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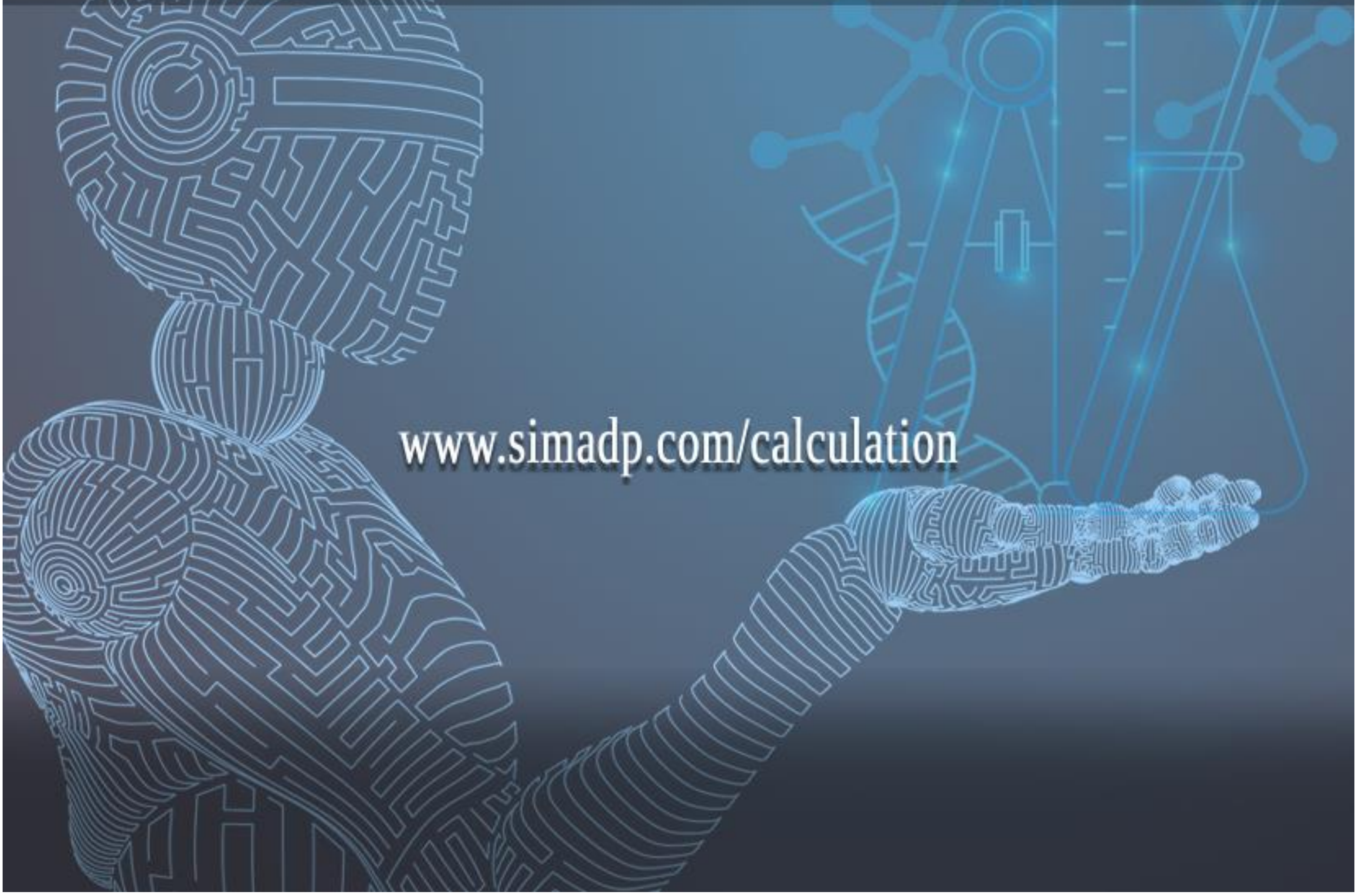

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ON THE TRINAJSTIC INDEX OF SOME ZERO DIVISOR GRAPHS

ALPARSLAN CENIKLI AND ARIF GÜRSOY *

Abstract. In this paper, the Trinajstic index, a novel topological index, is analyzed within the framework of basic concepts in Graph Theory, particularly focusing on Zero-Divisor Graphs, excluding trees. The Trinajstic index, initially developed in the context of Chemical Graph Theory, investigates chemical structures based on a distance-balance concept. After constructing a pseudocode to calculate the Trinajstic index, the relevant algorithms were implemented using MATLAB. Subsequently, MATLAB codes for generating graphs and calculating the Trinajstic index were combined to compute the index for various graphs. Formulas relating to prime-based Zero-Divisor Graphs were derived and proven.

