_2^2 < 1$ .

#### 4. CONCLUSION

In this paper, we focus on type-1 interpolating sesqui- $f$ -harmonic maps. Moreover, we obtain necessary and sufficient conditions for a differentiable curve in a Riemannian space form to be type-1 interpolating sesqui- $f$ -harmonic. While the results establish a consistent theoretical framework for sesqui- $f$ -harmonic maps, further investigations may focus on different space forms and possible applications in mathematical physics and mechanics, following the geometric approaches discussed in this paper. Additionally, future studies can investigate similar intermediate concepts between  $f$ -harmonic and  $f$ -biharmonic maps on Riemannian manifolds.

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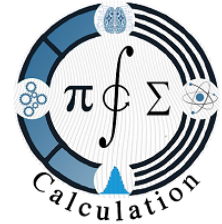
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## CLAIRAUT CONFORMAL HEMI-SLANT SUBMERSIONS FROM KÄHLER MANIFOLDS

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**Abstract.** In this paper, we introduce and study Clairaut conformal hemi-slant submersions from Kähler manifolds onto Riemannian manifolds. This class of maps combines the geometry of Clairaut conformal submersions with the hemi-slant decomposition of the vertical distribution in the almost Hermitian manifolds. We first establish a characterization theorem for Clairaut conformal hemi-slant submersions in terms of the geodesic behavior on the total manifold, the mean curvature of the fibers, and the behavior of the dilation along the fibers. We then derive equivalent formulations of the Clairaut condition adapted to the slant and anti-invariant components of the vertical distribution and obtain refined decompositions of the Clairaut relation and the harmonicity condition with respect to the hemi-slant splitting. Furthermore, we investigate the stability of the Clairaut conformal hemi-slant structure under conformal deformations of the total metric. We also study curvature properties of such submersions and obtain vertical sectional, scalar and Ricci curvature decomposition formulas compatible with the hemi-slant structure. In particular, the vertical curvature is decomposed into its slant, anti-invariant and mixed components, revealing the geometric influence of the hemi-slant splitting on the Clairaut and harmonic structures. Finally, we provide an explicit nontrivial example illustrating the theory.

**Keywords:** Kähler manifold, conformal hemi-slant submersion, Clairaut conformal submersion, Clairaut conformal hemi-slant submersion.

**2020 Mathematics Subject Classification:** 53C15, 53C43, 53C55, 53D15.

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### 1. INTRODUCTION

The study of smooth maps preserving geometric structures between Riemannian manifolds has long occupied a central position in differential geometry. Among such mappings, Riemannian submersions, introduced by O’Neill [23], provide a fundamental mechanism for relating the geometry of a manifold to that of its quotient through the interaction of vertical and horizontal distributions. O’Neill’s tensorial formalism and curvature identities [23] remain the basic tools in the subject and have been systematically developed in standard

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references such as [9, 24]. These mappings play an important role not only in global differential geometry but also in the study of foliations, fiber bundles and geometric reduction procedures.

A natural extension of Riemannian submersions is obtained by replacing the isometric condition on the horizontal distribution with a conformal one. This leads to the notion of horizontally conformal, or simply conformal, submersions. Such mappings arise naturally in the theory of harmonic morphisms and have been extensively studied from both geometric and analytic viewpoints [10]. In particular, the monograph of [5] shows that horizontally conformal submersions form the natural geometric background for harmonic morphisms between Riemannian manifolds. The geometry of conformal submersions is strongly influenced by the dilation function, whose behavior governs the interaction between the horizontal and vertical distributions and plays a decisive role in harmonicity and curvature questions [2, 3, 11, 12, 14].

On the other hand, submersion theory in the presence of an almost Hermitian structure has generated several important classes of geometrically distinguished mappings. The interaction between the almost complex structure and the vertical distribution gives rise to invariant, anti-invariant, slant, semi-invariant and semi-slant geometries. Anti-invariant, semi-invariant, slant and semi-slant constructions in almost Hermitian geometry were developed in the context of submersions and submanifolds by several authors, and their modern formulations in submersion theory may be traced through the works of Watson, Şahin, Park and Prasad, and others [34, 30, 31, 32, 25, 1, 29, 13]. In this setting, the vertical distribution carries additional geometric information inherited from the ambient complex structure, and this interaction has proved to be highly effective in the study of integrability, harmonicity and curvature properties.

Within this framework, hemi-slant submersions form a particularly natural class. Introduced in the almost Hermitian manifolds by [16], hemi-slant submersions are characterized by the orthogonal decomposition of the vertical distribution into a slant component and an anti-invariant component. This decomposition provides a natural common generalization of anti-invariant and slant geometries and yields a flexible setting in which the complex structure interacts nontrivially with the vertical bundle. The geometry of hemi-slant submersions and related distributions has subsequently been developed further in several directions, including hemi-slant submersions, conformal versions and harmonic aspects [33, 16, 17, 27]. In particular, conformal variants of complex-compatible submersions have been studied extensively and shown to produce rich geometric structures.

A different and equally influential direction in submersion theory originates in Clairaut's classical theorem for geodesics on surfaces of revolution. The extension of this idea to submersion theory was initiated by Bishop [7], who introduced Clairaut submersions and showed that the constancy of a suitable angular quantity along geodesics imposes strong geometric restrictions on the fibers. Since then, Clairaut-type conditions have become an important tool in understanding the relationship between the geodesic flow of the total manifold and the geometry of the submersion. In particular, Clairaut conditions encode a subtle compatibility between the geometry of the fibers and the behavior of geodesics.

In recent years, Clairaut-type structures have been studied intensively in broader settings, especially for conformal submersions. Meena and coauthors developed the theory of Clairaut

conformal submersions in a systematic way and obtained several characterization theorems in terms of totally umbilical fibers, mean curvature vector fields and dilation functions [19, 20, 22]. Their results show that the classical Clairaut relation extends naturally to the conformal category and remains closely tied to the geometry of the vertical foliation. Related developments for Clairaut-type conditions, conformal submersions and conformal maps in different geometric settings may also be found in [4, 21, 28, 26], where further curvature and structural consequences were investigated.

Although conformal submersions, Clairaut submersions and hemi-slant geometries have each been studied extensively, their simultaneous interaction has not yet been systematically examined. In particular, the effect of the hemi-slant splitting of the vertical distribution on the Clairaut condition remains essentially unexplored. This raises several natural questions: how does the decomposition

$$\ker \pi_* = D_\theta \oplus D^\perp$$

influence the Clairaut relation, how does the conformal factor interact with the slant and anti-invariant parts, and what additional curvature information can be extracted from this refined geometric structure?

The aim of the present paper is to initiate a systematic study of Clairaut conformal hemi-slant submersions from Kähler manifolds onto Riemannian manifolds. We introduce this class of maps and establish a characterization of the Clairaut condition in terms of the geodesic behavior on the total manifold, the geometry of the fibers, the behavior of the dilation function, and the hemi-slant decomposition of the vertical distribution. We then derive equivalent formulations of the Clairaut relation adapted to the slant and anti-invariant components of the vertical bundle and show that the hemi-slant splitting yields refined decompositions of both the curvature and harmonicity structures associated with the submersion. In particular, we obtain vertical sectional, scalar and Ricci curvature identities compatible with the decomposition of the vertical distribution into its slant, anti-invariant and mixed components, together with harmonicity criteria reflecting the contribution of each part of the hemi-slant splitting. These results show that the hemi-slant structure does not merely coexist with the Clairaut condition, but contributes directly to its geometric, curvature and harmonic properties.

Finally, we construct an explicit nontrivial example showing that the class introduced here is nonempty and geometrically natural.

## 2. PRELIMINARIES

Let  $(M^{2m}, g, J)$  be a Kähler manifold and  $(N^n, g_N)$  a Riemannian manifold. Then

$$J^2 = -I, \tag{2.1}$$

$$g(JE_1, JE_2) = g(E_1, E_2), \quad g(JE_1, E_2) = -g(E_1, JE_2), \tag{2.2}$$

and

$$(\nabla_{E_1} J)E_2 = 0 \tag{2.3}$$

for all  $E_1, E_2 \in \Gamma(TM)$ , where  $\nabla$  denotes the Levi-Civita connection of  $g$  [6, 15].

Let  $\pi : (M, g, J) \rightarrow (N, g_N)$  be a smooth submersion. We write

$$TM = \ker \pi_* \oplus (\ker \pi_*)^\perp \tag{2.4}$$

and denote the vertical and horizontal projections by  $\mathcal{V}$  and  $\mathcal{H}$ , respectively. The map  $\pi$  is called a horizontally conformal submersion if there exists a positive function  $\lambda$  on  $M$  such that

$$g_N(\pi_*X_1, \pi_*X_2) = \lambda^2g(X_1, X_2) \tag{2.5}$$

for all horizontal vector fields  $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$  [11, 5, 19]. The function  $\lambda$  is called the dilation of  $\pi$ . In particular,  $\pi$  is a Riemannian submersion whenever  $\lambda \equiv 1$  [23, 9].

A vector field  $E$  on  $M$  is called projectable if there exists a vector field  $\tilde{E}$  on  $N$  such that  $\pi_*E = \tilde{E} \circ \pi$  [9, 19]. A horizontal projectable vector field is called basic. For every vector field  $\tilde{Z}$  on  $N$ , there exists a unique basic vector field  $Z$  on  $M$  such that  $\pi_*Z = \tilde{Z} \circ \pi$  [9, 5].

Following O’Neill [23], we define tensor fields  $\mathcal{A}$  and  $\mathcal{T}$  by

$$\mathcal{A}_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}E_2, \tag{2.6}$$

$$\mathcal{T}_{E_1}E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}E_2 \tag{2.7}$$

for all  $E_1, E_2 \in \Gamma(TM)$ . Hence, for  $U_1, U_2 \in \Gamma(\ker \pi_*)$  and  $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$ , we have

$$\nabla_{U_1}U_2 = \mathcal{T}_{U_1}U_2 + \widehat{\nabla}_{U_1}U_2, \tag{2.8}$$

$$\nabla_{U_1}X_1 = \mathcal{H}\nabla_{U_1}X_1 + \mathcal{T}_{U_1}X_1, \tag{2.9}$$

$$\nabla_{X_1}U_1 = \mathcal{A}_{X_1}U_1 + \mathcal{V}\nabla_{X_1}U_1, \tag{2.10}$$

$$\nabla_{X_1}X_2 = \mathcal{H}\nabla_{X_1}X_2 + \mathcal{A}_{X_1}X_2, \tag{2.11}$$

where  $\widehat{\nabla}_{U_1}U_2 = \mathcal{V}\nabla_{U_1}U_2$  [23, 9, 19]. Moreover, for each  $p \in M$ ,  $U \in \ker \pi_{*p}$  and  $X \in (\ker \pi_{*p})^\perp$ , the endomorphisms  $\mathcal{T}_U$  and  $\mathcal{A}_X$  are skew-symmetric, that is,

$$g(\mathcal{A}_X E_1, E_2) = -g(E_1, \mathcal{A}_X E_2), \quad g(\mathcal{T}_U E_1, E_2) = -g(E_1, \mathcal{T}_U E_2) \tag{2.12}$$

for all  $E_1, E_2 \in T_pM$  [23, 9].

The fibers of  $\pi$  are said to be totally umbilical if there exists a horizontal vector field  $H$  such that

$$\mathcal{T}_{U_1}U_2 = g(U_1, U_2)H \tag{2.13}$$

for all  $U_1, U_2 \in \Gamma(\ker \pi_*)$  [5, 19]. The vector field  $H$  is the mean curvature vector field of the fibers and is given by

$$(2m - n)H = \sum_{i=1}^{2m-n} \mathcal{T}_{U_i}U_i, \tag{2.14}$$

where  $\{U_i\}_{i=1}^{2m-n}$  is a local orthonormal frame of  $\ker \pi_*$  [9, 19]. In particular, the fibers are minimal if and only if  $H = 0$  [5, 9].

For a conformal submersion, the tensor  $\mathcal{A}$  satisfies

$$\mathcal{A}_{X_1}X_2 = \frac{1}{2} \left\{ \mathcal{V}[X_1, X_2] - \lambda^2g(X_1, X_2) \nabla^\mathcal{V} \left( \frac{1}{\lambda^2} \right) \right\} \tag{2.15}$$

for all  $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$ , where  $\nabla^\mathcal{V}(1/\lambda^2) = \mathcal{V}(\text{grad}(1/\lambda^2))$  [19]. In particular,  $(\ker \pi_*)^\perp$  is totally geodesic if and only if  $\lambda$  is constant along the fibers.

The second fundamental form of  $\pi$  is defined by

$$(\nabla\pi_*)(E_1, E_2) = \nabla_{E_1}^\pi \pi_* E_2 - \pi_*(\nabla_{E_1} E_2), \quad (2.16)$$

where  $\nabla^\pi$  denotes the pull-back connection [5, 11]. If  $X_1, X_2$  are basic horizontal vector fields, then

$$\begin{aligned} \pi_*(\mathcal{H}\nabla_{X_1} X_2) = & \nabla_{\pi_* X_1}^N \pi_* X_2 + \frac{\lambda^2}{2} \left\{ X_1 \left( \frac{1}{\lambda^2} \right) \pi_* X_2 + X_2 \left( \frac{1}{\lambda^2} \right) \pi_* X_1 \right. \\ & \left. - g(X_1, X_2) \pi_* \left( \text{grad}^{\mathcal{H}} \frac{1}{\lambda^2} \right) \right\}. \end{aligned} \quad (2.17)$$

and consequently

$$(\nabla\pi_*)(X_1, X_2) = -\frac{\lambda^2}{2} \left\{ X_1 \left( \frac{1}{\lambda^2} \right) \tilde{X}_2 + X_2 \left( \frac{1}{\lambda^2} \right) \tilde{X}_1 - g(X_1, X_2) \pi_* \left( \text{grad}^{\mathcal{H}} \frac{1}{\lambda^2} \right) \right\}, \quad (2.18)$$

where  $\tilde{X}_i$  denotes the  $\pi$ -related vector field on  $N$  [19].

For later use, we also record

$$(\nabla\pi_*)(U_1, U_2) = -\pi_*(\mathcal{T}U_1 U_2), \quad (2.19)$$

$$(\nabla\pi_*)(X_1, U_1) = -\pi_*(\nabla_{X_1} U_1) = -\pi_*(\mathcal{A}_{X_1} U_1) \quad (2.20)$$

for  $X_1 \in \Gamma((\ker \pi_*)^\perp)$  and  $U_1, U_2 \in \Gamma(\ker \pi_*)$  [17].

Now we recall the notion of conformal hemi-slant submersion in the almost Hermitian manifolds [16]. A horizontally conformal submersion  $\pi : (M, g, J) \rightarrow (N, g_N)$  is called a conformal hemi-slant submersion if the vertical distribution  $\ker \pi_*$  admits two orthogonal complementary distributions  $D_\theta$  and  $D^\perp$  such that

$$\ker \pi_* = D_\theta \oplus D^\perp, \quad (2.21)$$

where  $D_\theta$  is a slant distribution with slant angle  $\theta$  and  $D^\perp$  is anti-invariant, that is,

$$J(D^\perp) \subset (\ker \pi_*)^\perp. \quad (2.22)$$

The angle  $\theta$  is called the hemi-slant angle. We say that  $\pi$  is proper if  $D^\perp \neq \{0\}$  and  $\theta \notin \{0, \frac{\pi}{2}\}$  [16].

For each  $U \in \Gamma(\ker \pi_*)$ , we write

$$U = PU + QU, \quad (2.23)$$

where  $PU \in \Gamma(D_\theta)$  and  $QU \in \Gamma(D^\perp)$ . Moreover, for  $U \in \Gamma(\ker \pi_*)$  and  $X \in \Gamma((\ker \pi_*)^\perp)$ , we set

$$JU = \phi U + \omega U, \quad (2.24)$$

$$JX = BX + CX, \quad (2.25)$$

where  $\phi U, BX \in \Gamma(\ker \pi_*)$  and  $\omega U, CX \in \Gamma((\ker \pi_*)^\perp)$  [16, 30, 31].

The horizontal distribution decomposes as

$$(\ker \pi_*)^\perp = \omega D_\theta \oplus JD^\perp \oplus \mu, \quad (2.26)$$

where  $\mu$  is the orthogonal complement of  $\omega D_\theta \oplus JD^\perp$  in  $(\ker \pi_*)^\perp$ ; moreover,  $\mu$  is  $J$ -invariant [16].

The following elementary relations will be used repeatedly:

$$\phi(D_\theta) = D_\theta, \quad \phi(D^\perp) = \{0\}, \quad B(\omega D_\theta) = D_\theta, \quad B(JD^\perp) = D^\perp. \tag{2.27}$$

Also, from (2.1), one obtains

$$\phi^2 U + B\omega U = -U, \tag{2.28}$$

$$\omega\phi U + C\omega U = 0, \tag{2.29}$$

$$\phi BX + BCX = 0, \tag{2.30}$$

$$\omega BX + C^2 X = -X \tag{2.31}$$

for all  $U \in \Gamma(\ker \pi_*)$  and  $X \in \Gamma((\ker \pi_*)^\perp)$  [16].

For the slant part, one has

$$\phi^2 W = -(\cos^2 \theta) W \tag{2.32}$$

for all  $W \in \Gamma(D_\theta)$  [16, 31]. Equivalently,

$$g(\phi W_1, \phi W_2) = \cos^2 \theta g(W_1, W_2), \quad g(\omega W_1, \omega W_2) = \sin^2 \theta g(W_1, W_2) \tag{2.33}$$

for all  $W_1, W_2 \in \Gamma(D_\theta)$  [16].

Finally, for  $f \in C^\infty(M)$ , the gradient, divergence and Laplacian are defined by

$$g(\text{grad } f, E) = E(f), \tag{2.34}$$

$$\text{div}(E) = \sum_{i=1}^{2m} g(\nabla_{E_i} E, E_i), \tag{2.35}$$

$$\Delta f = \text{div}(\text{grad } f), \tag{2.36}$$

where  $\{E_i\}_{i=1}^{2m}$  is a local orthonormal frame on  $M$  [18, 8].

### 3. CLAIRAUT CONFORMAL HEMI-SLANT SUBMERSIONS

In this section, we investigate Clairaut conformal hemi-slant submersions from Kähler manifolds onto Riemannian manifolds and establish their basic geometric characterizations.

**Definition 3.1.** *Let  $\pi : (M, g, J) \rightarrow (N, g_N)$  be a conformal hemi-slant submersion. Then  $\pi$  is called a Clairaut conformal hemi-slant submersion if there exists a positive smooth function  $r$  on  $M$  such that for every geodesic  $\alpha : I \rightarrow M$ , parametrized by arc length, the quantity*

$$(r \circ \alpha)(t) \sin \omega(t) \tag{3.37}$$

*is constant along  $\alpha$ , where  $\omega(t)$  denotes the angle between the tangent vector  $\dot{\alpha}(t)$  and the horizontal distribution  $(\ker \pi_*)^\perp$  at the point  $\alpha(t) \in M$ .*

We observe that Definition 3.1 is independent of the parametrization of the geodesic. Indeed, the angle  $\omega(t) \in [0, \pi/2]$  depends only on the direction of the velocity vector field  $\dot{\alpha}(t)$  relative to the horizontal distribution. Hence, in what follows, we may assume that  $\alpha$  is parametrized by arc length.

**Proposition 3.1.** *Let  $\pi : (M, g, J) \rightarrow (N, g_N)$  be a conformal hemi-slant submersion with dilation  $\lambda$ , and let  $\alpha : I \rightarrow M$  be a regular curve parametrized by arc length. Suppose that*

$$\dot{\alpha} = X + U, \tag{3.38}$$

where  $X \in \Gamma((\ker \pi_*)^\perp)$ ,  $U \in \Gamma(\ker \pi_*)$ . Then  $\alpha$  is a geodesic on  $M$  if and only if

$$\mathcal{V}\nabla_X U + \mathcal{T}_U X + \widehat{\nabla}_U U - \frac{\lambda^2}{2} g(X, X) \nabla^\mathcal{V} \left( \frac{1}{\lambda^2} \right) = 0 \quad (3.39)$$

and

$$\mathcal{H}\nabla_X X + 2\mathcal{A}_X U + \mathcal{T}_U U = 0. \quad (3.40)$$

*Proof.* Using (3.38), we compute

$$\nabla_{\dot{\alpha}} \dot{\alpha} = \nabla_X X + \nabla_X U + \nabla_U X + \nabla_U U. \quad (3.41)$$

By (2.11), (2.10), (2.9), and (2.8), we have

$$\begin{aligned} \nabla_{\dot{\alpha}} \dot{\alpha} &= (\mathcal{A}_X X + \mathcal{V}\nabla_X U + \mathcal{T}_U X + \widehat{\nabla}_U U) \\ &\quad + (\mathcal{H}\nabla_X X + \mathcal{A}_X U + \mathcal{H}\nabla_U X + \mathcal{T}_U U). \end{aligned} \quad (3.42)$$

Since  $\alpha$  is geodesic if and only if  $\nabla_{\dot{\alpha}} \dot{\alpha} = 0$ , the vertical and horizontal components of (3.42) must vanish. For the vertical component, using (2.15) with  $X_1 = X_2 = X$  and  $[X, X] = 0$ , we get

$$\mathcal{A}_X X = -\frac{\lambda^2}{2} g(X, X) \nabla^\mathcal{V} \left( \frac{1}{\lambda^2} \right). \quad (3.43)$$

Substituting (3.43) into the vertical component of (3.42), we obtain (3.39). For the horizontal component, by (2.9), we have  $\mathcal{H}\nabla_U X = \mathcal{A}_X U$ . Thus the horizontal component of (3.42) becomes (3.40). This completes the proof.  $\square$

**Theorem 3.1.** *Let  $\pi : (M, g, J) \rightarrow (N, g_N)$  be a conformal hemi-slant submersion with connected fibers and dilation  $\lambda$ . Then the following assertions are equivalent:*

- (1)  $\pi$  is a Clairaut conformal hemi-slant submersion with  $r = e^f$ .
- (2) For every geodesic  $\alpha : I \rightarrow M$  with (3.38), one has

$$g(U, U)g(\dot{\alpha}, \text{grad } f) + \frac{\lambda^2}{2} g\left(\nabla^\mathcal{V} \left( \frac{1}{\lambda^2} \right), U\right) g(X, X) + g(\mathcal{T}_U U, X) = 0. \quad (3.44)$$

- (3) Writing  $U = PU + QU$  as in (2.23), the identity

$$\begin{aligned} &g(\mathcal{T}_{PU} PU, X) + g(\mathcal{T}_{QU} QU, X) + 2g(\mathcal{T}_{PU} QU, X) \\ &\quad + \left( \sec^2 \theta g(\phi PU, \phi PU) + g(\omega QU, \omega QU) \right) g(\dot{\alpha}, \text{grad } f) \\ &\quad + \frac{\lambda^2}{2} g\left(\nabla^\mathcal{V} \left( \frac{1}{\lambda^2} \right), PU + QU\right) g(X, X) = 0 \end{aligned} \quad (3.45)$$

holds along every geodesic  $\alpha$ .