Keywords: Graph theory, Chemical graph theory, Topological index, Trinajstic index, Zero-divisor graphs

2020 Mathematics Subject Classification: 05C09, 05C25.

1. Introduction

Graph theory plays a crucial role in many areas of science. Nowadays, graph theory is particularly essential in chemistry for representing chemical molecules as graphs, enabling deeper analysis and a better understanding of their structures. This necessitated the development of chemical graph theory. In chemical graph theory, numerous topological indices have emerged, including the Wiener index, Szeged index, Harary index, and others. Some topological indices are computed using the degrees of a graph, while others are determined based on the distances between its vertices. Additionally, various features can be explored to understand how different topological indices are calculated and what aspects they are related to. In 2022, the Trinajstic index, which will briefly be referred to as NT, was introduced by Boris Furtula [11]. He provided information on this index in the context of complete graphs, cycle graphs, path graphs, star graphs, and trees. This topological index is based on distances between vertices to determine whether the structure is balanced. It is particularly relevant in chemical graph theory for understanding the balance of chemical structures.

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2. Preliminaries

The zero divisor graph is distinct from other types of graphs because of its construction. Zero divisor graph was studied on by I. Beck [4] and its construction is related to commutative rings that is related to algebraic combinatorics. Beck's definition for the zero divisor graph is that graph consists of vertices in R. If any two vertices of graph yields $xy = 0$, graph is zero divisor graph.

The Trinajstic index was defined for connected, undirected and simple graph G as follows:

$$NT(G) = \sum_{u,v \in V(G)} (n_u - n_v)^2 \tag{2.1}$$

where n_u is number of vertices closer to u than v, n_v is number of vertices closer to v than u.

This topological index is distance-based, and can also be referred to as distance-balance-based topological index to understand of graph structure.

3. Trinajstic topological index of $\Gamma(\mathbb{Z}_n)$

Zero divisor graph of $\mathbb{Z}$ is popular for especially in chemical graph theory. For that reason the Trinajstic index could be also considered on zero divisor graphs for $n = \rho^2$, $n = \rho^3$, $n = \rho q$, $n = \rho^2 q$ and $n = \rho q r$. In this section, we will focus on Trinajstic index of $\Gamma(\mathbb{Z}_n)$.

Theorem 3.1. *Let ρ be a prime number and be $n = \rho^3$. Trinajstic index of $\Gamma(\mathbb{Z}_{\rho^3})$ is as follows:*

$$NT\left(\Gamma\left(\mathbb{Z}_{\rho^3}\right)\right) = \rho(\rho^2 - \rho - 1)^2(\rho - 1)^2. \tag{3.2}$$

Proof. Vertex set of zero divisor graph could be partitioned as $V\ (\Gamma(Z_{\rho^3})) = V_1 \cup V_2$ and for $i,\ j \in 1, 2$ there are two subsets of $\left(\Gamma\left(\mathbb{Z}_{\rho^3}\right)\right)$ such that

$V_1 = \{\rho\alpha \mid \alpha = 1,\ 2,\ \ldots,\ \rho^2 - 1, \rho \nmid \alpha\}$,

$V_2 = \{\rho^2\alpha \mid \alpha = 1,\ 2,\ \ldots,\ \rho - 1\}$ where $V_1 \cap V_2 = \emptyset$.

In addition property of zero divisor graph with $n = \rho^3$, $\mid V_1 \mid = \rho(\rho - 1)$ and $\mid V_2 \mid = \rho - 1$ for all $u \in V_1$ and $v \in V_2$, n_u and n_v as follow:

$n_u = \mid V_2 \mid$ and

$n_v = 1$. Then,

$$\begin{aligned} NT\left(\Gamma\left(\mathbb{Z}_{\rho^3}\right)\right) &= \sum_{\{u,v\} \in V\left(\Gamma\left(\mathbb{Z}_{\rho^3}\right)\right)} (n_u - n_v)^2 \\ &= \mid V_1 \mid\mid V_2 \mid (\ |V_1| - 1)^2 \\ &= \rho(\rho^2 - \rho - 1)^2(\rho - 1)^2. \end{aligned}$$

□

The results for $\Gamma(\mathbb{Z}_{\rho^3})$ for prime $\rho < 20$ are listed in Table 3.1.