- (4)  $\text{grad } f$  is horizontal, the fibers of  $\pi$  are totally umbilical, and

$$H = -\text{grad } f, \quad (3.46)$$

while  $\lambda$  is constant along the fibers.

*Proof.* We first prove (i)  $\iff$  (ii). Let  $\alpha : I \rightarrow M$  be an arc-length geodesic and write  $\dot{\alpha}$  as in (3.38). Let  $\omega(t)$  be the angle between  $\dot{\alpha}$  and  $(\ker \pi_*)^\perp$ . Then

$$g(X, X) = \cos^2 \omega, \quad g(U, U) = \sin^2 \omega. \quad (3.47)$$

Using the vertical geodesic equation (3.39), together with (2.15), we get

$$g(U, U)g(\dot{\alpha}, \text{grad } f) = g(\mathcal{A}_X X + \mathcal{T}_U X, U) \tag{3.48}$$

if and only if

$$\frac{d}{dt}((e^f \circ \alpha) \sin \omega) = 0.$$

Indeed, differentiating  $g(U, U) = \sin^2 \omega$  and using (3.39) gives

$$g(\mathcal{A}_X X + \mathcal{T}_U X, U) = -\sin \omega \cos \omega \omega',$$

whereas the Clairaut relation gives

$$g(U, U)g(\dot{\alpha}, \text{grad } f) = -\sin \omega \cos \omega \omega'.$$

Thus the two identities are equivalent. Now, by (2.12),

$$g(\mathcal{T}_U X, U) = -g(\mathcal{T}_U U, X),$$

and by (2.15) with  $X_1 = X_2 = X$ ,

$$g(\mathcal{A}_X X, U) = -\frac{\lambda^2}{2}g(X, X)g\left(\nabla^\nu\left(\frac{1}{\lambda^2}\right), U\right).$$

Substituting these two identities into (3.48) gives precisely (3.44). Hence (i)  $\iff$  (ii).

Next we prove (ii)  $\iff$  (iii). Since the decomposition (2.21) is orthogonal and from (2.23), we have

$$g(U, U) = g(PU, PU) + g(QU, QU).$$

For  $PU \in D_\theta$ , (2.33) gives

$$g(PU, PU) = \sec^2 \theta g(\phi PU, \phi PU).$$

For  $QU \in D^\perp$ , (2.27), (2.24), and (2.2) give

$$g(QU, QU) = g(\omega QU, \omega QU).$$

Moreover, by bilinearity and symmetry of  $\mathcal{T}$  on vertical vector fields,

$$\mathcal{T}_U U = \mathcal{T}_{PU} PU + \mathcal{T}_{QU} QU + 2\mathcal{T}_{PU} QU.$$

Substituting these identities into (3.44) yields (3.45). Thus (ii)  $\iff$  (iii).

It remains to prove (ii)  $\iff$  (iv). Assume first that (3.44) holds. At a point  $p \in M$ , take a geodesic with initial vector  $U + X$ , where  $U$  is vertical and  $X$  is horizontal. Evaluating (3.44) at  $p$  gives

$$g(\mathcal{T}_U U, X) = -g(U, U)g(U + X, \text{grad } f) - \frac{\lambda^2}{2}g(X, X)g\left(\nabla^\nu\left(\frac{1}{\lambda^2}\right), U\right). \tag{3.49}$$

Taking  $X = 0$  in (3.49) gives  $g(U, \text{grad } f) = 0$  for every vertical  $U$ ; hence  $\text{grad } f$  is horizontal. Therefore (3.49) becomes

$$g(\mathcal{T}_U U, X) = -g(U, U)g(X, \text{grad } f) - \frac{\lambda^2}{2}g(X, X)g\left(\nabla^\nu\left(\frac{1}{\lambda^2}\right), U\right).$$

Replacing  $X$  by  $cX$  and comparing the coefficients of  $c$  and  $c^2$ , we obtain

$$g(\mathcal{T}_U U, X) = -g(U, U)g(X, \text{grad } f)$$

and

$$g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right) = 0.$$

Thus  $\lambda$  is constant along the fibers. Polarizing the first identity gives

$$g(\mathcal{T}_U V, X) = -g(U, V)g(X, \text{grad } f)$$

for all vertical  $U, V$  and horizontal  $X$ . Comparing this with (2.13), the fibers are totally umbilical and (3.46) hold. Conversely, assume (iv). Then (2.13) and (3.46) give

$$g(\mathcal{T}_U U, X) = -g(U, U)g(X, \text{grad } f).$$

Since  $\text{grad } f$  is horizontal, (3.38) implies

$$g(\dot{\alpha}, \text{grad } f) = g(X, \text{grad } f).$$

Since  $\lambda$  is constant along the fibers,

$$\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right) = 0.$$

Substitution into (3.44) gives zero identically. Hence (3.44) holds, and the proof is complete.  $\square$

**Corollary 3.1.** *Let  $\pi : (M, g, J) \rightarrow (N, g_N)$  be a Clairaut conformal hemi-slant submersion with connected fibers, dilation  $\lambda$ , hemi-slant angle  $\theta$ , and  $r = e^f$ . Define  $\psi : N \rightarrow (0, \infty)$  by*

$$\psi \circ \pi = \frac{1}{\lambda}. \quad (3.50)$$

*Then  $\pi : (M, g, J) \rightarrow (N, \psi^2 g_N)$  is a Clairaut Riemannian hemi-slant submersion with the same Clairaut function  $r = e^f$  and the same hemi-slant angle  $\theta$ .*

*Proof.* By Theorem 3.1,  $\lambda$  is constant along the fibers. Since the fibers are connected,  $\psi$  is well defined by (3.50). For  $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$ , (2.5) gives

$$(\psi^2 g_N)(\pi_* X_1, \pi_* X_2) = \frac{1}{\lambda^2} g_N(\pi_* X_1, \pi_* X_2) = g(X_1, X_2).$$

Thus  $\pi : (M, g, J) \rightarrow (N, \psi^2 g_N)$  is a Riemannian submersion. Since neither  $\pi_*$  nor the metric  $g$  on  $M$  is changed, the vertical distribution, the horizontal distribution, and the decomposition (2.21) are unchanged. Hence the anti-invariant condition (2.22) and the slant relation (2.33) remain valid with the same angle  $\theta$ . Therefore the resulting Riemannian submersion is hemi-slant. Finally, the Clairaut condition (3.37) is unchanged, since the geodesics of  $(M, g)$  and the angle  $\omega(t)$  with the horizontal distribution are not affected by the conformal change of the target metric. Hence the new Riemannian hemi-slant submersion is Clairaut with  $r = e^f$ .  $\square$

**Proposition 3.2.** *Let  $\pi : (M, g, J) \rightarrow (N, g_N)$  be a Clairaut conformal hemi-slant submersion and let  $\alpha : I \rightarrow M$  be a geodesic with decomposition (3.38). If  $U \in D_\theta$ , then along  $\alpha$  one has*

$$g(\mathcal{T}_U U, X) + \sec^2 \theta g(\phi U, \phi U) g(\dot{\alpha}, \nabla f) + \frac{\lambda^2}{2} g(X, X) g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right) = 0 \quad (3.51)$$

if and only if the Clairaut condition holds. Equivalently,

$$g(\mathcal{T}_U U, X) + \csc^2 \theta g(\omega U, \omega U) g(\dot{\alpha}, \nabla f) + \frac{\lambda^2}{2} g(X, X) g\left(\nabla^\nu \left(\frac{1}{\lambda^2}\right), U\right) = 0. \quad (3.52)$$

*Proof.* Since  $U \in D_\theta$ , the Clairaut condition (3.44) is

$$g(\mathcal{T}_U U, X) + g(U, U) g(\dot{\alpha}, \nabla f) + \frac{\lambda^2}{2} g(X, X) g\left(\nabla^\nu \left(\frac{1}{\lambda^2}\right), U\right) = 0.$$

Putting  $W_1 = W_2 = U$  in (2.33), we obtain

$$g(\phi U, \phi U) = \cos^2 \theta g(U, U), \quad g(\omega U, \omega U) = \sin^2 \theta g(U, U).$$

Hence

$$g(U, U) = \sec^2 \theta g(\phi U, \phi U), \quad g(U, U) = \csc^2 \theta g(\omega U, \omega U).$$

Substitution of these identities into (3.44) gives (3.51) and (3.52), respectively.  $\square$

**Corollary 3.2.** *Let  $\alpha : I \rightarrow M$  be a geodesic with decomposition (3.38). If  $U \in D^\perp$ , then the Clairaut condition (3.44) is equivalent to*

$$g(\mathcal{T}_U U, X) + g(\omega U, \omega U) g(\dot{\alpha}, \nabla f) + \frac{\lambda^2}{2} g(X, X) g\left(\nabla^\nu \left(\frac{1}{\lambda^2}\right), U\right) = 0. \quad (3.53)$$

*Proof.* Since  $U \in D^\perp$ , by (2.27) we have  $\phi U = 0$ . Thus, from (2.24), we get  $JU = \omega U$ . Using (2.2), it follows that

$$g(U, U) = g(JU, JU) = g(\omega U, \omega U).$$

Substituting this identity into (3.44), we obtain (3.53).  $\square$

**Remark 3.1.** *Proposition 3.2 and Corollary 3.2 show that the vertical contribution in the Clairaut condition is governed by different geometric quantities on the two summands of (2.21). On the slant distribution  $D_\theta$ , it is controlled by  $g(\phi U, \phi U)$  or equivalently by  $g(\omega U, \omega U)$  through the angle  $\theta$  via (2.33). On the anti-invariant distribution  $D^\perp$ , the same term is determined solely by  $g(\omega U, \omega U)$ , since  $\phi U = 0$ .*

**Theorem 3.2.** *Let  $\pi : (M, g, J) \rightarrow (N, g_N)$  be a Clairaut conformal hemi-slant submersion with connected fibers and  $r = e^f$ . Let  $\alpha : I \rightarrow M$  be a geodesic with decomposition (3.38) and (2.23). Then the Clairaut condition (3.44) is equivalent to*

$$\begin{aligned} &g(\mathcal{T}_{PU} PU, X) + g(\mathcal{T}_{QU} QU, X) + 2g(\mathcal{T}_{PU} QU, X) \\ &+ \left( \sec^2 \theta g(\phi PU, \phi PU) + g(\omega QU, \omega QU) \right) g(\dot{\alpha}, \nabla f) = 0. \end{aligned} \quad (3.54)$$

Equivalently,

$$\begin{aligned} &g(\mathcal{T}_{PU} PU, X) + g(\mathcal{T}_{QU} QU, X) + 2g(\mathcal{T}_{PU} QU, X) \\ &+ \left( \csc^2 \theta g(\omega PU, \omega PU) + g(\omega QU, \omega QU) \right) g(\dot{\alpha}, \nabla f) = 0. \end{aligned} \quad (3.55)$$

*Proof.* Since  $\pi$  is a Clairaut conformal hemi-slant submersion with connected fibers, Theorem 3.1 implies that  $\lambda$  is constant along the fibers. Hence

$$\nabla^\nu \left(\frac{1}{\lambda^2}\right) = 0. \quad (3.56)$$

From (3.44) and (3.56), we obtain

$$g(\mathcal{T}_U U, X) + g(U, U)g(\dot{\alpha}, \nabla f) = 0. \quad (3.57)$$

Now, bilinearity of  $\mathcal{T}$ , and the symmetry of  $\mathcal{T}$  on vertical vectors, we get

$$\begin{aligned} \mathcal{T}_U U &= \mathcal{T}_{PU+QU}(PU + QU) \\ &= \mathcal{T}_{PU}PU + \mathcal{T}_{PU}QU + \mathcal{T}_{QU}PU + \mathcal{T}_{QU}QU \\ &= \mathcal{T}_{PU}PU + 2\mathcal{T}_{PU}QU + \mathcal{T}_{QU}QU. \end{aligned} \quad (3.58)$$

Taking the inner product with  $X$ , we obtain

$$g(\mathcal{T}_U U, X) = g(\mathcal{T}_{PU}PU, X) + g(\mathcal{T}_{QU}QU, X) + 2g(\mathcal{T}_{PU}QU, X). \quad (3.59)$$

On the other hand, since the decomposition (2.21) is orthogonal,

$$g(U, U) = g(PU, PU) + g(QU, QU). \quad (3.60)$$

Because  $PU \in D_\theta$ , putting  $W_1 = W_2 = PU$  in (2.33), we have

$$g(\phi PU, \phi PU) = \cos^2 \theta g(PU, PU) \quad (3.61)$$

and

$$g(\omega PU, \omega PU) = \sin^2 \theta g(PU, PU). \quad (3.62)$$

Therefore

$$g(PU, PU) = \sec^2 \theta g(\phi PU, \phi PU) \quad (3.63)$$

and

$$g(PU, PU) = \csc^2 \theta g(\omega PU, \omega PU). \quad (3.64)$$

Since  $QU \in D^\perp$ , (2.27) gives

$$\phi QU = 0.$$

Hence, by (2.24),

$$JQU = \omega QU.$$

Using (2.2), we get

$$g(QU, QU) = g(\omega QU, \omega QU). \quad (3.65)$$

Substituting (3.63) and (3.65) into (3.60), we obtain

$$g(U, U) = \sec^2 \theta g(\phi PU, \phi PU) + g(\omega QU, \omega QU). \quad (3.66)$$

Similarly, from (3.64) and (3.65), we obtain

$$g(U, U) = \csc^2 \theta g(\omega PU, \omega PU) + g(\omega QU, \omega QU). \quad (3.67)$$

Finally, substituting (3.59) and (3.66) into (3.57), we obtain (3.54). Likewise, substituting (3.59) and (3.67) into (3.57), we obtain (3.55). This completes the proof.  $\square$

**Lemma 3.1.** *Let  $\pi : (M, g, J) \rightarrow (N, g_N)$  be a Clairaut conformal hemi-slant submersion with connected fibers, dilation  $\lambda$ , and  $r = e^f$ . Let  $\varphi$  be a positive smooth function on  $M$  such that*

$$\mathcal{V}(\text{grad } \varphi) = 0. \quad (3.68)$$

Then  $\tilde{\pi} : (M, \varphi^2 g, J) \rightarrow (N, g_N)$ ,  $\tilde{\pi}(x) = \pi(x)$ , is a Clairaut conformal hemi-slant submersion with dilation

$$\tilde{\lambda} = \frac{\lambda}{\varphi} \tag{3.69}$$

and Clairaut function

$$\tilde{r} = \varphi e^f. \tag{3.70}$$

*Proof.* Let  $\tilde{g} = \varphi^2 g$ . Since  $\tilde{\pi} = \pi$  as a smooth map, one has  $\ker \tilde{\pi}_* = \ker \pi_*$ . Because  $\tilde{g}$  is conformal to  $g$ , orthogonality is preserved, and hence the horizontal distribution  $(\ker \pi_*)^\perp$  is unchanged. Therefore the orthogonal decomposition (2.21) remains valid with respect to  $\tilde{g}$ . Moreover, since neither  $J$  nor the vertical distribution changes, the anti-invariant condition (2.22) is preserved. For  $W_1, W_2 \in \Gamma(D_\theta)$ , (2.33) yields

$$\tilde{g}(\phi W_1, \phi W_2) = \varphi^2 g(\phi W_1, \phi W_2) = \varphi^2 \cos^2 \theta g(W_1, W_2) = \cos^2 \theta \tilde{g}(W_1, W_2).$$

Hence the slant relation (2.33) is preserved with the same angle  $\theta$ . Thus  $\tilde{\pi}$  is again a conformal hemi-slant submersion.

Now let  $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$ . By (2.5),

$$g_N(\tilde{\pi}_* X_1, \tilde{\pi}_* X_2) = g_N(\pi_* X_1, \pi_* X_2) = \lambda^2 g(X_1, X_2) = \frac{\lambda^2}{\varphi^2} \tilde{g}(X_1, X_2).$$

Thus  $\tilde{\pi}$  is horizontally conformal with dilation (3.69).

By Theorem 3.1, the fibers are totally umbilical and satisfy (3.46). Under the conformal change  $\tilde{g} = \varphi^2 g$ , condition (3.68) implies

$$\tilde{H} = -\widetilde{\text{grad}}(f + \log \varphi).$$

Moreover, (3.68) and Theorem 3.1 imply that both  $\varphi$  and  $\lambda$  are constant along the fibers. Hence  $\tilde{\lambda} = \lambda/\varphi$  is constant along the fibers. Applying Theorem 3.1 once more, we conclude that  $\tilde{\pi}$  is Clairaut with

$$\tilde{r} = e^{f + \log \varphi} = \varphi e^f.$$

□

The following result refines the vertical curvature formula for Clairaut conformal submersions by taking into account the hemi-slant decomposition of the vertical distribution. While the global scalar curvature identity is the vertical trace of the usual Clairaut conformal submersion formula, the identities below split this trace into its slant, anti-invariant and mixed hemi-slant components.

**Theorem 3.3.** *Let  $\pi : (M, g, J) \rightarrow (N, g_N)$  be a Clairaut conformal hemi-slant submersion with connected fibers, dilation  $\lambda$ , and  $r = e^f$ . Put  $q = \dim(\ker \pi_*)$ ,  $p = \dim D_\theta$ ,  $s = \dim D^\perp$ , so that  $q = p + s$ . For any orthonormal vertical vector fields  $U, V \in \Gamma(\ker \pi_*)$ , one has*

$$K(U, V) = \widehat{K}(U, V) - \|\nabla f\|^2, \tag{3.71}$$

where  $K$  denotes the sectional curvature of  $M$  and  $\widehat{K}$  denotes the sectional curvature of the corresponding fiber. Consequently, with respect to the hemi-slant decomposition (2.21), the vertical scalar curvature splits as

$$K^\mathcal{V} = \widehat{K} - q(q - 1)\|\nabla f\|^2, \tag{3.72}$$

and more precisely,

$$K^{D_\theta} = \widehat{K}^{D_\theta} - p(p-1)\|\nabla f\|^2, \quad (3.73)$$

$$K^{D^\perp} = \widehat{K}^{D^\perp} - s(s-1)\|\nabla f\|^2, \quad (3.74)$$

$$K^{D_\theta, D^\perp} = \widehat{K}^{D_\theta, D^\perp} - 2ps\|\nabla f\|^2. \quad (3.75)$$

Hence

$$K^\mathcal{V} = K^{D_\theta} + K^{D^\perp} + K^{D_\theta, D^\perp}. \quad (3.76)$$

In particular, for every unit vector field  $W \in \Gamma(D_\theta)$  with  $\cos \theta \neq 0$ ,

$$K\left(W, \frac{1}{\cos \theta} \phi W\right) = \widehat{K}\left(W, \frac{1}{\cos \theta} \phi W\right) - \|\nabla f\|^2. \quad (3.77)$$

*Proof.* Since  $\pi$  is a Clairaut conformal hemi-slant submersion, Theorem 3.1 gives that  $\nabla f$  is horizontal, the fibers are totally umbilical, (3.46) holds, and  $\lambda$  is constant along the fibers. In particular, the conformal correction term in (2.15) vanishes for  $X_1 = X_2$  along the Clairaut fibers. By (2.13) and (3.46), for all vertical vector fields  $U, V \in \Gamma(\ker \pi_*)$  we have

$$\mathcal{T}_U V = -g(U, V)\nabla f. \quad (3.78)$$

Let  $U, V$  be orthonormal vertical vector fields. The Gauss equation for the fibers gives

$$K(U, V) = \widehat{K}(U, V) + g(\mathcal{T}_U V, \mathcal{T}_V U) - g(\mathcal{T}_U U, \mathcal{T}_V V). \quad (3.79)$$

Using (3.78), we obtain

$$\mathcal{T}_U V = 0, \quad \mathcal{T}_U U = -\nabla f, \quad \mathcal{T}_V V = -\nabla f.$$

Substitution into (3.79) gives (3.71). Now choose a local orthonormal frame adapted to (2.21),

$$\{E_1, \dots, E_p, Z_1, \dots, Z_s\},$$

where  $E_a \in \Gamma(D_\theta)$  and  $Z_r \in \Gamma(D^\perp)$ . Taking the trace of (3.71) over all ordered distinct vertical pairs gives (3.72). Restricting the same trace to the ordered pairs inside  $D_\theta$  gives (3.73), restricting it to the ordered pairs inside  $D^\perp$  gives (3.74), and taking the ordered mixed pairs  $(E_a, Z_r)$  and  $(Z_r, E_a)$  gives (3.75). Summing these three contributions yields (3.76). Finally, let  $W \in \Gamma(D_\theta)$  be unit. By (2.33),

$$g(\phi W, \phi W) = \cos^2 \theta.$$

Moreover, by (2.2),  $g(W, \phi W) = 0$ . Hence

$$\left\{W, \frac{1}{\cos \theta} \phi W\right\}$$

is an orthonormal pair in  $D_\theta$ . Applying (3.71) to this pair gives (3.77).  $\square$

**Corollary 3.3.** *Let  $\pi : (M, g, J) \rightarrow (N, g_N)$  be a Clairaut conformal hemi-slant submersion with connected fibers, dilation  $\lambda$ , and  $r = e^f$ . Let  $q = \dim(\ker \pi_*)$ ,  $p = \dim D_\theta$  and  $s = \dim D^\perp$ . Then, for any unit vector field  $E \in \Gamma(D_\theta)$ ,*

$$\text{Ric}^\mathcal{V}(E, E) = \widehat{\text{Ric}}(E, E) - (q-1)\|\nabla f\|^2. \quad (3.80)$$

More precisely,

$$\text{Ric}^{D_\theta}(E, E) = \widehat{\text{Ric}}^{D_\theta}(E, E) - (p - 1)\|\nabla f\|^2, \tag{3.81}$$

$$\text{Ric}^{D_\theta, D^\perp}(E, E) = \widehat{\text{Ric}}^{D_\theta, D^\perp}(E, E) - s\|\nabla f\|^2. \tag{3.82}$$

Similarly, for any unit vector field  $Z \in \Gamma(D^\perp)$ ,

$$\text{Ric}^V(Z, Z) = \widehat{\text{Ric}}(Z, Z) - (q - 1)\|\nabla f\|^2, \tag{3.83}$$

with the refined decomposition

$$\text{Ric}^{D^\perp}(Z, Z) = \widehat{\text{Ric}}^{D^\perp}(Z, Z) - (s - 1)\|\nabla f\|^2, \tag{3.84}$$

$$\text{Ric}^{D^\perp, D_\theta}(Z, Z) = \widehat{\text{Ric}}^{D^\perp, D_\theta}(Z, Z) - p\|\nabla f\|^2. \tag{3.85}$$

*Proof.* Let

$$\{E_1, \dots, E_p, Z_1, \dots, Z_s\}$$

be a local orthonormal frame adapted to the hemi-slant decomposition (2.21). By Theorem 3.3, for any orthonormal vertical pair  $U, V$ , we have

$$K(U, V) = \widehat{K}(U, V) - \|\nabla f\|^2.$$

Fix a unit vector field  $E \in \Gamma(D_\theta)$ . Taking the trace over all vertical basis vectors orthogonal to  $E$ , we obtain

$$\text{Ric}^V(E, E) = \widehat{\text{Ric}}(E, E) - (q - 1)\|\nabla f\|^2,$$

which gives (3.80). If the trace is restricted to the  $D_\theta$ -directions only, there are  $p - 1$  terms, and hence (3.81) follows. If the trace is taken over the  $D^\perp$ -directions, there are  $s$  terms, which gives (3.82).

The proof for a unit vector field  $Z \in \Gamma(D^\perp)$  is identical: tracing first over all vertical directions gives (3.83), tracing over  $D^\perp$  gives (3.84), and tracing over  $D_\theta$  gives (3.85).  $\square$

We first record the global harmonicity criterion in the present setting. This criterion is the Clairaut conformal submersion harmonicity condition specialized to the hemi-slant case. The contribution of the hemi-slant structure appears in the refined decomposition of the tension field given in the next theorem.

**Theorem 3.4.** *Let  $\pi : (M^{2m}, g, J) \rightarrow (N^n, g_N)$  be a Clairaut conformal hemi-slant submersion with connected fibers, dilation  $\lambda$ , and  $r = e^f$ . Put*

$$q = \dim(\ker \pi_*) = 2m - n.$$

*Then  $\pi$  is harmonic if and only if*

$$q \text{ grad } f = (n - 2) \text{ grad}^{\mathcal{H}}(\log \lambda). \tag{3.86}$$

*Proof.* Let  $\{U_1, \dots, U_q\}$  be a local orthonormal frame of  $\ker \pi_*$  and let  $\{X_1, \dots, X_n\}$  be a local orthonormal frame of  $(\ker \pi_*)^\perp$  consisting of basic vector fields. By (2.16), the tension field is

$$\tau(\pi) = \sum_{i=1}^q (\nabla \pi_*)(U_i, U_i) + \sum_{a=1}^n (\nabla \pi_*)(X_a, X_a).$$

For the vertical part, using (2.19) and (2.14), we obtain

$$\sum_{i=1}^q (\nabla \pi_*)(U_i, U_i) = -\pi_* \left( \sum_{i=1}^q \mathcal{T}_{U_i} U_i \right) = -q \pi_* H. \quad (3.87)$$

Since  $\pi$  is Clairaut, Theorem 3.1 gives (3.46). Hence

$$\sum_{i=1}^q (\nabla \pi_*)(U_i, U_i) = q \pi_*(\text{grad } f). \quad (3.88)$$

For the horizontal part, applying (2.17) with  $X_1 = X_2 = X_a$ , we have

$$\begin{aligned} \pi_*(\mathcal{H}\nabla_{X_a} X_a) &= \nabla_{\pi_* X_a}^N \pi_* X_a + \lambda^2 X_a \left( \frac{1}{\lambda^2} \right) \pi_* X_a \\ &\quad - \frac{\lambda^2}{2} \pi_* \left( \text{grad}^{\mathcal{H}} \frac{1}{\lambda^2} \right). \end{aligned} \quad (3.89)$$

On the other hand, by the definition of the second fundamental form (2.16),

$$(\nabla \pi_*)(X_a, X_a) = \nabla_{\pi_* X_a}^N \pi_* X_a - \pi_*(\nabla_{X_a} X_a).$$

Since  $\pi_*(\nabla_{X_a} X_a) = \pi_*(\mathcal{H}\nabla_{X_a} X_a)$ , it follows from (3.89) that

$$(\nabla \pi_*)(X_a, X_a) = -\lambda^2 X_a \left( \frac{1}{\lambda^2} \right) \pi_* X_a + \frac{\lambda^2}{2} \pi_* \left( \text{grad}^{\mathcal{H}} \frac{1}{\lambda^2} \right). \quad (3.90)$$

Equivalently, this is the special case  $X_1 = X_2 = X_a$  of (2.18). Summing (3.90) over  $a = 1, \dots, n$ , we get

$$\begin{aligned} \sum_{a=1}^n (\nabla \pi_*)(X_a, X_a) &= -\lambda^2 \sum_{a=1}^n X_a \left( \frac{1}{\lambda^2} \right) \pi_* X_a \\ &\quad + \frac{n\lambda^2}{2} \pi_* \left( \text{grad}^{\mathcal{H}} \frac{1}{\lambda^2} \right). \end{aligned} \quad (3.91)$$

By (2.34),

$$\text{grad}^{\mathcal{H}} \left( \frac{1}{\lambda^2} \right) = \sum_{a=1}^n X_a \left( \frac{1}{\lambda^2} \right) X_a.$$

Therefore

$$\sum_{a=1}^n X_a \left( \frac{1}{\lambda^2} \right) \pi_* X_a = \pi_* \left( \text{grad}^{\mathcal{H}} \frac{1}{\lambda^2} \right).$$

Substituting this into (3.91), we obtain

$$\sum_{a=1}^n (\nabla \pi_*)(X_a, X_a) = \frac{(n-2)\lambda^2}{2} \pi_* \left( \text{grad}^{\mathcal{H}} \frac{1}{\lambda^2} \right). \quad (3.92)$$

Since

$$\frac{\lambda^2}{2} \text{grad}^{\mathcal{H}} \left( \frac{1}{\lambda^2} \right) = -\text{grad}^{\mathcal{H}}(\log \lambda),$$

we get from (3.88) and (3.92)

$$\tau(\pi) = \pi_* (q \text{grad } f - (n-2) \text{grad}^{\mathcal{H}}(\log \lambda)).$$

By Theorem 3.1,  $\text{grad } f$  is horizontal. Hence both  $q \text{ grad } f$  and  $(n - 2) \text{ grad}^{\mathcal{H}}(\log \lambda)$  are horizontal vector fields. Since  $\pi_*$  is injective on  $(\ker \pi_*)^\perp$ , we conclude that

$$\tau(\pi) = 0$$

if and only if

$$q \text{ grad } f = (n - 2) \text{ grad}^{\mathcal{H}}(\log \lambda).$$

This proves (3.86). □

**Theorem 3.5.** *Let  $\pi : (M^{2m}, g, J) \rightarrow (N^n, g_N)$  be a Clairaut conformal hemi-slant submersion with connected fibers, dilation  $\lambda$ , and  $r = e^f$ . Put  $p = \dim D_\theta$ ,  $s = \dim D^\perp$ ,  $q = p + s$ . Let*

$$H_\theta = \frac{1}{p} \sum_{a=1}^p \mathcal{T}_{E_a} E_a, \quad H_\perp = \frac{1}{s} \sum_{r=1}^s \mathcal{T}_{Z_r} Z_r,$$

where  $\{E_1, \dots, E_p\}$  and  $\{Z_1, \dots, Z_s\}$  are local orthonormal frames of  $D_\theta$  and  $D^\perp$ , respectively. Then

$$H_\theta = H_\perp = -\text{grad } f. \tag{3.93}$$

Moreover, the tension field decomposes as

$$\tau(\pi) = p \pi_*(\text{grad } f) + s \pi_*(\text{grad } f) - (n - 2) \pi_*(\text{grad}^{\mathcal{H}} \log \lambda). \tag{3.94}$$

Consequently,  $\pi$  is harmonic if and only if

$$p \text{ grad } f + s \text{ grad } f = (n - 2) \text{ grad}^{\mathcal{H}} \log \lambda. \tag{3.95}$$

Equivalently,

$$q \text{ grad } f = (n - 2) \text{ grad}^{\mathcal{H}} \log \lambda. \tag{3.96}$$

*Proof.* Let  $\{E_1, \dots, E_p, Z_1, \dots, Z_s\}$  be a local orthonormal frame adapted to (2.21). Since  $\pi$  is Clairaut, Theorem 3.1 gives (3.46). Hence, by the totally umbilical condition (2.13),

$$\mathcal{T}_{E_a} E_a = g(E_a, E_a) H = H = -\text{grad } f$$

and

$$\mathcal{T}_{Z_r} Z_r = g(Z_r, Z_r) H = H = -\text{grad } f.$$

Taking traces over  $D_\theta$  and  $D^\perp$  separately gives

$$H_\theta = \frac{1}{p} \sum_{a=1}^p \mathcal{T}_{E_a} E_a = -\text{grad } f, \quad H_\perp = \frac{1}{s} \sum_{r=1}^s \mathcal{T}_{Z_r} Z_r = -\text{grad } f,$$

which proves (3.93).

Now, using (2.19), we split the vertical part of the tension field as

$$\begin{aligned} \sum_{i=1}^q (\nabla \pi_*)(U_i, U_i) &= \sum_{a=1}^p (\nabla \pi_*)(E_a, E_a) + \sum_{r=1}^s (\nabla \pi_*)(Z_r, Z_r) \\ &= -\pi_* \left( \sum_{a=1}^p \mathcal{T}_{E_a} E_a \right) - \pi_* \left( \sum_{r=1}^s \mathcal{T}_{Z_r} Z_r \right) \\ &= p \pi_*(\text{grad } f) + s \pi_*(\text{grad } f). \end{aligned} \tag{3.97}$$

On the other hand, the horizontal trace computed in Theorem 3.4 gives

$$\sum_{\alpha=1}^n (\nabla \pi_*)(X_\alpha, X_\alpha) = -(n-2)\pi_*(\text{grad}^{\mathcal{H}} \log \lambda). \quad (3.98)$$

Combining (3.97) and (3.98) yields (3.94). Since  $\text{grad} f$  is horizontal by Theorem 3.1, and  $\text{grad}^{\mathcal{H}} \log \lambda$  is horizontal, the injectivity of  $\pi_*$  on  $(\ker \pi_*)^\perp$  gives (3.95).  $\square$

As immediate consequences of the global harmonicity criterion, we obtain the following special cases. They are included here for completeness and to clarify the behavior of the Clairaut function in the hemi-slant setting.

**Corollary 3.4.** *Let  $\pi : (M^{2m}, g, J) \rightarrow (N^2, g_N)$  be a Clairaut conformal hemi-slant submersion with connected fibers and  $r = e^f$ . Then  $\pi$  is harmonic if and only if  $\text{grad} f = 0$ . In particular,  $\pi$  is harmonic if and only if the Clairaut function  $r$  is constant.*

**Corollary 3.5.** *Let  $\pi : (M^{2m}, g, J) \rightarrow (N^n, g_N)$  be a Clairaut conformal hemi-slant submersion with connected fibers and constant dilation  $\lambda$ . Then  $\pi$  is harmonic if and only if  $\text{grad} f = 0$ . Equivalently, a homothetic Clairaut conformal hemi-slant submersion is harmonic if and only if its fibers are minimal.*

**Corollary 3.6.** *Let  $\pi : (M^{2m}, g, J) \rightarrow (N^n, g_N)$  be a harmonic Clairaut conformal hemi-slant submersion with connected fibers, dilation  $\lambda$ , and  $r = e^f$ . Then*

$$q \Delta f = (n-2) \text{div}(\text{grad}^{\mathcal{H}} \log \lambda), \quad (3.99)$$

where  $q = \dim(\ker \pi_*) = 2m - n$ .

*Proof.* Since  $\pi$  is harmonic, Theorem 3.4 gives  $q \text{grad} f = (n-2) \text{grad}^{\mathcal{H}} \log \lambda$ . Taking divergence on both sides and using (2.35), we obtain

$$q \text{div}(\text{grad} f) = (n-2) \text{div}(\text{grad}^{\mathcal{H}} \log \lambda).$$

By (2.36), this gives (3.99).  $\square$

**Example 3.1.** *Let  $M = \mathbb{R}^6 \setminus \{\rho = 0\}$  with standard coordinates  $(x_1, y_1, x_2, y_2, x_3, y_3)$  and standard Euclidean metric  $g$ . Let  $J$  be the standard Kähler structure defined by*

$$J \frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \quad J \frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}, \quad i = 1, 2, 3.$$

Fix  $0 < \theta < \frac{\pi}{2}$  and define

$$\xi_1 = -\sin \theta y_1 + \cos \theta x_2, \quad \xi_2 = y_2, \quad \xi_3 = y_3,$$

with  $\rho^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$ . Let  $(N, g_N) = (\mathbb{R}^3, g_{\mathbb{R}^3})$  and define

$$\pi : M \rightarrow N, \quad \pi = \frac{1}{\rho^2}(\xi_1, \xi_2, \xi_3).$$

Put

$$E_1 = \frac{\partial}{\partial x_1}, \quad E_2 = \frac{\partial}{\partial y_1}, \quad E_3 = \frac{\partial}{\partial x_2}, \quad E_4 = \frac{\partial}{\partial y_2}, \quad E_5 = \frac{\partial}{\partial x_3}, \quad E_6 = \frac{\partial}{\partial y_3}.$$

Since the Euclidean inversion

$$F(\xi) = \frac{\xi}{\|\xi\|^2}$$

is a local diffeomorphism on  $\mathbb{R}^3 \setminus \{0\}$ , the vertical distribution of  $\pi = F \circ (\xi_1, \xi_2, \xi_3)$  coincides with the kernel of  $d(\xi_1, \xi_2, \xi_3)$ . Therefore

$$\ker \pi_* = \text{span}\{U_1, U_2, U_3\},$$

where

$$U_1 = E_1, \quad U_2 = \cos \theta E_2 + \sin \theta E_3, \quad U_3 = E_5.$$

An orthonormal frame of the horizontal distribution is

$$X_1 = -\sin \theta E_2 + \cos \theta E_3, \quad X_2 = E_4, \quad X_3 = E_6.$$

Indeed,

$$X_i(\xi_j) = \delta_{ij}, \quad i, j = 1, 2, 3.$$

Let

$$D_\theta = \text{span}\{U_1, U_2\}, \quad D^\perp = \text{span}\{U_3\}.$$

Then

$$\ker \pi_* = D_\theta \oplus D^\perp.$$

Moreover,

$$JU_1 = E_2 = \cos \theta U_2 - \sin \theta X_1,$$

and

$$JU_2 = J(\cos \theta E_2 + \sin \theta E_3) = -\cos \theta U_1 + \sin \theta X_2.$$

Thus, for every nonzero  $U \in D_\theta$ , the angle between  $JU$  and  $D_\theta$  is constant and equal to  $\theta$ . Hence  $D_\theta$  is a slant distribution with slant angle  $\theta$ . Also,

$$JU_3 = JE_5 = E_6 = X_3 \in (\ker \pi_*)^\perp,$$

so  $D^\perp$  is anti-invariant. Since  $0 < \theta < \frac{\pi}{2}$  and  $D^\perp \neq \{0\}$ ,  $\pi$  is proper hemi-slant. We now verify the conformality. The inversion  $F(\xi) = \xi/\|\xi\|^2$  satisfies

$$\langle dF_\xi(a), dF_\xi(b) \rangle = \frac{1}{\rho^4} \langle a, b \rangle.$$

Since  $X_i(\xi_j) = \delta_{ij}$ , for horizontal vector fields  $X, Y$  we obtain

$$g_N(\pi_*X, \pi_*Y) = \frac{1}{\rho^4} g(X, Y).$$

Therefore  $\pi$  is a conformal hemi-slant submersion with dilation  $\lambda = \frac{1}{\rho^2}$ . Since  $\lambda$  is not constant on  $M$ ,  $\pi$  is not a Riemannian submersion. Next, the fibers of  $\pi$  coincide with the affine fibers of the linear map

$$(x_1, y_1, x_2, y_2, x_3, y_3) \mapsto (\xi_1, \xi_2, \xi_3).$$

Hence the fibers are totally geodesic in the Euclidean space, and so  $\mathcal{T}_U V = 0$  for all vertical vector fields  $U, V$ . Thus  $H = 0$ . Taking  $f = 0$ ,  $r = e^f = 1$ , we have  $H = 0 = -\text{grad } f$ . Moreover,

$$\frac{1}{\lambda^2} = \rho^4$$

depends only on the horizontal variables  $\xi_1, \xi_2, \xi_3$ . Since

$$U_i(\xi_j) = 0, \quad i = 1, 2, 3, \quad j = 1, 2, 3,$$

we get

$$U_i(\rho^4) = 0.$$

Therefore (3.56) holds. By Theorem 3.1,  $\pi$  is a Clairaut conformal hemi-slant submersion with Clairaut function  $r = 1$ . Finally, we check the harmonicity. Since  $q = \dim(\ker \pi_*) = 3$  and  $n = 3$ , Theorem 3.4 gives

$$3 \operatorname{grad} f = \operatorname{grad}^{\mathcal{H}}(\log \lambda).$$

Here  $\operatorname{grad} f = 0$ , while

$$\log \lambda = -2 \log \rho.$$

Hence  $\operatorname{grad}^{\mathcal{H}}(\log \lambda)$  is not identically zero. Therefore  $\pi$  is not harmonic. Thus this example provides a non-harmonic, non-Riemannian, proper Clairaut conformal hemi-slant submersion. Since  $f = 0$ , the curvature formula in Theorem 3.3 reduces to  $K^{\mathcal{V}} = \widehat{K}$ . Indeed, the fibers are affine Euclidean subspaces, so both  $K^{\mathcal{V}}$  and  $\widehat{K}$  vanish.

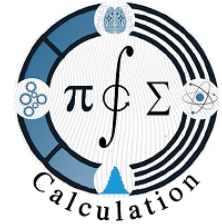
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SECOND VARIATION OF  $F$ -EINSTEIN-HILBERT FUNCTIONAL

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**Abstract.** This article describes a formula for second variation of generalized Einstein-Hilbert functional on Riemannian manifolds. This work extends the definition of stable Einstein manifolds, and we present some properties.

**Keywords:** Einstein manifolds, Einstein-Hilbert functional.

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1. INTRODUCTION

The Einstein-Hilbert functional  $\mathcal{E}$  associates to each Riemannian metric  $g$  the integral of its scalar curvature  $S$ , that is

$$\mathcal{E} : \mathcal{M} \longrightarrow \mathbb{R}, \quad g \longmapsto \mathcal{E}(g) = \int_M S v^g, \tag{1.1}$$

where  $\mathcal{M}$  is the set of smooth Riemannian metrics on  $M$ , and  $v^g$  the volume form with respect to  $g$ . It is the action functional that defines the dynamics of gravity in general relativity [3, 4, 5, 6, 8, 16].

One of the simplest modifications to general relativity is the  $F(S)$  gravity in which the Lagrangian density  $F$  is an arbitrary smooth function of the scalar curvature  $S$  of a Riemannian manifold  $(M, g)$ . When  $F(s) = s$ , gives the classical Einstein-Hilbert functional, therefore the Einstein gravity, corresponds to  $F(S) = S$ . The Euler-Lagrange equation of the generalized Einstein-Hilbert functional (it is known by Einstein-Hilbert functional in  $f(R)$  gravity, or briefly  $F$ -Einstein-Hilbert functional) with respect to  $g$  is proved by A. D. Felice, S. Tsujikawa in [7], and T. P. Sotiriou, V. Faraoni in [17].

The second variation of Einstein-Hilbert functional at Einstein metrics was considered in [11]. In [9], K. Kröncke study the second variation of the Einstein-Hilbert functional on Einstein metrics, he find some conditions for stability of Einstein manifolds with respect to the Einstein-Hilbert functional, i.e., that the second variation of the Einstein-Hilbert functional at the metric is nonpositive in the direction of transverse-traceless tensors. Stability properties of compact Riemannian Einstein manifold play a role in mathematical general relativity [1], and in geometric analysis to understand rigidity of Riemannian structures, for example

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the dynamical behaviour of the Ricci flow.

In this paper, we extend the definition of the Einstein tensor, where we calculate the first variation of the  $F$ -Einstein-Hilbert functional, and we conclude the generalized Einstein tensor. We prove that the generalized Einstein tensor is divergence-free. We study the second variation of the  $F$ -Einstein-Hilbert functional on the Riemannian manifold. The second variation formula gives a tool/is a prerequisite for the study the stability of any generalized Einstein manifold, and to see if the  $F$ -Einstein-Hilbert functional has extremality properties at some critical points. The smooth function  $F$  can be chosen for the existence and the stability of such Riemannian metrics which provide additional information on Riemannian manifolds.

## 2. $F$ -EINSTEIN-HILBERT FUNCTIONAL

First, we give some definitions. Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold, and let  $X, X_1, \dots, X_{q-1}, Y, Z \in \Gamma(TM)$ . By  $R$ , Ric and  $S$  we denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of  $(M, g)$ . Thus  $R$ , Ric and  $S$  are defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \tag{2.2}$$

$$\text{Ric}(X, Y) = g(R(X, e_i)e_i, Y), \quad S = \text{Ric}(e_i, e_i), \tag{2.3}$$

where  $\nabla$  is the Levi-Civita connection with respect to  $g$ ,  $\{e_1, \dots, e_n\}$  is an orthonormal frame. Given a smooth function  $f$  on  $M$ , the gradient of  $f$  is defined by

$$g(\text{grad } f, X) = X(f), \tag{2.4}$$

the Hessian of  $f$  is defined by

$$(\text{Hess } f)(X, Y) = g(\nabla_X \text{grad } f, Y), \tag{2.5}$$

the Laplacian of  $f$  is defined by

$$\Delta f = -\text{Tr}(\text{Hess } f). \tag{2.6}$$

The divergence of  $(0, q)$ -tensor  $\alpha$  on  $M$  is defined by

$$(\delta\alpha)(X_1, \dots, X_{q-1}) = -(\nabla_{e_i}\alpha)(e_i, X_1, \dots, X_{q-1}). \tag{2.7}$$

The formal adjoint of the divergence  $\delta : \Gamma(\otimes^2 T^*M) \rightarrow \Gamma(T^*M)$  is the map  $\delta^* : \Gamma(T^*M) \rightarrow \Gamma(\otimes^2 T^*M)$  defined by

$$(\delta^*\alpha)(X, Y) = \frac{1}{2}((\nabla_X\alpha)Y + (\nabla_Y\alpha)X). \tag{2.8}$$

The formal adjoint of the Levi-civita connection  $\nabla$  is given by

$$(\nabla^*\alpha)(X_1, \dots, X_{q-1}) = -(\nabla_{e_i}\alpha)(e_i, X_1, \dots, X_{q-1}), \tag{2.9}$$

where  $\alpha \in \Gamma(T^*M \otimes T^{(p,q)}M)$ , and  $\{e_1, \dots, e_n\}$  is an orthonormal frame. The composition of  $T, Q \in \Gamma(\odot^2 T^*M)$  is defined by

$$(T \circ Q)(X, Y) = T(X, e_i)Q(Y, e_i), \tag{2.10}$$

where  $\{e_1, \dots, e_n\}$  is an orthonormal frame on  $M$ .

For  $T \in \Gamma(\otimes^2 T^*M)$ , we define  $T^\sigma \in \Gamma(\odot^2 T^*M)$  by

$$T^\sigma(X, Y) = \frac{1}{2}(T(X, Y) + T(Y, X)). \quad (2.11)$$

We define an endomorphism  $\overset{\circ}{R} : \Gamma(\odot^2 T^*M) \rightarrow \Gamma(\odot^2 T^*M)$  by

$$(\overset{\circ}{R}T)(X, Y) = T(R(e_i, X)Y, e_i). \quad (2.12)$$

For  $T \in \Gamma(\odot^2 T^*M)$ , we define the Lichnerowicz Laplacian by

$$\Delta_L T = \nabla^* \nabla T + \text{Ric} \circ T + T \circ \text{Ric} - 2\overset{\circ}{R}T. \quad (2.13)$$

(For more details, see for example [4], [15]).

**Definition 2.1** ([7], [17]). *We let  $\mathcal{M}$  denote the space of Riemannian metrics on a closed orientable manifold  $M$ . The generalized Einstein-Hilbert functional (or  $F$ -Einstein-Hilbert functional) is defined by*

$$\mathcal{E}_F : \mathcal{M} \rightarrow \mathbb{R}, \quad g \mapsto \mathcal{E}_F(g) = \int_M F(S)v^g, \quad (2.14)$$

where  $S$  is the scalar curvature of  $(M, g)$ , and  $F : \mathbb{R} \rightarrow \mathbb{R}$  is a non-constant smooth function.

The Definition 2.1, is a natural generalization of Einstein-Hilbert functional or the total scalar curvature, when  $F$  is the identity map, then  $\mathcal{E}_F$  reduces to the usual Einstein-Hilbert functional whose second order infinitesimal behaviour is well understood (see [3, 4, 5, 6, 8, 10, 11, 12, 13, 16]).

Let  $(M, g)$  be a closed orientable Riemannian manifold. Consider a smooth one-parameter variation of the metric  $g$ , i.e., a smooth family of metrics  $(g_t)$  with  $-\epsilon < t < \epsilon$ , such that  $g_0 = g$ . Take local coordinates  $(x^i)$  on  $M$ , and write the metric on  $M$  in the usual way as  $g_t = g_{i,j}(t, x)dx^i \otimes dx^j$ . Write  $h = (\partial g_t / \partial t)_{t=0}$ , then  $h \in \Gamma(\odot^2 T^*M)$  is a symmetric 2-covariant tensor field on  $M$ , we get the following.

**Theorem 2.1** ([7], [17]). *The first variation of the  $F$ -Einstein-Hilbert functional in the direction of  $h$  is given by the formula*

$$\left. \frac{d}{dt} \mathcal{E}_F(g_t) \right|_{t=0} = - \int_M \langle E_F(g), h \rangle v^g, \quad (2.15)$$

where  $\langle \cdot, \cdot \rangle$  is the induced Riemannian metric on  $\otimes^2 T^*M$ ,

$$E_F(g) = F'(S) \text{Ric} - \text{Hess } F'(S) - (\Delta F'(S) + \frac{1}{2} F'(S))g, \quad (2.16)$$

and  $F'$  is the derivative of the function  $F$ .

**Definition 2.2.**  $E_F(g)$  is called the generalized Einstein tensor (or  $F$ -Einstein tensor).

For the proof of Theorem 2.1, we need the following lemma.

**Lemma 2.1** ([14], [18]). *Let  $(M, g)$  be a Riemannian manifold. Then, the differential at  $g$ , in the direction of  $h$ , of the volume element and the scalar curvature are given by the following formulas*

$$\left. \frac{\partial v^{g_t}}{\partial t} \right|_{t=0} = \frac{1}{2} (\text{Tr } h) v^g = \frac{1}{2} \langle g, h \rangle v^g, \quad (2.17)$$

$$\left. \frac{\partial S_t}{\partial t} \right|_{t=0} = \Delta(\text{Tr } h) + \delta(\delta h) - \langle \text{Ric}, h \rangle. \quad (2.18)$$

*Proof of Theorem 2.1.* First note that

$$\left. \frac{d}{dt} \mathcal{E}_F(g_t) \right|_{t=0} = \int_M \left[ \frac{\partial F(S_t)}{\partial t} v^{g_t} + F(S_t) \frac{\partial v^{g_t}}{\partial t} \right]_{t=0}, \quad (2.19)$$

for all  $t \in (-\epsilon, \epsilon)$ , we have

$$\frac{\partial F(S_t)}{\partial t} = F'(S_t) \frac{\partial S_t}{\partial t},$$

by the Lemma 2.1, we obtain

$$\begin{aligned} \left. \frac{\partial F(S_t)}{\partial t} \right|_{t=0} &= F'(S) \Delta(\text{Tr } h) + F'(S) \delta(\delta h) \\ &\quad - F'(S) \langle \text{Ric}, h \rangle. \end{aligned} \quad (2.20)$$

Calculating in a normal frame at  $x \in M$  we have

$$\begin{aligned} F'(S) \Delta(\text{Tr } h) &= -F'(S) e_i(e_i(\text{Tr } h)) \\ &= -e_i(F'(S) e_i(\text{Tr } h)) + e_i(F'(S)) e_i(\text{Tr } h) \\ &= -e_i(F'(S) e_i(\text{Tr } h)) + e_i(e_i(F'(S)) \text{Tr } h) \\ &\quad - e_i(e_i(F'(S))) \text{Tr } h, \end{aligned} \quad (2.21)$$

so, the first term in the right-hand side of (2.20), is given by

$$\begin{aligned} F'(S) \Delta(\text{Tr } h) &= \delta(F'(S) d(\text{Tr } h)) - \delta((\text{Tr } h) dF'(S)) \\ &\quad + \Delta(F'(S)) \langle g, h \rangle. \end{aligned} \quad (2.22)$$

If  $f \in C^\infty(M)$  and  $\alpha \in \Gamma(T^*M)$ , then (see [18], [15])

$$\delta(f\alpha) = -\langle df, \alpha \rangle + f\delta\alpha, \quad (2.23)$$

with  $\langle df, \alpha \rangle = \alpha(\text{grad } f)$ . Applying this formula, gives

$$F'(S) \delta(\delta h) = \delta(F'(S) \delta h) + \langle dF'(S), \delta h \rangle, \quad (2.24)$$

by using the following formula (see [18])

$$(\delta T)(Z) = \delta(T(\cdot, Z)) + \frac{1}{2} \langle T, \mathcal{L}_Z g \rangle, \quad (2.25)$$

where  $\mathcal{L}_Z g$  is the Lie-derivative of  $g$  along  $Z \in \Gamma(TM)$  (see [15]), and  $T \in \Gamma(\odot^2 T^*M)$ , we get

$$\begin{aligned} \langle dF'(S), \delta h \rangle &= (\delta h)(\text{grad } F'(S)) \\ &= \delta(h(\cdot, \text{grad } F'(S))) + \frac{1}{2} \langle h, \mathcal{L}_{\text{grad } F'(S)} g \rangle \\ &= \delta(h(\cdot, \text{grad } F'(S))) + \langle h, \text{Hess } F'(S) \rangle, \end{aligned} \quad (2.26)$$

by equations (2.24) and (2.26), the second term on the left-hand side of (2.20) is

$$\begin{aligned} F'(S) \delta(\delta h) &= \delta(F'(S) \delta h) + \delta(h(\cdot, \text{grad } F'(S))) \\ &\quad + \langle h, \text{Hess } F'(S) \rangle. \end{aligned} \quad (2.27)$$

Substituting (2.22) and (2.27) in (2.20), we obtain

$$\begin{aligned} \left. \frac{\partial F(S_t)}{\partial t} \right|_{t=0} &= \delta(F'(S)d(\text{Tr } h)) - \delta((\text{Tr } h)dF'(S)) \\ &\quad + \Delta(F'(S))\langle g, h \rangle + \delta(F'(S)\delta h) \\ &\quad + \delta(h(\cdot, \text{grad } F'(S))) + \langle h, \text{Hess } F'(S) \rangle \\ &\quad - F'(S)\langle \text{Ric}, h \rangle. \end{aligned} \quad (2.28)$$

From equation (2.28) and the Lemma 2.1, we have

$$\begin{aligned} \left[ \frac{\partial F(S_t)}{\partial t} v^{gt} + F(S_t) \frac{\partial v^{gt}}{\partial t} \right]_{t=0} &= \left\{ \delta(F'(S)d(\text{Tr } h)) - \delta((\text{Tr } h)dF'(S)) \right. \\ &\quad + \Delta(F'(S))\langle g, h \rangle + \delta(F'(S)\delta h) \\ &\quad + \delta(h(\cdot, \text{grad } F'(S))) + \langle h, \text{Hess } F'(S) \rangle \\ &\quad \left. - F'(S)\langle \text{Ric}, h \rangle \right\} v^g + \frac{F(S)}{2} \langle g, h \rangle v^g. \end{aligned} \quad (2.29)$$

Substituting the formula (2.29) in (2.19), and consider the divergence theorem (see [2]), the Theorem 2.1 follows.  $\square$

**Remark 2.1.** Let  $X, Y \in \Gamma(TM)$ , we have

$$\begin{aligned} \text{Hess } F'(S)(X, Y) &= X(Y(F'(S))) - (\nabla_X Y)(F'(S)) \\ &= X(F''(S)Y(S)) - F''(S)(\nabla_X Y)(S) \\ &= X(F''(S))Y(S) + F''(S)X(Y(S)) - F''(S)(\nabla_X Y)(S) \\ &= F'''(S)X(S)Y(S) + F''(S)(\text{Hess } S)(X, Y). \end{aligned}$$

According to this formula, the  $F$ -Einstein tensor is given by

$$\begin{aligned} E_F(g) &= F'(S) \text{Ric} - F''(S) \text{Hess } S - F'''(S) dS \otimes dS \\ &\quad - (F''(S) \Delta S + F'''(S) |\text{grad } S|^2 + \frac{1}{2} F(S)) g. \end{aligned} \quad (2.30)$$

**Remark 2.2.** Let  $(M, g)$  be a Riemannian manifold, we get the following

- If  $F(s) = s$ , for all  $s \in \mathbb{R}$ , the  $F$ -Einstein tensor is given by the formula (see [4], [14])

$$E_F(g) = E(g) = \text{Ric} - \frac{S}{2} g, \quad (2.31)$$

is the Einstein tensor.

- If  $F(s) = s^2$ , for all  $s \in \mathbb{R}$ , the  $F$ -Einstein tensor is given by (see [4], [5], [6])

$$E_F(g) = 2S \text{Ric} - 2 \text{Hess } S - (2\Delta S + \frac{S^2}{2}) g. \quad (2.32)$$

From Theorem 2.1, we deduce.

**Theorem 2.2** ([7], [17]). A Riemannian metric  $g$  is a critical point of the  $F$ -Einstein-Hilbert functional if and only if

$$F'(S) \text{Ric} - \text{Hess } F'(S) - (\Delta F'(S) + \frac{1}{2} F(S)) g = 0, \quad (2.33)$$

where  $F : \mathbb{R} \rightarrow \mathbb{R}$  is a non-constant smooth function.

By taking traces in (2.33), we obtain

$$SF'(S) + (1 - n)\Delta F'(S) - \frac{n}{2}F(S) = 0. \tag{2.34}$$

**Theorem 2.3.** *Let  $(M, g)$  be a Riemannian manifold. Then, the divergence of the generalized Einstein tensor is zero (that is,  $\delta E_F(g) = 0$ ).*

*Proof.* Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function, calculating in a normal frame  $\{e_i\}$  at  $x \in M$ , with  $X = e_j$ , we have

$$\delta E_F(g)(X) = -(\nabla_{e_i} E_F(g))(e_i, X) = -e_i(E_F(g)(e_i, X)), \tag{2.35}$$

by the definitions of generalized Einstein tensor, and the Hessian tensor, we get

$$\begin{aligned} E_F(g)(e_i, X) &= F'(S) \operatorname{Ric}(e_i, X) - g(\nabla_{e_i} \operatorname{grad} F'(S), X) \\ &\quad - (\Delta F'(S) + \frac{1}{2}F(S))g(e_i, X), \end{aligned} \tag{2.36}$$

substituting (2.36) in (2.35), and consider the definition of gradient operator, we obtain

$$\begin{aligned} \delta E_F(g)(X) &= -\operatorname{Ric}(\operatorname{grad} F'(S), X) - F'(S)e_i(\operatorname{Ric}(e_i, X)) \\ &\quad + g(\nabla_{e_i} \nabla_X \operatorname{grad} F'(S), e_i) + X(\Delta F'(S)) + \frac{1}{2}X(F(S)), \end{aligned}$$

by the definitions of the divergence, and the curvature tensor, with  $[e_i, X] = 0$ , we conclude that

$$\begin{aligned} \delta E_F(g)(X) &= -\operatorname{Ric}(\operatorname{grad} F'(S), X) + F'(S)(\delta \operatorname{Ric})(X) \\ &\quad + g(R(e_i, X) \operatorname{grad} F'(S), e_i) + g(\nabla_X \nabla_{e_i} \operatorname{grad} F'(S), e_i) \\ &\quad + X(\Delta F'(S)) + \frac{1}{2}X(F(S)), \end{aligned} \tag{2.37}$$

note that

$$\operatorname{Ric}(\operatorname{grad} F'(S), X) = g(R(e_i, X) \operatorname{grad} F'(S), e_i), \tag{2.38}$$

$$F'(S)(\delta \operatorname{Ric})(X) = -\frac{1}{2}X(F(S)) = -\frac{1}{2}F'(S)X(S), \tag{2.39}$$

$$g(\nabla_X \nabla_{e_i} \operatorname{grad} F'(S), e_i) = -X(\Delta F'(S)). \tag{2.40}$$

Substituting the formulas (2.38), (2.39) and (2.40) in (2.37), the Theorem 2.3 follows.  $\square$

**Remark 2.3.**

- If  $E_F(g) = fg$  for some function  $f$  on  $M$ , then  $f$  is constant function on  $M$  (because  $\delta E_F(g) = 0$ ).
- The condition  $E_F(g) = \lambda g$  is equivalent to

$$F'(S) \operatorname{Ric} - \operatorname{Hess} F'(S) = \mu g, \tag{2.41}$$

for some function  $\mu$  on  $M$ , it is also equivalent to

$$F'(S) \operatorname{Ric} - F''(S) \operatorname{Hess} S - F'''(S)dS \otimes dS = \mu g, \tag{2.42}$$

(see equation (2.30)).

- If  $F(s) = s$ , for all  $s \in \mathbb{R}$ , then  $E_F(g) = \lambda g$  if  $(M, g)$  is Einstein manifold, that is  $\text{Ric} = \mu g$  for some constant  $\mu$  (see [4]).

**Example 2.1.** Let  $M = (0, \infty) \times \mathbb{R}^3$  equipped with the Riemannian metric  $g = dt^2 + t^2(dx^2 + dy^2 + dz^2)$ . Let  $F(s) = s^\alpha$  for some constant  $\alpha$ . Then,  $E_F(g) = 0$  if and only if  $\alpha = \frac{1 \pm \sqrt{3}}{2}$ .

**Example 2.2.** Let  $M = \mathbb{S}^n \subset \mathbb{R}^{n+1}$  and  $F : \mathbb{R} \rightarrow \mathbb{R}$  a non-constant smooth function. Then, the induced Riemannian metric  $g^{\mathbb{S}^n}$  is a critical point of the  $F$ -Einstein-Hilbert functional if and only if  $F(s_0) = 0$  and  $F'(s_0) = 0$  where  $s_0 = n(n-1)$  is the scalar curvature of  $(\mathbb{S}^n, g^{\mathbb{S}^n})$ .

**Remark 2.4.** The previous examples prove the following results; There is no equivalence between  $E_F(g) = 0$  and  $E(g) = 0$  where  $F$  is a non-constant smooth function. There exist Riemannian Einstein metrics which are critical points of the  $F$ -Einstein-Hilbert functional where  $F(s) \neq s$ .

### 3. THE SECOND VARIATION OF $\mathcal{E}_F$

Let  $M$  be a closed orientable manifold. We denote by

$$\mathcal{M}_c = \{g \in \mathcal{M} \mid \text{Vol}(M, g) = \int_M v^g = c\},$$

for some constant  $c > 0$ . This is a submanifold of  $\mathcal{M}$  of codimension 1, and its tangent space at  $g \in \mathcal{M}_c$  is given by

$$T_g \mathcal{M}_c = \{T \in \Gamma(\odot^2 T^* M) \mid \int_M \langle g, T \rangle v^g = 0\}.$$

A Riemannian metric  $g$  is a critical point of  $\mathcal{E}_F|_{\mathcal{M}_c}$  if and only if  $E_F(g)$  is orthogonal to  $T_g \mathcal{M}_c$ , that is  $E_F(g) = \lambda g$  for some constant  $\lambda$ . In the following Theorem, we calculate the second derivative of  $\mathcal{E}_F(g_t)$  at  $t = 0$  where  $(g_t)$  ( $-\epsilon < t < \epsilon$ ) is a smooth one-parameter variation of such Riemannian metric  $g$  which enables us to know the extremality properties of  $\mathcal{E}_F$ . Write

$$h = \frac{\partial g_t}{\partial t} \Big|_{t=0}, \quad k = \frac{\partial^2 g_t}{\partial t^2} \Big|_{t=0}, \quad (3.43)$$

then  $h, k \in \Gamma(\odot^2 T^* M)$ . Under the notation above we have the following.

**Theorem 3.1.** Let  $(M, g)$  be a closed orientable Riemannian manifold with volume  $c$ . Suppose that  $E_F(g) = \lambda g$ , for some constant  $\lambda$ , then the second variation of  $\mathcal{E}_F|_{\mathcal{M}_c}$  at  $g$  in the direction of  $h$  is given by

$$\frac{d^2}{dt^2} \mathcal{E}_F(g_t) \Big|_{t=0} = \int_M \langle T_0(h) + T_1(h), h \rangle v^g,$$

where  $T_0(h), T_1(h)$  are defined by

$$\begin{aligned} T_0(h) &= -\frac{F'(S)}{2} \nabla^* \nabla h + F'(S) \overset{\circ}{R} h + F'(S) \delta^*(\delta h) + \frac{1}{2} F'(S) \text{Hess}(\text{Tr } h) \\ &\quad + \frac{F'(S)}{2} [\Delta(\text{Tr } h) + \delta(\delta h)] g - \frac{1}{2} [\lambda + \frac{1}{2} F'(S)] (\text{Tr } h) g, \end{aligned}$$

$$\begin{aligned}
 T_1(h) &= -f \operatorname{Ric} + \operatorname{Hess} f + (\Delta f)g - h(\nabla \cdot \operatorname{grad} F'(S), \cdot)^\sigma - (\nabla \cdot h)(\cdot, \operatorname{grad} F'(S))^\sigma \\
 &\quad + \frac{1}{2} \nabla_{\operatorname{grad} F'(S)} h - \langle \delta h + \frac{1}{2} d(\operatorname{Tr} h), dF'(S) \rangle g - \frac{1}{2} (\Delta F'(S))(\operatorname{Tr} h)g \\
 &\quad + \frac{1}{2} \langle \operatorname{Hess} F'(S), h \rangle g,
 \end{aligned}$$

and  $f = F''(S)[\Delta(\operatorname{Tr} h) + \delta(\delta h) - \langle \operatorname{Ric}, h \rangle]$ .

For the proof of Theorem 3.1, we need the following lemmas.

**Lemma 3.1.** *Let  $T_t, Q_t \in \Gamma(\odot^2 T^*M)$  all dependent of time  $t \in (-\epsilon, \epsilon)$  with  $T_0 = T$  and  $Q_0 = Q$ . Then*

$$\frac{\partial}{\partial t} \Big|_{t=0} \langle T_t, Q_t \rangle_t = \left\langle \frac{\partial T_t}{\partial t} \Big|_{t=0}, Q \right\rangle + \left\langle T, \frac{\partial Q_t}{\partial t} \Big|_{t=0} \right\rangle - 2 \langle T, h \circ Q \rangle,$$

where  $\langle \cdot, \cdot \rangle_t$  is the induced Riemannian metric (with respect to  $g_t$ ) on  $\odot^2 T^*M$ .

*Proof.* We have

$$\langle T_t, Q_t \rangle_t = T_t^{ij} Q_t^{ab} g_t^{ia} g_t^{jb},$$

so that

$$\begin{aligned}
 \frac{\partial}{\partial t} \Big|_{t=0} \langle T_t, Q_t \rangle_t &= \frac{\partial T_t^{ij}}{\partial t} \Big|_{t=0} Q^{ab} g^{ia} g^{jb} + T^{ij} \frac{\partial Q_t^{ab}}{\partial t} \Big|_{t=0} g^{ia} g^{jb} \\
 &\quad + T^{ij} Q^{ab} \frac{\partial g_t^{ia}}{\partial t} \Big|_{t=0} g^{jb} + T^{ij} Q^{ab} g^{ia} \frac{\partial g_t^{jb}}{\partial t} \Big|_{t=0},
 \end{aligned}$$

since  $\frac{\partial g_t^{ia}}{\partial t} \Big|_{t=0} = -g^{iu} g^{av} h_{uv}$  and  $\frac{\partial g_t^{jb}}{\partial t} \Big|_{t=0} = -g^{ju} g^{bv} h_{uv}$  (see [14]), we get

$$\begin{aligned}
 \frac{\partial}{\partial t} \Big|_{t=0} \langle T_t, Q_t \rangle_t &= \left\langle \frac{\partial T_t}{\partial t} \Big|_{t=0}, Q \right\rangle + \left\langle T, \frac{\partial Q_t}{\partial t} \Big|_{t=0} \right\rangle \\
 &\quad - T^{ij} Q^{ab} g^{iu} g^{av} h_{uv} g^{jb} - T^{ij} Q^{ab} g^{ia} g^{ju} g^{bv} h_{uv},
 \end{aligned}$$

note that

$$-T^{ij} Q^{ab} g^{iu} g^{av} h_{uv} g^{jb} - T^{ij} Q^{ab} g^{ia} g^{ju} g^{bv} h_{uv} = -2T^{ij} Q^{ab} g^{iu} g^{av} h_{uv} g^{jb},$$

on the other hand

$$\begin{aligned}
 -2 \langle T, h \circ Q \rangle &= -2T^{ij} (h \circ Q)^{ub} g^{iu} g^{jb} \\
 &= -2T^{ij} g^{av} h_{uv} Q^{ab} g^{iu} g^{jb}.
 \end{aligned}$$

□

**Lemma 3.2.** *Let  $(f_t)$  ( $-\epsilon < t < \epsilon$ ) be a time dependent family of smooth functions on  $M$  with  $f_0 = f$ . Then, the first variation of the Hessian and the Laplacian are given by*

$$\frac{\partial \operatorname{Hess}_t f_t}{\partial t} \Big|_{t=0} = \operatorname{Hess} \left( \frac{\partial f_t}{\partial t} \Big|_{t=0} \right) - (\nabla \cdot h)(\cdot, \operatorname{grad} f)^\sigma + \frac{1}{2} \nabla_{\operatorname{grad} f} h,$$

$$\frac{\partial \Delta_t f_t}{\partial t} \Big|_{t=0} = \Delta \left( \frac{\partial f_t}{\partial t} \Big|_{t=0} \right) - \langle \delta h + \frac{1}{2} d(\operatorname{Tr} h), df \rangle + \langle \operatorname{Hess} f, h \rangle,$$

where  $\operatorname{Hess}_t f_t$  (resp.  $\Delta_t f_t$ ) is the Hessian (resp. Laplacian) of  $f_t$  with respect to the metric  $g_t$ .

*Proof.* By the definition of Hessian (2.5), we obtain

$$\begin{aligned}
\frac{\partial \text{Hess}_t f_t}{\partial t}(X, Y) &= \frac{\partial}{\partial t} \left[ X(Y(f_t)) - (\nabla_X^t Y)(f_t) \right] \\
&= \frac{\partial}{\partial t} \left[ X(Y(f_t)) - g(\nabla_X^t Y, \text{grad } f_t) \right] \\
&= X(Y(\frac{\partial f_t}{\partial t})) - g(\frac{\partial}{\partial t} \nabla_X^t Y, \text{grad } f_t) - g(\nabla_X^t Y, \text{grad } \frac{\partial f_t}{\partial t}),
\end{aligned} \tag{3.44}$$

where  $\nabla^t$  is the Levi-Civita connection with respect to  $g_t$ . The first variation of the Levi-Civita connection in the direction of  $h$  is given by the formula

$$g(\frac{\partial}{\partial t} \nabla_X^t Y \Big|_{t=0}, Z) = \frac{1}{2} [(\nabla_X h)(Y, Z) + (\nabla_Y h)(X, Z) - (\nabla_Z h)(X, Y)], \tag{3.45}$$

(see [4]). Here  $X, Y, Z \in \Gamma(TM)$  (all independent of time  $t$ ). We conclude that

$$\frac{\partial \text{Hess}_t f_t}{\partial t} \Big|_{t=0} = \text{Hess} \left( \frac{\partial f_t}{\partial t} \Big|_{t=0} \right) - (\nabla \cdot h)(\cdot, \text{grad } f)^\sigma + \frac{1}{2} \nabla_{\text{grad } f} h. \tag{3.46}$$

By the Lemma 3.1, the first variation of  $\Delta_t f_t$  is given by

$$\begin{aligned}
\frac{\partial \Delta_t f_t}{\partial t} \Big|_{t=0} &= -\frac{\partial}{\partial t} \Big|_{t=0} \langle \text{Hess}_t f_t, g_t \rangle_t \\
&= -\langle \frac{\partial}{\partial t} \text{Hess}_t f_t \Big|_{t=0}, g \rangle - \langle \text{Hess } f, h \rangle + 2\langle \text{Hess } f, h \circ g \rangle,
\end{aligned} \tag{3.47}$$

by equations (3.46), (3.47), with  $h \circ g = h$ , we have

$$\begin{aligned}
\frac{\partial \Delta_t f_t}{\partial t} \Big|_{t=0} &= \Delta \left( \frac{\partial f_t}{\partial t} \Big|_{t=0} \right) + \text{Tr}(\nabla \cdot h)(\cdot, \text{grad } f)^\sigma \\
&\quad - \frac{1}{2} \text{Tr} \nabla_{\text{grad } f} h + \langle \text{Hess } f, h \rangle,
\end{aligned} \tag{3.48}$$

and note that

$$\text{Tr}(\nabla \cdot h)(\cdot, \text{grad } f)^\sigma = -\langle \delta h, df \rangle, \tag{3.49}$$

$$-\frac{1}{2} \text{Tr} \nabla_{\text{grad } f} h = -\frac{1}{2} \langle d(\text{Tr } h), df \rangle. \tag{3.50}$$

The proof is completed.  $\square$

*Proof of Theorem 3.1.* First note that

$$\frac{d^2}{dt^2} \mathcal{E}_F(g_t) \Big|_{t=0} = -\frac{d}{dt} \Big|_{t=0} \int_M \langle E_F(g_t), \frac{\partial g_t}{\partial t} \rangle_t v^{g_t}, \tag{3.51}$$

by the variational formulas in Lemma 3.1, we have

$$\begin{aligned}
\frac{d^2}{dt^2} \mathcal{E}_F(g_t) \Big|_{t=0} &= -\int_M \langle \frac{\partial}{\partial t} E_F(g_t) \Big|_{t=0}, h \rangle v^g \\
&\quad - \int_M \langle E_F(g), k \rangle v^g \\
&\quad + 2 \int_M \langle E_F(g), h \circ h \rangle v^g \\
&\quad - \frac{1}{2} \int_M \langle E_F(g), h \rangle (\text{Tr } h) v^g.
\end{aligned} \tag{3.52}$$

Since  $E_F(g) = \lambda g$ , we obtain

$$2 \int_M \langle E_F(g), h \circ h \rangle v^g = 2\lambda \int_M |h|^2 v^g, \quad (3.53)$$

$$-\frac{1}{2} \int_M \langle E_F(g), h \rangle (\text{Tr } h) v^g = -\frac{\lambda}{2} \int_M (\text{Tr } h)^2 v^g. \quad (3.54)$$

Since  $\text{Vol}(M, g_t) = c$ , we have

$$\frac{d^2}{dt^2} \Big|_{t=0} \text{Vol}(M, g_t) = \int_M \frac{\partial^2 v^{g_t}}{\partial t^2} \Big|_{t=0} = 0, \quad (3.55)$$

by equation (3.55), and Lemma 2.1, we get

$$\frac{1}{2} \int_M \frac{\partial}{\partial t} \Big|_{t=0} \left[ (\text{Tr}_t \frac{\partial g_t}{\partial t}) v^{g_t} \right] = 0, \quad (3.56)$$

where  $\text{Tr}_t \frac{\partial g_t}{\partial t}$  is the trace of  $\frac{\partial g_t}{\partial t}$  with respect to  $g_t$ , from equation (3.56), and the Lemmas 2.1 and 3.1, we obtain

$$\begin{aligned} 0 &= \int_M \left[ \frac{\partial}{\partial t} (\text{Tr}_t \frac{\partial g_t}{\partial t}) \Big|_{t=0} v^g + (\text{Tr } h) \frac{\partial v^{g_t}}{\partial t} \Big|_{t=0} \right] \\ &= \int_M \left[ \frac{\partial}{\partial t} \langle g_t, \frac{\partial g_t}{\partial t} \rangle \Big|_{t=0} v^g + \frac{1}{2} (\text{Tr } h)^2 v^g \right] \\ &= \int_M \left[ |h|^2 + (\text{Tr } k) - 2\langle g, h \circ h \rangle + \frac{1}{2} (\text{Tr } h)^2 \right] v^g \\ &= \int_M \left[ -|h|^2 + (\text{Tr } k) + \frac{1}{2} (\text{Tr } h)^2 \right] v^g, \end{aligned} \quad (3.57)$$

by equation (3.57), the second term on the left-hand side of (3.52) is

$$\begin{aligned} - \int_M \langle E_F(g), k \rangle v^g &= -\lambda \int_M (\text{Tr } k) v^g \\ &= \int_M \left[ -\lambda |h|^2 + \frac{\lambda}{2} (\text{Tr } h)^2 \right] v^g. \end{aligned} \quad (3.58)$$

We compute

$$\begin{aligned} \frac{\partial}{\partial t} E_F(g_t) \Big|_{t=0} &= \frac{\partial F'(S_t)}{\partial t} \Big|_{t=0} \text{Ric} + F'(S) \frac{\partial \text{Ric}_t}{\partial t} \Big|_{t=0} \\ &\quad - \frac{\partial \text{Hess}_t F'(S_t)}{\partial t} \Big|_{t=0} - \frac{\partial \Delta_t F'(S_t)}{\partial t} \Big|_{t=0} g \\ &\quad - \Delta F'(S) h - \frac{1}{2} \frac{\partial F(S_t)}{\partial t} \Big|_{t=0} g - \frac{1}{2} F(S) h, \end{aligned} \quad (3.59)$$

note that, from the Lemma 2.1, we have

$$\frac{\partial F(S_t)}{\partial t} \Big|_{t=0} = F'(S) [\Delta(\text{Tr } h) + \delta(\delta h) - \langle \text{Ric}, h \rangle], \quad (3.60)$$

$$\frac{\partial F'(S_t)}{\partial t} \Big|_{t=0} = F''(S) [\Delta(\text{Tr } h) + \delta(\delta h) - \langle \text{Ric}, h \rangle], \quad (3.61)$$

by Lemma 3.1, and the definition of Lichnerowicz Laplacian, we get

$$\begin{aligned} \left. \frac{\partial \text{Ric}_t}{\partial t} \right|_{t=0} &= \frac{1}{2} \Delta_L h - \delta^*(\delta h) - \frac{1}{2} \text{Hess}(\text{Tr } h) \\ &= \frac{1}{2} \nabla^* \nabla h - \overset{\circ}{R}h + \frac{1}{2} \text{Ric} \circ h + \frac{1}{2} h \circ \text{Ric} \\ &\quad - \delta^*(\delta h) - \frac{1}{2} \text{Hess}(\text{Tr } h), \end{aligned} \quad (3.62)$$

from equations (3.59), (3.60), (3.61), (3.62), and the Lemma 3.2, we have

$$\begin{aligned} \left. \frac{\partial}{\partial t} E_F(g_t) \right|_{t=0} &= f \text{Ric} + F'(S) \left[ \frac{1}{2} \nabla^* \nabla h - \overset{\circ}{R}h + \frac{1}{2} \text{Ric} \circ h + \frac{1}{2} h \circ \text{Ric} \right. \\ &\quad \left. - \delta^*(\delta h) - \frac{1}{2} \text{Hess}(\text{Tr } h) \right] - \text{Hess } f + (\nabla \cdot h)(\cdot, \text{grad } F'(S))^\sigma \\ &\quad - \frac{1}{2} \nabla_{\text{grad } F'(S)} h - (\Delta f)g + \langle \delta h + \frac{1}{2} d(\text{Tr } h), dF'(S) \rangle g \\ &\quad - \langle \text{Hess } F'(S), h \rangle g - (\Delta F'(S))h - \frac{F'(S)}{2} \left[ \Delta(\text{Tr } h) \right. \\ &\quad \left. + \delta(\delta h) - \langle \text{Ric}, h \rangle \right] g - \frac{1}{2} F'(S)h, \end{aligned} \quad (3.63)$$

where  $f = F''(S) [\Delta(\text{Tr } h) + \delta(\delta h) - \langle \text{Ric}, h \rangle]$ .

Note that, from the definitions of the composition (2.10) and the  $F$ -Einstein tensor (2.16), and the condition  $E_F(g) = \lambda g$ , we get

$$\begin{aligned} \frac{F'(S)}{2} (\text{Ric} \circ h + h \circ \text{Ric}) &= [\lambda + \Delta F'(S) + \frac{1}{2} F'(S)]h \\ &\quad + h(\nabla \cdot \text{grad } F'(S), \cdot)^\sigma, \end{aligned} \quad (3.64)$$

$$\begin{aligned} \frac{F'(S)}{2} \langle \text{Ric}, h \rangle g &= \frac{1}{2} [\lambda + \Delta F'(S) + \frac{1}{2} F'(S)] (\text{Tr } h)g \\ &\quad + \frac{1}{2} \langle \text{Hess } F'(S), h \rangle g. \end{aligned} \quad (3.65)$$

From equations (3.52), (3.53), (3.54), (3.58), (3.63), (3.64) and (3.65), we have

$$\begin{aligned} \left. \frac{d^2}{dt^2} \mathcal{E}_F(g_t) \right|_{t=0} &= \int_M \left\langle -f \text{Ric} - \frac{F'(S)}{2} \nabla^* \nabla h + F'(S) \overset{\circ}{R}h \right. \\ &\quad \left. - h(\nabla \cdot \text{grad } F'(S), \cdot)^\sigma + F'(S) \delta^*(\delta h) \right. \\ &\quad \left. + \frac{1}{2} F'(S) \text{Hess}(\text{Tr } h) + \text{Hess } f - (\nabla \cdot h)(\cdot, \text{grad } F'(S))^\sigma \right. \\ &\quad \left. + \frac{1}{2} \nabla_{\text{grad } F'(S)} h + (\Delta f)g - \langle \delta h + \frac{1}{2} d(\text{Tr } h), dF'(S) \rangle g \right. \\ &\quad \left. + \frac{F'(S)}{2} [\Delta(\text{Tr } h) + \delta(\delta h)]g - \frac{1}{2} [\lambda + \Delta F'(S) \right. \\ &\quad \left. + \frac{1}{2} F'(S)] (\text{Tr } h)g + \frac{1}{2} \langle \text{Hess } F'(S), h \rangle g, h \right\rangle v^g, \end{aligned} \quad (3.66)$$

the Theorem follows from equation (3.66).  $\square$

**Remark 3.1.** *If  $F(s) = s$ , for all  $s \in \mathbb{R}$ . Note that, the condition  $E_F(g) = \lambda g$  is equivalent to  $\text{Ric} = [\lambda + \frac{S}{2}]g$ . That is,  $g$  is Einstein Riemannian metric with constant  $\mu = \lambda + \frac{S}{2}$ . In*

this case, we have

$$T_0(h) = -\frac{1}{2}\nabla^*\nabla h + \overset{\circ}{R}h + \delta^*(\delta h) + \frac{1}{2}\text{Hess}(\text{Tr } h) + \frac{1}{2}[\Delta(\text{Tr } h) + \delta(\delta h)]g - \frac{\mu}{2}(\text{Tr } h)g,$$

and  $T_1(h) = 0$ . From the formula

$$(\text{Tr } h)\delta(\delta h) = \delta((\text{Tr } h)\delta h) + \delta(h(\cdot, \text{grad}(\text{Tr } h))) + \langle \text{Hess}(\text{Tr } h), h \rangle,$$

and the divergence theorem (see [2]), the second variation of  $\mathcal{E}_F|_{\mathcal{M}_c}$  at  $g$  in the direction of  $h$  is given by (see [4], [9])

$$\begin{aligned} \frac{d^2}{dt^2}\mathcal{E}_F(g_t)\Big|_{t=0} &= \int_M \left\langle -\frac{1}{2}\nabla^*\nabla h + \overset{\circ}{R}h + \delta^*(\delta h) \right. \\ &\quad \left. + \frac{1}{2}\Delta(\text{Tr } h)g + \delta(\delta h)g - \frac{\mu}{2}(\text{Tr } h)g, h \right\rangle v^g. \end{aligned}$$

**Definition 3.1.** A Riemannian manifold  $(M, g)$  is said to be  $F$ -Einstein if  $E_F(g) = \lambda g$  for some constant  $\lambda$ , where  $F : \mathbb{R} \rightarrow \mathbb{R}$  is a non-constant smooth function. We call  $\lambda$  the  $F$ -Einstein constant of  $g$ . We say that a closed orientable  $F$ -Einstein manifold is stable (resp. strictly stable) if for any  $h \in TT = \text{Tr}^{-1}(0) \cap \delta^{-1}(0)$  (such tensors are called transverse traceless or  $TT$ -tensors)

$$\mathcal{E}_F''(h) = \int_M \langle \widehat{T}_0(h) + \widehat{T}_1(h), h \rangle v^g \leq 0 \quad (\text{resp. } < 0),$$

where  $\widehat{T}_0, \widehat{T}_1$  are the restrictions of  $T_0, T_1$  to  $TT$  respectively, given by

$$\widehat{T}_0(h) = -\frac{F'(S)}{2}[\nabla^*\nabla h - 2\overset{\circ}{R}h],$$

$$\widehat{T}_1(h) = -\widehat{f}\text{Ric} + \frac{1}{2}\nabla_{\text{grad } F'(S)}h,$$

and  $\widehat{f} = f|_{TT} = -F''(S)\langle \text{Ric}, h \rangle$ .

**Remark 3.2.**

- In the Definition 3.1,  $\text{Tr}^{-1}(0)$  (resp.  $\delta^{-1}(0)$ ) denotes the space of symmetric  $(0, 2)$ -tensor fields, whose trace (resp. divergence) vanishes on  $(M, g)$ .
- By using  $\delta h = 0$  and symmetry of  $h$ , we obtain the following formulas

$$\begin{aligned} \langle \text{Hess } \widehat{f}, h \rangle &= -\delta[h(\text{grad } \widehat{f}, \cdot)]; \\ -\langle h(\nabla \cdot \text{grad } F'(S), \cdot)^\sigma, h \rangle - \langle (\nabla \cdot h)(\cdot, \text{grad } F'(S))^\sigma, h \rangle \\ &= \delta[(h \circ h)(\text{grad } F'(S), \cdot)]. \end{aligned}$$

This explains the disappearance of these terms in  $\langle \widehat{T}_1(h), h \rangle$  after integration over  $M$ .

- The Definition 3.1, is a natural generalization of stable Einstein manifold (see [4, 9, 11, 12]).
- We call the operator  $\Delta_E^F(h) = -2(\widehat{T}_0(h) + \widehat{T}_1(h))$  the  $F$ -Einstein operator. Thus, an  $F$ -Einstein manifold  $(M, g)$  is stable, if the  $F$ -Einstein operator is nonnegative on  $TT$ -tensors, and strictly stable if it is positive on  $TT$ -tensors. If  $F(s) = s$  for all

$s \in \mathbb{R}$ , then the  $F$ -Einstein operator reduces to the usual Einstein operator  $\Delta_E(h) = \widehat{\nabla^* \nabla} h - 2\widehat{R}h$ .

**Theorem 3.2.** *Let  $F \in C^\infty(\mathbb{R})$ . We assume that  $F'(s) \geq 0$  (resp.  $F'(s) > 0$ ) for all  $s \in \mathbb{R}$ . Then, any Einstein manifold of negative sectional curvature is stable (resp. strictly stable)  $F$ -Einstein manifold.*

*Proof.* Let  $(M, g)$  be an Einstein manifold with Einstein constant  $\mu$ , i.e.,  $\text{Ric} = \mu g$ . Thus,  $(M, g)$  is  $F$ -Einstein manifold with  $F$ -Einstein constant  $\lambda = \mu F'(S) - \frac{1}{2}F(S)$ . Moreover,  $\widehat{f} = 0$  and the  $F$ -Einstein operator becomes

$$\Delta_E^F(h) = F'(S) [\widehat{\nabla^* \nabla} h - 2\widehat{R}h].$$

Hence, if  $F'(S) \geq 0$  (resp.  $F'(S) > 0$ ) and the sectional curvature of  $(M, g)$  is negative, then  $(M, g)$  is stable (resp. strictly stable)  $F$ -Einstein manifold (see [9, 10]).  $\square$

**Remark 3.3.** *Let  $(M, g)$  be an  $F$ -Einstein manifold with  $F$ -Einstein constant  $\lambda$ . We assume that  $(M, g)$  has constant scalar curvature. If  $F'(S) > 0$ , according to (2.16), the Riemannian manifold  $(M, g)$  is Einstein with Einstein constant  $\mu = F'(S)^{-1} (\lambda + \frac{1}{2}F(S))$ . Moreover, if the sectional curvature of  $(M, g)$  is negative, then  $(M, g)$  is strictly stable. Here, if the manifold  $M$  is even-dimensional, by using (2.34) with  $E_F(g) = \lambda g$ , we can consider the smooth function  $F(s) = -2\lambda + c s^{n/2}$  for some  $c \in \mathbb{R}$ .*

**Theorem 3.3.** *Let  $F \in C^\infty(\mathbb{R})$  and  $(M, g)$  be a closed orientable  $F$ -Einstein manifold of constant sectional curvature  $c > 0$ . We assume that  $F'(s) \geq 0$  and  $F''(s) \leq 0$  for all  $s \in \mathbb{R}$ . Then,  $(M, g)$  is stable. Moreover, if  $F'(s) > 0$  for all  $s \in \mathbb{R}$ , then  $(M, g)$  is strictly stable.*

*Proof.* A straightforward calculation shows that if  $h \in TT$ ,

$$\begin{aligned} -\frac{F'(S)}{2} \langle \widehat{\nabla^* \nabla} h, h \rangle &= -\frac{1}{2} \delta [F'(S) \langle \nabla \cdot h, h \rangle] - \frac{1}{2} \langle \nabla_{\text{grad } F'(S)} h, h \rangle \\ &\quad - \frac{F'(S)}{2} \text{Tr} \langle \nabla \cdot h, \nabla \cdot h \rangle. \end{aligned} \quad (3.67)$$

By using  $\text{Tr } h = 0$ , we find that

$$F'(S) \langle \widehat{R}h, h \rangle = -c F'(S) |h|^2. \quad (3.68)$$

From equations (3.67) and (3.69), we conclude that

$$\begin{aligned} \mathcal{E}_F''(h) &= \int_M \left[ -\frac{1}{2} F'(S) \text{Tr} \langle \nabla \cdot h, \nabla \cdot h \rangle - c F'(S) |h|^2 \right. \\ &\quad \left. + F''(S) \langle \text{Ric}, h \rangle^2 \right] v^g. \end{aligned} \quad (3.69)$$

Theorem 3.3 follows from equation (3.69), the assumptions  $F' \geq 0$ ,  $F'' \leq 0$ , and  $c > 0$ .  $\square$

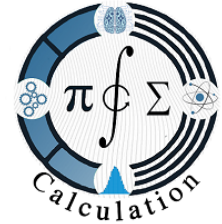
**Corollary 3.1.** *The  $n$ -dimensional unit sphere  $\mathbb{S}^n$  is a strictly stable  $F$ -Einstein manifold for all  $F \in C^\infty(\mathbb{R})$  such that  $F' > 0$  and  $F'' \leq 0$ .*

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**SOME PROPERTIES OF GEODESICS AND  $F$ -GEODESICS ON TANGENT BUNDLE WITH GRADIENT SASAKI METRIC**

ABDERRAHIM ZAGANE  \*

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**Abstract.** In this paper, we investigate the properties of geodesics and  $F$ -geodesics on the tangent bundle equipped with the gradient Sasaki metric. First, we establish necessary and sufficient conditions for a curve to be a geodesic. We then study the behavior of  $F$ -geodesics and  $F$ -planar curves on the tangent bundle with respect to the induced Levi-Civita connection. Our theoretical results are supported by explicit examples that illustrate the behavior of these curves.

**Keywords:** Tangent bundle, gradient Sasaki metric, geodesics,  $F$ -geodesics,  $F$ -planar curves.

**2020 Mathematics Subject Classification:** 53C22, 58E10, 53C05, 53A04.

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1. INTRODUCTION

Natural Riemannian metrics on the tangent bundle of a Riemannian manifold are typically constructed using the Levi–Civita connection of the base manifold. Among these constructions, the Sasaki metric [10] occupies a central position and has been rigorously investigated across numerous geometric contexts. However, due to the inherent rigidity of the Sasaki metric, research has increasingly shifted toward various natural deformations. Prominent examples include the Cheeger–Gromoll metric [8], the Berger-type deformed Sasaki metric [3, 13], and the gradient Sasaki metric [4]. To date, the geometry of tangent bundles remains a highly active and fertile domain within modern differential geometry.

The characterization of geodesics on tangent bundles has attracted substantial attention, particularly concerning the analysis of oblique (non-vertical) geodesics and their projections onto the base manifold. Sasaki [11] and Sato [12] provided a comprehensive description of the curves and associated vector fields that generate non-vertical geodesics on the tangent bundle and the unit tangent bundle, respectively. Their findings established that the projected curves exhibit constant geodesic curvatures (i.e., constant Frenet curvatures). Subsequently, Nagy [9] extended these results to the case of locally symmetric base manifolds. Furthermore, Yampolsky [13] conducted analogous investigations for the tangent and unit tangent bundles

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endowed with a Berger-type deformed Sasaki metric over Kähler manifolds, considering both locally symmetric spaces and manifolds of constant holomorphic sectional curvature.

In parallel, the exploration of  $F$ -planar curves and  $F$ -geodesics has emerged as a significant research frontier. These structures are of particular interest as  $F$ -planar curves generalize both magnetic curves and standard geodesics [6, 7]. It is essential to distinguish the concept of  $F$ -geodesics, as introduced in [5], which constitutes a related but formally distinct framework from that of  $F$ -planar curves. Recently, several studies have focused on the behavior of magnetic curves,  $F$ -planar curves, and  $F$ -geodesics within the geometry of tangent and unit tangent bundles (see, e.g., [1, 2, 16, 17, 18]). These contributions have substantially broadened the theoretical understanding of these geometric configurations.

The primary objective of the present paper is to investigate several problems concerning geodesics and  $F$ -geodesics on the tangent bundle endowed with the gradient Sasaki metric. The manuscript is organized as follows.

In Section 2, we recall some notions and results concerning tangent bundles with the gradient Sasaki metric .

Section 3 is devoted to the derivation of the necessary and sufficient conditions for a curve to constitute a geodesic on the tangent bundle with respect to the Levi–Civita connection of the gradient Sasaki metric. We also elucidate several specific properties of these geodesic trajectories.

In the final section, we analyze the behavior of  $F$ -geodesics and  $F$ -planar curves on the tangent bundle. To conclude, we provide illustrative examples of geodesics and  $F$ -geodesics to support the theoretical framework developed in the preceding sections.

Our work aims to deepen the understanding of Riemannian and pseudo-Riemannian structures on tangent bundles, effectively extending classical results associated with the Sasaki metric and its natural deformations to the case of the gradient Sasaki metric.

## 2. TANGENT BUNDLE WITH GRADIENT SASAKI METRIC

Let  $TM$  be the tangent bundle of an  $n$ -dimensional Riemannian manifold  $(M^n, g)$ , with  $\pi : TM \rightarrow M$  the natural projection. If  $(U, x^j), j = 1, \dots, n$  is a local chart on  $M$ , then it induces a local chart  $(\pi^{-1}(U), x^j, \xi^j), j = 1, \dots, n$  on  $TM$ . Here each  $\xi^j$  corresponds to the coordinate of a tangent vector  $\xi$  in the direction  $\frac{\partial}{\partial x^j}$ .

We recall the standard notion of vertical and horizontal lifts on the tangent bundle. For a vector field  $Z = Z^j \frac{\partial}{\partial x^j}$  on  $M$ , its *vertical lift*  ${}^VZ$  and *horizontal lift*  ${}^HZ$  are defined on  $TM$  by

$${}^VZ_{(p,\xi)} = Z^j \frac{\partial}{\partial \xi^j} \Big|_{(p,\xi)}, \quad {}^HZ_{(p,\xi)} = Z^j \left( \frac{\partial}{\partial x^j} - \xi^i \Gamma_{ji}^k \frac{\partial}{\partial \xi^j} \right) \Big|_{(p,\xi)}.$$

Equivalently,

$$d\pi({}^HZ) = Z, \quad d\pi({}^VZ) = 0,$$

for all vector field  $Z$  on  $M$ , where  $(p, \xi) \in TM$ .

Consider a smooth strictly positive function  $f : M \rightarrow \mathbb{R}_+^*$ . We define the gradient Sasaki metric on  $TM$ , denoted by  $g^f$ , as follows:

$$\begin{aligned} g^f(HX, HY) &= g(X, Y), \\ g^f(VX, HY) &= g^f(HX, VY) = 0, \\ g^f(VX, VY) &= g(X, Y) + X(f)Y(f), \end{aligned}$$

for all vector fields  $X, Y$  on  $M$ .

**Theorem 2.1.** [4] *Let  $(M^n, g)$  be a Riemannian manifold and  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric. The Levi-Civita connection  $\nabla^f$  of  $(TM, g^f)$  satisfies*

$$\begin{aligned} \nabla_{HX}^f HY &= H(\nabla_X Y) - \frac{1}{2}V(R(X, Y)\xi), \\ \nabla_{HX}^f VY &= \frac{1}{2}H(R(\xi, Y)X) + \frac{1}{2}Y(f)H(R(\xi, \text{grad}f)X) + \frac{1}{2}Y(f)V(\nabla_X \text{grad}f) \\ &\quad + (\nabla_X Y)^V + \frac{1}{2\alpha}(Hess_f(X, Y) - \frac{1}{2}X(\alpha)Y(f))^V(\text{grad}f), \\ \nabla_{VX}^f HY &= \frac{1}{2}H(R(\xi, X)Y) + \frac{1}{2}X(f)H(R(\xi, \text{grad}f)Y) + \frac{1}{2}X(f)V(\nabla_Y \text{grad}f) \\ &\quad + \frac{1}{2\alpha}(Hess_f(X, Y) - \frac{1}{2}Y(\alpha)X(f))^V(\text{grad}f), \\ \nabla_{VX}^f VY &= -\frac{1}{2}X(f)H(\nabla_Y \text{grad}f) - \frac{1}{2}Y(f)H(\nabla_X \text{grad}f), \end{aligned}$$

for all vector fields  $X, Y$  on  $M$ , where  $\alpha = 1 + |\text{grad}f|^2$  and  $Hess_f$  is the Hessian of  $f$  with respect to  $g$ .

### 3. GEODESICS ON TANGENT BUNDLE WITH THE GRADIENT SASAKI METRIC

Let  $\Gamma = (\gamma(t), \xi(t))$  be a naturally parameterized curve on the tangent bundle  $TM$  (i.e.  $t$  is an arc length parameter on  $\Gamma$ ), where  $\gamma$  is a curve on  $M$  and  $\xi$  is a vector field along this curve.

We denote

$$\gamma' = \frac{d\gamma}{dt}, \quad \xi' = \nabla_{\gamma'}\xi, \quad \Gamma' = \frac{d\Gamma}{dt}.$$

Then, according to [15], we have

$$\Gamma' = H\gamma' + V\xi'. \quad (3.1)$$

Furthermore, we set

$$\gamma'' = \nabla_{\gamma'}\gamma', \quad \xi'' = \nabla_{\gamma'}\xi'.$$

Note that  $(\prime)$  denotes the covariant derivative along  $\gamma$  with respect to parameter  $t$ .

The curve  $\gamma = \pi \circ \Gamma$  is called the projection (or projected curve) of  $\Gamma$  onto  $M$ , where  $\pi : TM \rightarrow M$  denotes the canonical bundle projection.

A curve  $\Gamma = (\gamma(t), \xi(t))$  on  $TM$  is said to be the horizontal lift of the curve  $\gamma$  to  $TM$  if and only if  $\xi' = 0$  [14].

A curve  $\Gamma = (\gamma(t), \gamma'(t))$  is called a natural lift of the curve  $\gamma$  to  $TM$ [14].

**Theorem 3.1.** *Let  $(M^n, g)$  be a Riemannian manifold and  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric. Let  $\Gamma = (\gamma(t), \xi(t))$  be a curve on  $TM$ . Then  $\Gamma$  is*

a geodesic if and only if

$$\left\{ \begin{array}{l} \gamma'' = R(\xi', \xi)\gamma' - g(\xi', \text{grad}f)R(\xi, \text{grad}f)\gamma' + g(\xi', \text{grad}f)\nabla_{\xi'}\text{grad}f, \\ \xi'' = -g(\xi', \text{grad}f)\nabla_{\gamma'}\text{grad}f - \frac{1}{\alpha}\text{Hess}_f(\gamma', \xi')\text{grad}f \\ \quad + \frac{1}{\alpha}\text{Hess}_f(\gamma', \text{grad}f)g(\xi', \text{grad}f)\text{grad}f. \end{array} \right. \quad (3.2)$$

*Proof.* From (3.1) and Theorem 2.1, we obtain

$$\begin{aligned} \nabla_{\Gamma'}^f \Gamma' &= \nabla_{(H\gamma' + V\xi')}^f ({}^H\gamma' + V\xi') \\ &= \nabla_{H\gamma'}^f {}^H\gamma' + \nabla_{H\gamma'}^f V\xi' + \nabla_{V\xi'}^f {}^H\gamma' + \nabla_{V\xi'}^f V\xi' \\ &= {}^H\gamma'' + {}^H(R(\xi, \xi')\gamma') + \xi'(f)({}^H(R(\xi, \text{grad}f)\gamma') + V\xi'' + \xi'(f)^V(\nabla_{\gamma'}\text{grad}f)) \\ &\quad + \frac{1}{\alpha}(\text{Hess}_f(\gamma', \xi') - \frac{1}{2}\gamma'(\alpha)\xi'(f))^V(\text{grad}f) - \xi'(f)({}^H(\nabla_{\xi'}\text{grad}f)) \\ &= {}^H(\gamma'' + R(\xi, \xi')\gamma' + g(\xi', \text{grad}f)R(\xi, \text{grad}f)\gamma' - g(\xi', \text{grad}f)(\nabla_{\xi'}\text{grad}f)) \\ &\quad + {}^V(\xi'' + g(\xi', \text{grad}f)\nabla_{\gamma'}\text{grad}f + \frac{1}{\alpha}\text{Hess}_f(\gamma', \xi')\text{grad}f \\ &\quad - \frac{1}{2\alpha}\gamma'(\alpha)g(\xi', \text{grad}f)\text{grad}f) \\ &= {}^H(\gamma'' + R(\xi, \xi')\gamma' + g(\xi', \text{grad}f)R(\xi, \text{grad}f)\gamma' - g(\xi', \text{grad}f)(\nabla_{\xi'}\text{grad}f)) \\ &\quad + {}^V(\xi'' + g(\xi', \text{grad}f)\nabla_{\gamma'}\text{grad}f + \frac{1}{\alpha}\text{Hess}_f(\gamma', \xi')\text{grad}f \\ &\quad - \frac{1}{\alpha}\text{Hess}_f(\gamma', \text{grad}f)g(\xi', \text{grad}f)\text{grad}f). \end{aligned} \quad (3.3)$$

If we put  $\nabla_{\Gamma'}^f \Gamma'$  equal to zero, we find (3.2).  $\square$

**Corollary 3.1.** *Let  $(M^n, g)$  be a Riemannian manifold and  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric. The natural lift  $\Gamma = (\gamma(t), \gamma'(t))$  of any geodesic  $\gamma$  is a geodesic on  $(TM, g^f)$ .*

**Corollary 3.2.** *Let  $(M^n, g)$  be a Riemannian manifold and  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric. The horizontal lift  $\Gamma = (\gamma(t), \xi(t))$  of any geodesic  $\gamma$  is a geodesic on  $(TM, g^f)$ .*

**Theorem 3.2.** *Let  $(M^n, g)$  be a Riemannian manifold and  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric. Let  $\Gamma = (\gamma(t), \xi(t))$  be a geodesic on  $TM$  such that  $\gamma$  is a geodesic on  $M$ . Then,  $\xi'$  is orthogonal to either  $\text{grad}f$  or  $\nabla_{\gamma'}\text{grad}f$ .*

*Proof.* Using the first equation of (3.2), we obtain

$$\begin{aligned} g(\gamma'', \gamma') &= g(R(\xi', \xi)\gamma', \gamma') - g(\xi', \text{grad}f)g(R(\xi, \text{grad}f)\gamma', \gamma') \\ &\quad + g(\xi', \text{grad}f)g(\nabla_{\xi'}\text{grad}f, \gamma') \\ &= g(\xi', \text{grad}f)g(\nabla_{\gamma'}\text{grad}f, \xi'). \end{aligned}$$

Since  $\gamma$  is a geodesic, we have  $\gamma'' = 0$ , and thus

$$g(\xi', \text{grad}f)g(\nabla_{\gamma'}\text{grad}f, \xi') = 0,$$

completing the proof.  $\square$

**Theorem 3.3.** *Let  $(M^n, g)$  be a Riemannian manifold and  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric. Let  $\Gamma = (\gamma(t), \xi(t))$  be a curve on  $TM$  satisfying  $\xi' \perp \text{grad} f$ . Then  $\Gamma$  is a geodesic on  $TM$  if and only if*

$$\begin{cases} \gamma'' = \mathbf{R}(\xi', \xi)\gamma', \\ \xi'' = -\frac{1}{\alpha}\text{Hess}_f(\gamma', \xi')\text{grad} f, \end{cases} \quad (3.4)$$

moreover,

$$\begin{cases} |\gamma'| = \text{const}, \\ |\xi'| = \text{const}. \end{cases}$$

*Proof.* Since  $\xi' \perp \text{grad} f$ , it follows that  $g(\xi', \text{grad} f) = 0$ . Hence, equation (3.2) reduces to

$$\begin{cases} \gamma'' = \mathbf{R}(\xi', \xi)\gamma', \\ \xi'' = -\frac{1}{\alpha}\text{Hess}_f(\gamma', \xi')\text{grad} f. \end{cases}$$

Furthermore,

$$(|\gamma'|^2)' = 2g(\gamma'', \gamma') = g(\mathbf{R}(\xi', \xi)\gamma', \gamma') = 0,$$

which implies  $|\gamma'| = \text{const}$ .

Using the second equation of (3.4), we obtain

$$(|\xi'|^2)' = 2g(\xi'', \xi') = -\frac{2}{\alpha}\text{Hess}_f(\gamma', \xi')g(\text{grad} f, \xi').$$

Since  $g(\text{grad} f, \xi') = 0$  by hypothesis, hence  $|\xi'| = \text{const}$ .  $\square$

**Theorem 3.4.** *Let  $(M^n, g)$  be a Riemannian manifold and  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric. Let  $\Gamma = (\gamma(t), \xi(t))$  be a geodesic on  $TM$  and assume that  $\nabla \text{grad} f = 0$ . Then we have*

$$\begin{cases} \gamma'' = \mathbf{R}(\xi', \xi)\gamma' - g(\xi', \text{grad} f)\mathbf{R}(\xi, \text{grad} f)\gamma', \\ \xi'' = 0, \end{cases}$$

and moreover,

$$\begin{cases} |\gamma'| = \text{const}, \\ |\xi'| = \text{const}. \end{cases}$$

**Remark 3.1.** *As a reminder, note that locally we have:*

$$\gamma'' = \sum_{k=1}^m \left( \frac{d^2 \gamma^k}{dt^2} + \sum_{i,j=1}^m \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k}, \quad (3.5)$$

and

$$\xi' = \sum_{k=1}^m \left( \frac{d\xi^k}{dt} + \sum_{i,j=1}^m \frac{d\gamma^j}{dt} \xi^i \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k}. \quad (3.6)$$

**Example 3.1.** Consider the upper half-plane

$$H = \{(x, y) \in \mathbb{R}^2, y > 0\}$$

equipped with the Riemannian metric  $g$  defined by

$$g_{11} = 1, \quad g_{22} = y^2, \quad g_{12} = g_{21} = 0.$$

The non-vanishing Christoffel symbols of the associated Levi-Civita connection are

$$\Gamma_{22}^2 = \frac{1}{y}.$$

The curve  $\gamma(t) = (x(t), y(t))$  is a geodesic curve if and only if  $\gamma'' = 0$ , hence from (3.5), we have

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^2 \frac{d\gamma^i}{dt} \frac{d\gamma^j}{dt} \Gamma_{ij}^k(\gamma(t)) = 0 \iff \begin{cases} x'' = 0, \\ y'' + \frac{1}{y}(y')^2 = 0. \end{cases}$$

Solving these equations yields

$$\begin{cases} x(t) = c_1t + c_2, \\ y(t) = \sqrt{c_3t + c_4}, \quad c_3t + c_4 > 0, \end{cases}$$

where  $c_i$  are real constants. Consequently,

$$\gamma(t) = (c_1t + c_2, \sqrt{c_3t + c_4}).$$

From Corollary 3.1, the natural lift  $\Gamma_1 = (\gamma(t), \gamma'(t))$  of  $\gamma$  is a geodesic on  $TH$ .

2) Let  $\Gamma = (\gamma(t), \xi(t))$  be the horizontal lift of  $\gamma$ , where  $\xi(t) = (\lambda(t), \mu(t))$ . Then  $\xi$  satisfies  $\xi' = 0$ , hence from (3.6), we have

$$\frac{d\xi^k}{dt} + \sum_{i,j=1}^2 \xi_i \frac{d\gamma^j}{dt} \Gamma_{ij}^k(\gamma(t)) = 0 \iff \begin{cases} \lambda' = 0, \\ \mu' + \frac{y'}{y}\mu = 0. \end{cases}$$

Solving these equations, we obtain

$$\begin{cases} \lambda(t) = c_5, \\ \mu(t) = \frac{c_6}{\sqrt{c_3t + c_4}}, \end{cases}$$

where  $c_i$  are real constants. Consequently,

$$\xi(t) = \left( c_5, \frac{c_6}{\sqrt{c_3t + c_4}} \right).$$

By Corollary 3.2, the horizontal lift  $\Gamma = (\gamma(t), \xi(t))$  of  $\gamma$  is a geodesic on  $TH$ .

#### 4. $F$ -GEODESICS ON THE TANGENT BUNDLE WITH THE GRADIENT SASAKI METRIC

Let  $(M^m, g)$  be an  $m$ -dimensional Riemannian manifold and let  $F$  be a  $(1, 1)$ -tensor field on  $M$ .

A curve  $\gamma : I \subset \mathbb{R} \rightarrow M$  is called  $F$ -planar if its tangent vector field, when subjected to parallel transport along  $\gamma$ , remains confined to the two-dimensional distribution spanned by

the tangent vector  $\gamma'$  and its image  $F\gamma'$ . Analytically, this geometric condition is equivalent to the requirement that the covariant acceleration of  $\gamma$  satisfies

$$\gamma'' = \varrho_1(t)\gamma' + \varrho_2(t)F\gamma',$$

where  $\varrho_1(t)$  and  $\varrho_2(t)$  are smooth scalar functions of the parameter  $t \in I$  [6, 7]. Because this definition relies entirely on the distribution spanned by the vectors rather than the specific timing of the trajectory,  $F$ -planarity is an intrinsic, purely geometric property of the curve that is invariant under reparameterization.

A curve  $\gamma$  on  $M$  is defined as an  $F$ -geodesic if it satisfies:

$$\gamma'' = F\gamma'.$$

Unlike  $F$ -planar curves, an  $F$ -geodesic represents a dynamic trajectory that dictates a highly specific affine parameterization. While every  $F$ -geodesic trivially constitutes an  $F$ -planar curve (where  $\varrho_1 = 0$  and  $\varrho_2 = 1$ ), the converse is generally false [5].

In the sequel, let  $\tilde{\nabla}$  denote the Levi-Civita connection of the the gradient Sasaki metric on tangent bundle  $TM$  with the condition

$$\nabla \text{grad} f = 0,$$

**Theorem 4.1.** *Let  $(M^n, g)$  be a Riemannian manifold,  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric and  $F$  be a  $(1, 1)$ -tensor field on  $M$ . A curve  $\Gamma = (\gamma(t), \xi(t))$  on  $TM_0^f$  is an  ${}^H F$ -planar with respect to  $\tilde{\nabla}$  if and only if the following system holds:*

$$\begin{cases} \gamma'' = R(\xi', \xi)\gamma' - g(\xi', \text{grad} f)R(\xi, \text{grad} f)\gamma' + \varrho_1\gamma' + \varrho_2F\gamma' \\ \xi'' = \varrho_1\xi' + \varrho_2F\xi' \end{cases} \quad (4.7)$$

where  $\varrho_1$  and  $\varrho_2$  are some functions of the parameter  $t$ .

*Proof.* The curve  $\Gamma$  is  ${}^H F$ -planar with respect to  $\tilde{\nabla}$  if and only if it satisfies the condition:

$$\tilde{\nabla}_{\Gamma'}\Gamma' = \varrho_1\Gamma' + \varrho_2{}^H F\Gamma',$$

where  $\varrho_1$  and  $\varrho_2$  are functions of  $t$ . Utilizing the decomposition provided in (3.1), we have:

$$\begin{aligned} \tilde{\nabla}_{\Gamma'}\Gamma' &= \varrho_1({}^H\gamma' + {}^V\xi') + \varrho_2{}^H F({}^H\gamma' + {}^V\xi') \\ &= \varrho_1{}^H\gamma' + \varrho_2{}^H F{}^H\gamma' + \varrho_1{}^V\xi' + \varrho_2{}^H F{}^V\xi' \\ &= {}^H(\varrho_1\gamma' + \varrho_2F\gamma') + {}^V(\varrho_1\xi' + \varrho_2F\xi') \\ &= {}^H(\varrho_1\gamma' + \varrho_2F\gamma') + {}^V(\varrho_1\xi' + \varrho_2F\xi'). \end{aligned} \quad (4.8)$$

By comparing the horizontal and tangential components of (3.3) with those in (4.8), the system (4.7) follows immediately.  $\square$

**Corollary 4.1.** *Let  $(M^n, g)$  be a flat Riemannian manifold,  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric and  $F$  be a  $(1, 1)$ -tensor field on  $M$ . A curve  $\Gamma = (\gamma(t), \xi(t))$  on  $TM_0^f$  is an  ${}^H F$ -planar with respect to  $\tilde{\nabla}$  if and only if the following*

system holds:

$$\begin{cases} \gamma'' = \varrho_1\gamma' + \varrho_2F\gamma' \\ \xi'' = \varrho_1\xi' + \varrho_2F\xi' \end{cases}$$

By setting  $\varrho_1 = 0$  and  $\varrho_2 = 1$  in Theorem 4.1, we obtain the following result for  ${}^HF$ -geodesics.

**Theorem 4.2.** *Let  $(M^n, g)$  be a Riemannian manifold,  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric and  $F$  be a  $(1, 1)$ -tensor field on  $M$ . A curve  $\Gamma = (\gamma(t), \xi(t))$  on  $TM_0^f$  is an  ${}^HF$ -geodesic with respect to  $\tilde{\nabla}$  if and only if the following system holds:*

$$\begin{cases} \gamma'' = R(\xi', \xi)\gamma' - g(\xi', \text{grad} f)R(\xi, \text{grad} f)\gamma' + F\gamma' \\ \xi'' = F\xi' \end{cases}$$

**Corollary 4.2.** *Let  $(M^n, g)$  be a flat Riemannian manifold,  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric and  $F$  be a  $(1, 1)$ -tensor field on  $M$ . A curve  $\Gamma = (\gamma(t), \xi(t))$  on  $TM_0^f$  is an  ${}^HF$ -geodesic with respect to  $\tilde{\nabla}$  if and only if the following system holds:*

$$\begin{cases} \gamma'' = F\gamma' \\ \xi'' = F\xi' \end{cases}$$

**Theorem 4.3.** *Let  $(M^n, g)$  be a Riemannian manifold,  $(TM, g^f)$  its tangent bundle endowed with the gradient Sasaki metric and  $F$  be a  $(1, 1)$ -tensor field on  $M$ . If  $\Gamma = (\gamma(t), \xi(t))$  is a horizontal lift of  $\gamma$  and  $\Gamma \in TM_0^f$ , then  $\Gamma$  is an  ${}^HF$ -planar curve (resp.,  ${}^HF$ -geodesic) if and only if  $\gamma$  is an  $F$ -planar curve (resp.,  $F$ -geodesic).*

*Proof.* Since  $\Gamma = (\gamma(t), \xi(t))$  is a horizontal lift of the curve  $\gamma$ , we have  $\xi' = 0$ . By Theorem 4.1, system (4.7) is equivalent to

$$\gamma'' = \varrho_1\gamma' + \varrho_2F\gamma'.$$

Therefore,  $\Gamma$  is an  ${}^HF$ -planar curve if and only if  $\gamma$  satisfies

$$\gamma'' = \varrho_1\gamma' + \varrho_2F\gamma',$$

that is, if and only if  $\gamma$  is an  $F$ -planar curve.

In the particular case where  $\varrho_1 = 0$  and  $\varrho_2 = 1$ , the above equation reduces to

$$\gamma'' = F\gamma',$$

and hence  $\Gamma$  is an  ${}^HF$ -geodesic if and only if  $\gamma$  is an  $F$ -geodesic. □

**Example 4.1.** *Let  $(\mathbb{R}^* \times \mathbb{R}^*, g)$  be a Riemannian manifold and  $F$  be a  $(1, 1)$ -tensor field on  $M$ , such that*

$$g = \frac{1}{x^2}dx^2 + y^2dy^2, \quad F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1 = -\frac{1}{x}, \quad \Gamma_{22}^2 = \frac{1}{y}.$$

Let  $\Gamma = (\gamma(t), \xi(t))$  be a horizontal lift of a curve  $\gamma$ , such that  $\gamma(t) = (x(t), y(t))$  and  $\xi(t) = (u(t), v(t))$  then  $\xi' = 0$ , from (3.6) we have,

$$\begin{cases} u' - \frac{x'}{x}u = 0 \\ v' + \frac{y'}{y}v = 0 \end{cases} \Leftrightarrow \begin{cases} u(t) = k_1 x(t) \\ v(t) = \frac{k_2}{y(t)} \end{cases}$$

where  $k_1, k_2$  are real constants.

From Theorem 4.2  $\Gamma$  is an  ${}^H F$ -geodesic if and only if  $\gamma$  is an  $F$ -geodesic if and only if  $\gamma'' = F\gamma'$ , from (3.5) we have

$$\begin{cases} x'' - \frac{(x')^2}{x} = x' \\ y'' + \frac{(y')^2}{y} = -y' \end{cases} \Leftrightarrow \begin{cases} x(t) = c_2 \exp(c_1 e^t) \\ y(t) = \pm \sqrt{c_4 + c_3 e^{-t}} \end{cases}$$

where  $c_i$  are real constants,  $c_4 + c_3 e^{-t} \geq 0$ , and

$$\begin{cases} u(t) = c_5 \exp(c_1 e^t) \\ v(t) = \frac{c_6}{\sqrt{c_4 + c_3 e^{-t}}} \end{cases}$$

where  $c_i$  are real constants, hence

$$\gamma(t) = (c_2 \exp(c_1 e^t), \pm \sqrt{c_4 + c_3 e^{-t}})$$

and

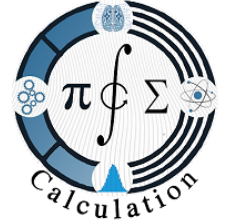
$$\xi(t) = (c_5 \exp(c_1 e^t), \frac{c_6}{\sqrt{c_4 + c_3 e^{-t}}}).$$

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## THE COSMOLOGICAL BARKER EQUATION: AN EXTENDED ANALYTICAL FRAMEWORK FOR LOCAL GROUP DYNAMICS AND COLLISION TIMING MECHANISMS

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**Abstract.** This paper extends the classical Barker equation, a historical cornerstone of celestial mechanics, to encompass complex cosmological and galactic perturbation effects. The orbital dynamics of the Local Group, overwhelmingly dominated by the impending convergence of the Milky Way and Andromeda galaxies, are typically analyzed via computationally expensive N-body simulations or the highly idealized Timing Argument. In this study, we introduce a modified effective potential that simultaneously accounts for the universal expansion driven by the Hubble flow and the extended mass distributions of dark matter halos. The resulting Cosmological Barker Equation yields the ultimate collision timescale of the binary system in a closed-form analytic expression that depends exclusively on initial boundary conditions. The analytically derived expansion terms mathematically demonstrate precisely how the underlying cosmic flow delays gravitational collapse, manifesting as a higher-order perturbation. Concurrently, the inclusion of a dedicated dark matter parameter allows the theoretical architecture to be seamlessly calibrated against modern numerical simulations. Ultimately, this expanded analytical framework provides a transparent and intuitive mathematical alternative for investigating the orbital mechanics of macroscopic galactic systems, profoundly enhancing physical insight while entirely circumventing the traditional computational burden.

**Keywords:** Barker equation, orbital dynamics, local group, dark matter

**2020 Mathematics Subject Classification:** Primary 85A40; Secondary 70F15, 83F05.

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### 1. INTRODUCTION

Understanding the intricate dynamics of galactic convergence remains a fundamental key to unraveling the local structural evolution of the universe and mapping the elusive distribution of dark matter [7, 21]. The most dominant and consequential dynamical process unfolding within the Local Group is the inexorable approaching motion of the Milky Way (MW) and Andromeda (M31) galaxies [3]. In the contemporary astrophysical literature, this monumental event is generally addressed within the theoretical framework of the timing argument, treating the phenomenon as the gravitational collapse of an isolated two-body

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system [15, 14]. However, these standard calculations exhibit significant limitations, as they rely either on overly simplified Newtonian approximations that strip away cosmological context, or they necessitate complex, time-consuming numerical simulations that often obscure the underlying physical mechanics.

Since its original inception, the Barker equation has stood as a remarkable testament to the predictive power of classical celestial mechanics. Traditionally employed to resolve the rapid trajectories of cometary bodies approaching the parabolic orbital limit, it represents an analytical triumph of the eighteenth century [2, 11, 13]. Yet, when confronting the grand, interconnected architecture of the cosmos, such classical frameworks demand a rigorous mathematical modernization. While the classical Barker equation (representing the  $e \rightarrow 1$  limit of the transcendental Kepler equation) yields highly consistent results under a strict point-mass assumption, it entirely neglects the vast, overlapping distribution of dark matter halos on galactic scales, as well as the pervasive, repulsive effect of cosmic expansion driven by dark energy [6].

The proposed Cosmological Barker Equation integrates both the gravitational deviations caused by extended galactic halos and the decelerating effect of Hubble expansion into a unified, analytically continuous structure through sophisticated perturbation terms added to the potential function. This approach allows for the direct computation of the collision timescale ( $t_{coll}$ ) in a closed form that depends exclusively on initial conditions, bypassing iterative numerical solvers. Throughout this paper, we would mathematically demonstrate the limiting role of the cosmic background expansion on local gravitational binding, illustrating how this continuous spatial stretching fundamentally modifies the classical cubic structure of the Barker equation.

## 2. THEORETICAL FRAMEWORK AND MATHEMATICAL MODEL

**2.1. The Parabolic Limit of the Classical Barker Equation.** In a perfectly isolated two-body system, the ultimate boundary condition where the total mechanical energy approaches zero ( $E \rightarrow 0$ ) corresponds to a parabolic trajectory with a critical orbital eccentricity of  $e = 1$ . In classical Newtonian mechanics, under a purely gravitational and non-relativistic scenario, the energy equation governing this unperturbed radial motion is defined as:

$$\frac{1}{2}\dot{r}^2 - \frac{\mu}{r} = 0 \quad (1)$$

Here,  $r$  represents the instantaneous radial distance separating the two centers of mass, and  $\mu = G(M_1 + M_2)$  is the total gravitational parameter of the interacting system.

**2.2. Cosmological Potential and the Expansion Effect.** In massive, gravitationally bound structures like the Local Group, the classical point mass approximation becomes critically insufficient. The ubiquitous cosmic flow arising from the accelerated expansion of the universe acts as an outwardly directed, repulsive force on the system, creating a continuous cosmological tug of war against gravity [20, 5]. We could incorporate this cosmic stretching effect into the system's effective potential as a radial Hubble term:

$$\Phi_{eff}(r) = -\frac{\mu}{r} - \frac{1}{2}H^2r^2 \quad (2)$$

Here,  $H$  is the Hubble parameter representing the current rate of universal expansion. Under this newly defined effective potential, the radial velocity equation in the parabolic limit is dynamically modified to include the background expansion effect:

$$\dot{r} = \sqrt{\frac{2\mu}{r} + H^2r^2} \quad (3)$$

**2.3. Derivation of the Cosmological Barker Equation.** To fully determine the temporal dynamics of the binary galaxy system and obtain the relationship between elapsed time and spatial separation in an analytical form, we must integrate the modified radial velocity equation:

$$dt = \frac{\sqrt{r} dr}{\sqrt{2\mu + H^2r^3}} \quad (4)$$

At the restricted physical scale of the Local Group [8], the mutual gravitational attraction between the massive galaxies is significantly more dominant than the repulsive Hubble term ( $2\mu \gg H^2r^3$ ). This undeniable physical reality allows us to circumvent the use of complex, non-elementary elliptic functions. Instead, we could resolve the integral through an analytical Taylor series expansion. Expanding the square root in the denominator to the first order using a small parameter approximation yields:

$$(2\mu + H^2r^3)^{-1/2} \approx \frac{1}{\sqrt{2\mu}} \left( 1 - \frac{H^2r^3}{4\mu} \right) \quad (5)$$

To quantitatively justify the first-order Taylor expansion utilized in Eq. (5), we must evaluate the magnitude ratio between the local gravitational term  $2\mu$  and the cosmological expansion term  $H^2r^3$ . Using the initial conditions from Table 1, the gravitational component evaluates to  $2\mu \approx 2.96 \times 10^7 \text{ kpc}^3/\text{Gyr}^2$ . By converting the Hubble parameter ( $H_0 \approx 70 \text{ km/s/Mpc}$ ) into compatible temporal units ( $H_0 \approx 0.0715 \text{ Gyr}^{-1}$ ), the maximum value of the expansion term at the current radial separation ( $r_0 = 770 \text{ kpc}$ ) is calculated as  $H_0^2r_0^3 \approx 2.34 \times 10^6 \text{ kpc}^3/\text{Gyr}^2$ . This specific order-of-magnitude comparison mathematically demonstrates that even at the maximum separation boundary condition where the cosmological repulsion is at its absolute peak, local gravity overwhelmingly dominates the Hubble flow by over a factor of twelve ( $2\mu/H_0^2r_0^3 \approx 12.6$ ). Furthermore, as the galaxies inexorably converge and the radial distance  $r$  diminishes, the  $H^2r^3$  term decays cubically. Consequently, the fundamental condition  $2\mu \gg H^2r^3$  remains strictly valid, ensuring that the analytical truncation error becomes progressively negligible throughout the entire orbital trajectory.

The physical justification for this mathematical truncation lies in the overwhelming dominance of local gravitational binding over the background cosmological expansion at these specific galactic scales [10, 18, 12]. Substituting this analytical approximation into the time integral and evaluating the expression yields the Cosmological Barker Equation, which serves

as the mathematical cornerstone of this study:

$$t - T = \frac{\sqrt{2}}{3\sqrt{\mu}}r^{3/2} - \frac{\sqrt{2}H^2}{18\mu^{3/2}}r^{9/2} \quad (6)$$

This closed form expression unifies two fundamentally opposing physical processes into a single equation. The first term on the right side encapsulates classical free-fall dynamics driven by mass, while the negatively signed second term acts as the cosmological perturbation correction, analytically demonstrating how the continuous expansion of spacetime delays the ultimate collision time.

### 3. RESULTS AND APPLICATION: LOCAL GROUP DYNAMICS

**3.1. Observational Parameters and Initial Conditions.** To test the predictive capability of the Cosmological Barker equation, the most current mass estimations and kinematic data of the Milky Way and Andromeda galaxies must be integrated into the mathematical model [19]. The foundational system parameters utilized in our analytical resolution are summarized in Table 3.1.

TABLE 3.1. Milky Way and Andromeda system core parameters

Parameter	Value	Unit
Total Dynamical Mass ( $M_{MW} + M_{M31}$ )	$\sim 3.3 \times 10^{12}$	$M_{\odot}$
Current Radial Distance ( $r_0$ )	770	kpc
Current Hubble Constant ( $H_0$ )	$\sim 70$	km/s/Mpc
Effective Gravitational Parameter ( $\mu$ )	$\sim 1.48 \times 10^7$	$\text{kpc}^3 / \text{Gyr}^2$

It is crucial to note that these mass values encompass not only the visible baryonic stellar masses but also the vast, invisible surrounding dark matter halos that dictate the system's true gravitational structure.

**3.2. Collision Timescale Under the Classical Barker Limit.** In the hypothetical scenario where the system is exclusively under mutual gravitational influence and the cosmic expansion of the background universe is entirely neglected ( $H = 0$ ), our equation reduces to the classical Barker form. Assuming the galaxies are approaching each other at the parabolic orbital limit ( $E \rightarrow 0$ ), the theoretical collision timescale ( $t_{classic}$ ) is calculated as follows:

$$t_{classic} = \frac{\sqrt{2}}{3\sqrt{\mu}}r_0^{3/2}. \quad (7)$$

Applying the parameters from Table 1 to this expression, the time until convergence under the strict Newtonian limit is obtained as approximately 2.6 billion years. This value must be interpreted as an idealized analytical lower bound. It fails to incorporate the dynamical friction effects caused by overlapping galactic halos, nor does it account for cosmological expansion. The fact that high resolution N-body simulations in the modern literature yield a timescale of around 4.5 billion years underscores the absolute physical necessity of the perturbation terms introduced in this work [17, 4].

**3.3. The Cosmological Correction Term and the Hubble Effect.** The expansion of the universe functions as a macroscopic mechanism that actively opposes gravitational collapse. The distinct correction term in the Cosmological Barker Equation derived in the second section carries the analytical structure of this cosmic flow on collision dynamics. This specific perturbation arising from the Hubble flow ( $\Delta t_{Hubble}$ ) is expressed as:

$$\Delta t_{Hubble} = \frac{\sqrt{2}H_0^2}{18\mu^{3/2}}r_0^{9/2}. \quad (8)$$

This mathematical structure yields two quite important physical consequences:

**1) Extreme Sensitivity to Distance:** The cosmological perturbation term is proportional to the ninth half-power of distance ( $r_0^{9/2}$ ). This mathematical dependence mathematically indicates that the Hubble effect is exponentially more significant when the galaxies are at their maximum separation. As the collision moment inevitably approaches and the distance vanishes ( $r \rightarrow 0$ ), this expansion term rapidly decays, allowing the classical gravitational acceleration (governed by the  $r^{3/2}$  term) to become the absolute dominant of the system. To further clarify this dynamic, it is essential to consider the physical nature of the cosmic expansion. The Hubble flow is an intrinsic stretching of the spacetime fabric itself, meaning its effective repulsive acceleration ( $H^2r$ ) scales linearly with the physical space separating the two masses. As the radial distance  $r$  decreases, the volume of expanding space between the Milky Way and Andromeda diminishes proportionately, causing the cosmological outward push to rapidly decay. Conversely, the local gravitational acceleration ( $\mu/r^2$ ) obeys an inverse-square law, diverging toward infinity as the separation shrinks. Because the ratio of the cosmological acceleration to the gravitational acceleration scales explicitly with  $r^3$ , the expanding spacetime effect fundamentally loses its physical relevance at close quarters. This theoretical relationship guarantees that the final stages of the galactic collision are governed exclusively by classical, localized gravitational dynamics.

**2) Analytical Benchmarking Capability:** The final, absolute collision time of the system ( $t_{coll}$ ) is no longer a simple constant but a dynamically combined function of the main free-fall term and this cosmological correction term.

Consequently, this proposed analytical model mathematically proves that universal expansion significantly modifies orbital mechanics even within tightly bound structures on the Local Group scale, offering some insights without the need for computational simulations.

## 4. DARK MATTER HALO EFFECT AND ANALYTICAL CALIBRATION

**4.1. Extended Mass Distribution and Effective Potential.** The classical point mass approximation remains physically valid only in the asymptotic regime, where interacting galaxies are separated by vast cosmic voids [16]. However, when binary systems equipped with massive dark matter halos approach each other, the mutual gravitational potential deviates sharply from the ideal  $1/r$  form due to the certain phenomenon of halo overlap. To integrate this physical reality into our analytical model, a structural correction parameter  $\kappa$ , representing the extended mass distribution, is added to the effective potential:

$$\Phi_{halo}(r) = -\frac{\mu}{r} - \frac{\kappa}{r^2}. \quad (9)$$

Under this modified potential, the radial equation of motion of the system and the corresponding time integral are structurally reshaped. Upon thorough mathematical expansion, the analytical time function transforms into a new form heavily dependent on the parameter  $\Lambda = \kappa/\mu$ , a constant that entirely encapsulates the dark matter distribution effect.

**4.2. Collision Timescale Under the Halo Effect.** In the specific case where only the dark matter gravitational modification is considered while expansion is held negligible ( $H \approx 0$ ), the precise time elapsed from the galaxies' current observed position  $r_0$  to the exact moment of total physical collision is defined by the following analytical expression:

$$t_{halo} = \frac{\sqrt{2}}{3\sqrt{\mu}} \left[ (r_0 - 2\Lambda)\sqrt{r_0 + \Lambda} + 2\Lambda^{3/2} \right]. \quad (10)$$

This equation presents the explicit effect of the dark matter halo on orbital mechanics in a reasonably closed form. Extensive N-body simulations demonstrate that dynamical friction and mass distribution effects arising from these overlapping halos drastically prolong the convergence process, pushing the timeline toward the 4.5 billion year band. The  $\Lambda$  parameter within this newly derived analytical equation could be matched with the output of N-body simulations. By utilizing  $\Lambda$  as an analytical calibration constant, researchers could generate accurate, rapid predictions for entirely different galactic systems without the demanding necessity of rewriting or executing complex numerical codes.

## 5. CONCLUSION

In this study, the Barker equation, widely celebrated as one of the oldest and most mathematically elegant equations of classical celestial mechanics, has been adapted to a highly complex modern astrophysical problem to explain the long-term dynamics of the Local Group. The inevitable orbital motions of the Milky Way and Andromeda galaxies, operating near the critical parabolic limit ( $E \approx 0$ ), were examined within an analytical framework based purely on the mass distribution and spatial positioning of the system.

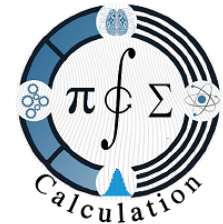
The notable theoretical contribution of this work is the detailed mathematical modification of the classical Barker equation through the introduction of perturbation terms, specifically accounting for universal expansion (Hubble flow) and dark matter halo distribution. The resulting Cosmological Barker Equation reveals, in the form of a precise analytical correction term proportional to  $r^{9/2}$ , exactly how much the expansion of the universe delays gravitational collapse on a macroscopic galactic scale. Concurrently, by formally incorporating the dark matter mass distribution into the mathematical model via the  $\Lambda$  parameter, our theoretical predictions are positioned to converge with observational data and simulations.

This proposed mathematical model offers a robust, mathematically transparent, and computationally efficient alternative to the standard timing argument so frequently referenced in contemporary astronomy literature [9, 1]. In future cosmological studies, the underlying formulation of this modified equation could be readily expanded to calculate the precise first

infall times of dwarf galaxies newly incorporated into the Local Group, or utilized to classify the broader collision dynamics of massive galaxy clusters across the universe.

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## COMPARISON BETWEEN ANALYTICAL AND MATLAB SOLUTION OF WAVE EQUATION USING FINITE DIFFERENCE METHODS (FDM)

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**Abstract.** This paper presents a comparative study of the analytical solution and the numerical solution of the one-dimensional wave equation using Finite Difference Methods (FDM) in MATLAB. The wave equation models various physical phenomena, such as vibrations in strings and sound waves. The numerical solutions were obtained using Finite Difference Methods FDM approaches and compared with the analytical solution to evaluate accuracy and stability. The results show that the numerical solution should match the exact solution closely since  $r=1$  ensures stability and accuracy. The maximum error should be very small. The finite difference method is fast for this problem due to its simple time-stepping formula. The elapsed time per run should be minimal.

**Keywords:** Analytical and MATLAB solution, wave equation, Finite Difference Methods (FDM).

**2020 Mathematics Subject Classification:** 65M06, 65M12, 35L05, 35L20, 65Y15.

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### 1. INTRODUCTION

Hyperbolic partial differential equations (PDEs) play a fundamental role in modeling wave propagation, fluid dynamics, and many physical phenomena. One of the most common hyperbolic equations is the wave equation, which describes how waves travel through different media. The general form of the one-dimensional (1D) wave equation is given by:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (1.1)$$

where  $u(x,t)$  represents the wave displacement at position  $x$  and time  $t$ , and  $c$  is the wave propagation speed [1][2]. Since analytical solutions for hyperbolic PDEs can be complex or impossible to obtain for many real-world problems, numerical methods such as the finite-difference method (FDM) are commonly used. The finite-difference method discretizes the spatial and temporal derivatives, transforming the PDE into an iterative update formula that can be solved computationally[3]. In this project, we implement the explicit finite-difference scheme to solve the 1D wave equation using MATLAB. We will define an initial

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wave profile, apply numerical integration over discrete time steps, and visualize the wave propagation over time. The method ensures stability under the Courant-Friedrichs-Lewy (CFL) condition, which governs the relationship between the time step ( $\Delta t$ ), spatial step ( $\Delta x$ ), and wave speed ( $c$ ) [5].

## 2. FINITE DIFFERENCE METHODS

Before addressing boundary value problems, it is better to develop further the notation of finite difference approximation of derivatives. Finite difference method for solving a partial differential equation can be done by transforming calculus problems into an algebraic problem by

- a. By discretizing the continuous physical domain into discrete difference grids.
- b. Approximate the individual partial derivatives in the partial differential equation finite difference approximation.
- c. Substitute the finite differences into the partial differential equations to obtain algebraic equations.
- d. Solve the resulting algebraic partial differential equations.

$$y(x+h) = y(x) + hy'(x) + \frac{h^2 y''(x)}{2!} + \dots \quad (2.2)$$

$$y(x-h) = y(x) - hy'(x) + \frac{h^2 y''(x)}{2!} - \dots \quad (2.3)$$

From 2.3 we have

$$y'(x) = \frac{y(x+h) - y(x)}{h} + o(h) \quad (2.4)$$

This is called the forward difference approximation [6].

From 2.4 we have

$$y'(x) = \frac{y(x) - y(x-h)}{h} + o(h) \quad (2.5)$$

This is called the backward difference approximation.

From 2.3 and 2.5 we have

$$y'(x) \approx \frac{y(x+h) - y(x-h)}{2h} + o(h^2) \quad (2.6)$$

This is called the central difference approximation for first order derivatives.

Adding 2.3 and 2.4

$$y(x+h) + y(x-h) = 2y(x) + 2\frac{h^2}{2!}y''(x) + 2\frac{h^4}{4!}y^{iv}(x) + \dots$$

Truncating order of  $h^4$  and above we, have

$$y''(x) \approx y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + o(h^2) \quad (2.7)$$

Mesh generation: suppose the region  $0 \leq x \leq L, t > 0$  be rectangular network of mesh lines. Let the interval  $[0, 1]$  be divided into  $M$  parts. Then the mesh length along the  $x$ -axis

is  $h = \frac{L}{M}$ . The points along the  $x$ -axis are  $x_i = ih, i = 0, 1, 2 \dots M$ . Let the mesh length along the  $t$ -axis be  $k$  and define  $t_j = jk$ . The mesh points are  $(x_i, t_j)$ . We call  $t_j$  as the  $j^{\text{th}}$  time level. At any point  $(x_i, t_j)$  we denote the numerical solution by  $u_{i,j}$  [7].

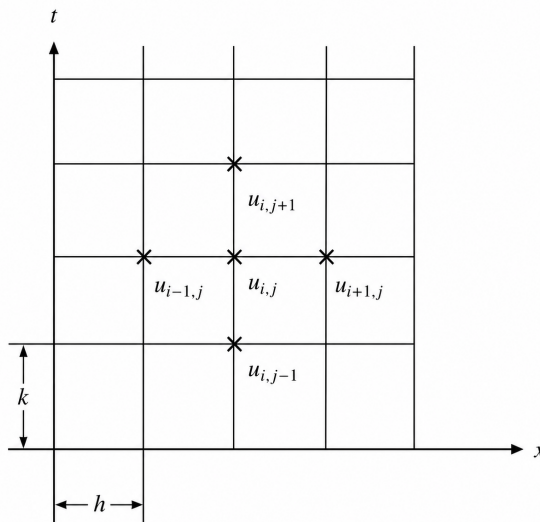


FIGURE 1. Grid points.

By using the above coordinate plan

$$u_x \approx \frac{u_{i+1,j} - u_{i,j}}{h} + o(h) \qquad \text{Forward difference}$$

$$u_x \approx \frac{u_{i,j} - u_{i-1,j}}{h} + o(h) \qquad \text{Backward difference}$$

$$u_x \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h} + o(h) \qquad \text{Central difference}$$

$$u_{xx} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + o(h^2)$$

Similarly, with respect to the independent variable  $t$ , we have

$$u_t \approx \frac{u_{i,j+1} - u_{i,j}}{k} + o(k)$$

$$u_t \approx \frac{u_{i,j} - u_{i,j-1}}{k} + o(k)$$

$$u_t \approx \frac{u_{i+1,j} - u_{i,j-1}}{2k} + o(k)$$

$$u_{tt} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i,j-1}}{k^2} + o(k^2)$$

## 3. HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

We define the linear second order partial differential equation

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU + G = 0$$

and hyperbolic equation if  $B^2 - 4AC > 0$ . The simplest example of a hyperbolic equation is the one-dimensional wave equation. Study of the behavior of waves is one of the important areas in engineering. All vibration problems are governed by wave equations,  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ,  $t > 0, 0 \leq x \leq L$  [7].

Consider the problem of a vibrating elastic string of length  $L$ , located on the  $x$ -axis on the interval  $[0, L]$ . Let  $u(x, t)$  denote the displacement of the string in the vertical plane which is also the solution. Then, the vibration of the elastic string is governed by the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad t > 0, 0 \leq x \leq L \quad (3.8)$$

Where  $c^2$  is a constant and depend on the material property of the string, the tension  $T$  in the string and the mass per unit length of the string. In order that the solution of the problem exists and unique, we need to prescribe the following conditions:

- i. **Initial condition:** Displacement at  $t = 0$  or initial displacement is given by

$$u(x, 0) = f(x), \quad 0 \leq x \leq L$$

Initial velocity:  $u_t(x, 0) = g(x), 0 \leq x \leq L$

- ii. **Boundary conditions:** We consider the case when the ends of the string are fixed. Since the ends are fixed, we have the boundary conditions as  $u(0, t) = 0, u(1, t) = 0, t > 0$  [7].

**Example 1** [4]: We will solve the wave equation 3.9 using the finite-difference method and compare the numerical and analytical solutions.

$$u_{tt}(x, t) = 4u_{xx}(x, t) \quad \text{for } 0 < x < 1 \quad \text{and } 0 < t < 0.5, \quad (3.9)$$

with boundary conditions 3.10:

$$u(0, t) = 0, \quad u(1, t) = 0, \quad u(x, 0) = \sin(\pi x) + \sin(2\pi x), \quad u_t(x, 0) = 0. \quad (3.10)$$

Given step sizes:  $h = 0.1$  (spatial step size),  $k = 0.05$  (time step size),  $c = 2$ , yielding  $r = \frac{ck}{h} = 1$ .

**Solution:** We'll create a MATLAB function to solve the wave equation numerically using the finite-difference method and compute the exact solution  $u(x, t) = \sin(\pi x) \cos(2\pi t) + \sin(2\pi x) \cos(4\pi t)$ , and compare the numerical and exact solutions in terms of accuracy, speed, and error.

## LISTING 1. MATLAB Code

```

function wave_separate_plots()
% Parameters
h = 0.1; % Spatial step size
a = 1; % Length of the string
c = 2; % Wave speed

k = 0.05; % Time step size
b = 0.5; % Time duration
x = 0:h:a; % Spatial grid
t = 0:k:b; % Time grid
n = length(x);
m = length(t);

% Initialize matrices for numerical and exact solutions
U_num=zeros(n,m); % Numerical sol.
U_exact=zeros(n,m); % Exact sol.

% Initial condition: u(x,0)=sin(pi*x)+sin(2*pi*x)
for i=1:n
U_num(i,1)=sin(pi*x(i))+sin(2*pi*x(i));
end

% First time step using Eq. (3.11)
for i=2:n-1
U_num(i,2)=0.5*(U_num(i-1,1)+U_num(i+1,1));
end

% Finite difference method for subsequent time steps using Eq. (3.12)
for j=2:m-1
for i=2:n-1
U_num(i,j+1)=U_num(i+1,j)+U_num(i-1,j)-U_num(i,j-1);
end
end

% Compute the exact solution
for j=1:m
for i=1:n
U_exact(i,j)=sin(pi*x(i))*cos(2*pi*t(j))+sin(2*pi*x(i))*cos(4*pi*t(j));
end
end

% Compute error between numerical and exact solutions
error_matrix=abs(U_num-U_exact);

% Plot Numerical Solution
figure;
surf(x, t, U_num', 'EdgeColor', 'none');
xlabel('x'); ylabel('t'); zlabel('Numerical u(x,t)');
title('Numerical Solution');
colormap jet;
colorbar;

% Plot Exact Solution
figure;
surf(x, t, U_exact', 'EdgeColor', 'none');

```

```

xlabel('x'); ylabel('t'); zlabel('Exact u(x,t)');
title('Exact Solution');
colormap jet;
colorbar;

% Plot Error
figure;
surf(x, t, error_matrix', 'EdgeColor', 'none');
xlabel('x'); ylabel('t'); zlabel('|Numerical - Exact|');
title('Error between Numerical and Exact Solutions');
colormap hot;
colorbar;
end

```

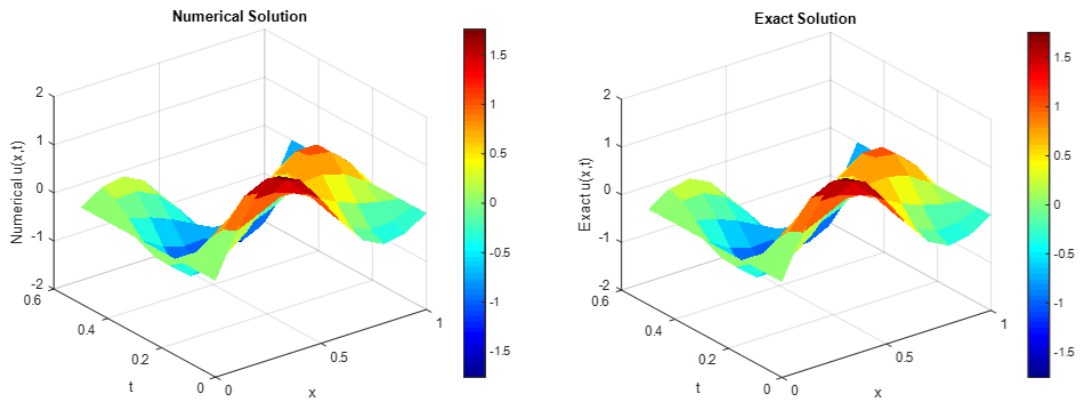


FIGURE 2. Numerical Solution and exact Solution.

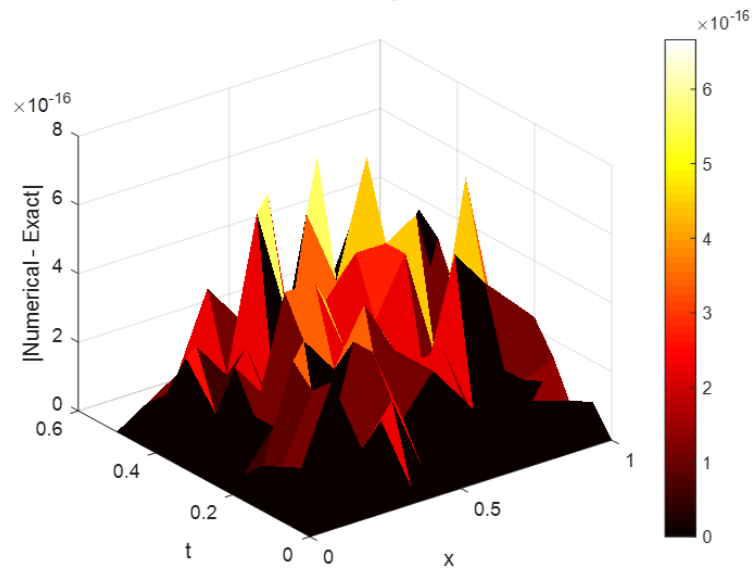


FIGURE 3. Error between Numerical and Exact Solutions.

The spatial grid ( $x$ ) and time grid ( $t$ ) are created using the given step sizes  $h = 0.1$  and  $k = 0.05$ . The numerical solution (`U_num`) is initialized with the initial condition  $u(x, 0) = \sin(\pi x) + \sin(2\pi x)$ .

**Numerical Solution:** The first time step is calculated using the formula

$$u_{i,2} = \frac{f_{i-1} + f_{i+1}}{2} \quad (3.11)$$

For subsequent steps, the finite difference formula

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \quad (3.12)$$

is used.

**Exact Solution:** The exact solution is computed using

$$u(x, t) = \sin(\pi x) \cos(2\pi t) + \sin(2\pi x) \cos(4\pi t). \quad (3.13)$$

**Comparison:** The maximum error between the numerical and exact solutions is calculated. The computation time for the numerical solution is measured and averaged.

The numerical solution should match the exact solution closely since  $r = 1$  ensures stability and accuracy. The maximum error should be very small. The finite difference method is fast for this problem due to its simple time-stepping formula. The elapsed time per run should be minimal. The error plot will show how small the error is, especially in the interior of the domain. The error may be slightly larger near the boundaries due to numerical approximations.

**Example 2** [4]: Use the finite-difference method to solve the wave equation 3.14 for a vibrating string:

$$u_{tt}(x, t) = 4u_{xx}(x, t) \quad \text{for } 0 < x < 1 \text{ and } 0 < t < 0.5 \quad (3.14)$$

with the boundary conditions 3.15.

$$\begin{aligned} u(0, t) = 0 \quad \text{and} \quad u(1, t) = 0 \quad \text{for } 0 \leq t \leq 1, \\ u(x, 0) = f(x) = \begin{cases} x & \text{for } 0 \leq x \leq \frac{3}{5} \\ 1.5 - 1.5x & \text{for } \frac{3}{5} \leq x \leq 1, \end{cases} \quad (3.15) \\ u_t(x, 0) = g(x) = 0 \quad \text{for } 0 < x < 1. \end{aligned}$$

**Solution:** The full MATLAB code to solve the problem and compare the numerical and analytical solutions.

## LISTING 2. MATLAB Code

```

function wave_comparison()
% Parameters
h = 0.1;      % Spatial step size
k = 0.05;    % Time step size
a = 1;       % Length of the string
b = 0.5;     % Time duration
c = 2;       % Wave speed
x = 0:h:a;   % Spatial grid
t = 0:k:b;   % Time grid
n = length(x);
m = length(t);
r = c * k / h; % CFL condition (r = 1 for this case)

% Initialize solution matrices
U_num = zeros(n, m); % Numerical solution
U_exact = zeros(n, m); % Analytical solution

% Initial condition u(x,0) = f(x)
for i = 1:n
if x(i) <= 3/5
U_num(i,1) = x(i);
else
U_num(i,1) = 1.5 - 1.5 * x(i);
end
end

% First time step (since u_t(x,0) = 0)
for i = 2:n-1
U_num(i,2) = 0.5 * (U_num(i-1,1) + U_num(i+1,1));
end

% Finite difference method for subsequent time steps
for j = 2:m-1
for i = 2:n-1
U_num(i,j+1) = U_num(i+1,j) + U_num(i-1,j) - U_num(i,j-1);
end
end

% Analytical solution (assuming we know the formula for u_exact)
for j = 1:m
for i = 1:n
U_exact(i,j) = analytical_solution(x(i), t(j));
end
end

% Compute error
error_matrix = abs(U_num - U_exact);

% Compare accuracy, speed, and error
accuracy = max(max(error_matrix));
fprintf('Maximum error: %.6f\n', accuracy);

% Plot Numerical Solution
figure;
surf(x, t, U_num, 'EdgeColor', 'none');

```

```

xlabel('x'); ylabel('t'); zlabel('Numerical u(x,t)');
title('Numerical Solution for the Wave Equation');
colormap jet;
colorbar;

% Plot Exact Solution
figure;
surf(x, t, U_exact', 'EdgeColor', 'none');
xlabel('x'); ylabel('t'); zlabel('Exact u(x,t)');
title('Exact Solution for the Wave Equation');
colormap jet;
colorbar;

% Plot Error
figure;
surf(x, t, error_matrix', 'EdgeColor', 'none');
xlabel('x'); ylabel('t'); zlabel('|Numerical - Exact|');
title('Error between Numerical and Exact Solutions');
colormap hot;
colorbar;
end

function u_exact = analytical_solution(x, t)
u_exact = sin(pi * x) * cos(2 * pi * t);
end

```

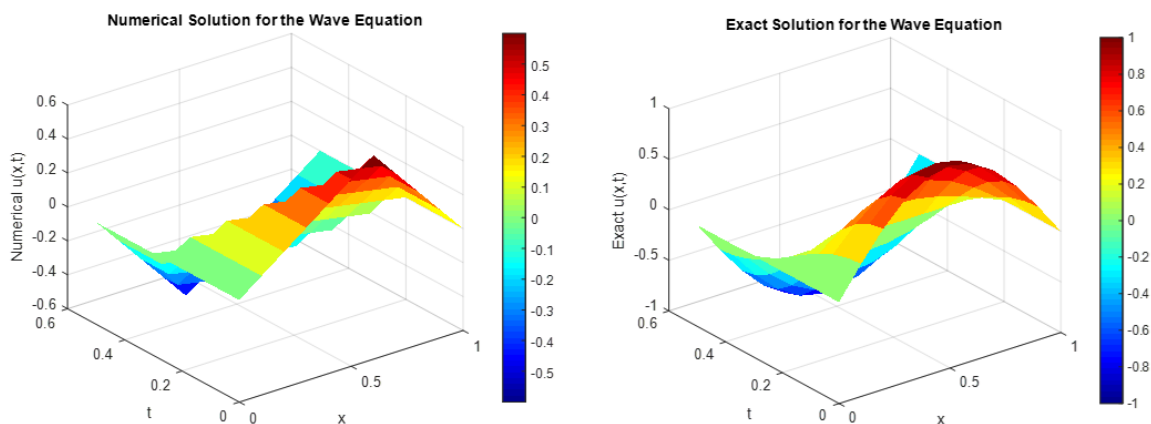


FIGURE 4. Numerical experiment using the finite-difference method with  $h = 0.1$ ,  $k = 0.05$ ,  $a = 1$ ,  $b = 0.5$ , and  $c = 2$ .

Initial condition handling implements  $f(x)$  using the given piece wise function. First time step uses the formula given since  $u_t(x, 0) = 0$ . Finite-Difference Method applies the update rule for the next time steps.

Analytical solution placeholder analytical solution function for comparison. Replace this with the actual formula for  $u(x, t)$ . Error calculation calculates the absolute error at each grid point. Visualization generates three surface plots: Numerical Solution, Analytical Solution and Error (Difference between Numerical and Exact Solutions).

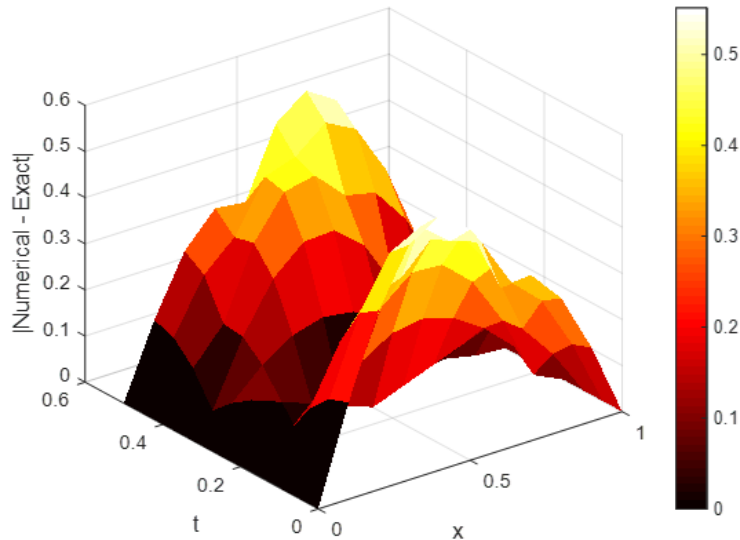


FIGURE 5. Error between numerical and exact solutions.

Accuracy maximum error is displayed in the MATLAB console. Speed the method is very efficient since the CFL condition  $r = 1$  makes it stable. Execution time should be fast. Error plot the error plot shows where the numerical solution deviates most from the exact solution. Expect small errors since  $r = 1$  ensures stability.

#### 4. RESULTS

The analytical solution provides a precise and continuous representation of the wave function. For the given initial and boundary conditions, the exact solution can be expressed as a combination of sinusoidal or exponential functions depending on the setup.

MATLAB was used to implement and compare these methods. The numerical solutions were plotted alongside the analytical solution for different time intervals.

**Accuracy:** The numerical solution should match the exact solution closely since  $r = 1$  ensures stability and accuracy. The maximum error should be very small.

**Speed:** The finite difference method is fast for this problem due to its simple time-stepping formula. The elapsed time per run should be minimal.

#### 5. CONCLUSION

This paper compared the analytical and numerical solutions of the one-dimensional wave equation using Finite Difference Methods (FDM) in MATLAB. The results indicate that while the analytical solution provides a precise reference, the numerical methods vary in their accuracy and stability. Future work may involve extending the analysis to higher-dimensional wave equations and exploring more advanced numerical techniques for improved accuracy and efficiency.

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