TABLE 3.1. Results of $\Gamma(\mathbb{Z}_{\rho^3})$

ρ	$n=\rho^3$	NT
2	8	2
3	27	300
5	125	28880
7	343	423612
11	1331	13069100
13	2197	44974800
17	4913	319615232
19	6859	715825836

Theorem 3.2. *Let ρ and q be prime numbers and $n=\rho q$. Trinajstic index of $\Gamma(\mathbb{Z}_{\rho q})$ is as follows:*

$$NT\left(\Gamma\left(\mathbb{Z}_{\rho q}\right)\right)=(\rho-1)(q-1)\left(\rho-q\right)^2 \tag{3.3}$$

Proof. Vertex set of zero divisor graph could be partitioned as $V\ (\Gamma(Z_{\rho q})) = V_1 \cup V_2$ and for $i,\ j\ \in\ 1,2$ there are two subsets of $(\Gamma\left(\mathbb{Z}_{\rho q}\right))$ such that

$V_1 = \{\rho\alpha \mid \alpha = 1,\ 2,\ \ldots,\ q-1\}$,

$V_2 = \{q\alpha \mid \alpha = 1,\ 2,\ \ldots,\ \rho-1\}$ where $V_1 \cap V_2 = \emptyset$.

Since $\mid V_1 \mid = q-1$ and $\mid V_2 \mid = \rho-1$, $\Gamma(Z_{\rho q})$ has $(\rho-1)(q-1)$ vertices. For set of sets pair $V_1,\ V_2$, we are able to construct graph as follow:

FIGURE 1. Structure of $\Gamma\left(\mathbb{Z}_{\rho q}\right)$

for all $u\in V_1$ and $v\in V_2$, n_u and n_v are as follow: $n_u = \mid V_1\mid$ and $n_v = \mid V_2 \mid$. Therefore $(n_u-n_v)^2 = (\mid V_1\mid - \mid V_2\mid)^2$. Now, we can calculate NT as

$$\begin{aligned} NT\left(\Gamma\left(\mathbb{Z}_{\rho q}\right)\right) = &\sum_{\{u,v\}\in V(\Gamma(\mathbb{Z}_{\rho q}))} (n_u-n_v)^2 \\ =&\mid V_1 \mid\mid V_2 \mid(\mid V_2\mid - \mid V_1\mid)^2 \\ =&(\rho-1)(q-1)(\rho-q)^2. \end{aligned}$$

□

Results pertaining to $\Gamma(\mathbb{Z}_{\rho q})$ for primes $\rho<50$ and $q<50$ are summarized in Table 5.2.

Theorem 3.3. *Let ρ and q be distinct prime numbers and $n=\rho^2 q$. Trinajstic index of $\Gamma(\mathbb{Z}_{\rho^2 q})$ is as follows:*

$$\begin{aligned} NT\left(\Gamma\left(\mathbb{Z}_{\rho^2 q}\right)\right) =&\rho\left(\rho-1\right)\left(\rho^5 q-\rho^5-2\rho^4 q^2+\rho^4 q+\rho^4+\rho^3 q^3+13\rho^3 q^2-19\rho^3 q+\right. \\ &\left.6\rho^3-3\rho^2 q^3-24\rho^2 q^2+36\rho^2 q-14\rho^2+4\rho q^3+6\rho q^2-3\rho q+\rho-q^2-5q+2\right) \end{aligned} \tag{3.4}$$

Proof. Since ρ, ρ^2 and q are divisors of $n = \rho^2 q$, Vertex set of zero divisor graph could be partitioned as $V\ (\Gamma(\mathbb{Z}_n)) = V_1 \cup V_2 \cup V_3 \cup V_4$, $i \neq j$ and for $i, j \in 1, \ldots, 4$ there are four subsets of $\left(\Gamma\left(\mathbb{Z}_{\rho^2 q}\right)\right)$ such that

$V_1 = \{\rho\alpha \mid \alpha = 1, 2, \ldots, \rho q - 1, \rho \nmid \alpha, q \nmid \alpha\}$,
$V_2 = \{q\alpha \mid \alpha = 1, 2, \ldots, \rho^2 - 1, \rho \nmid \alpha\}$,
$V_3 = \{\rho^2\alpha \mid \alpha = 1, 2, \ldots, q - 1\}$ and
$V_4 = \{\rho q\alpha \mid \alpha = 1, 2, \ldots, \rho - 1\}$.

Size of each subsets are $\mid V_1 \mid = (\rho - 1)(q - 1)$, $\mid V_2 \mid = \rho(\rho - 1)$, $\mid V_3 \mid = (q - 1)$, $\mid V_4 \mid = (\rho - 1)$, respectively. Graph was constructed by using V_1, V_2, V_3 and V_4 is below:

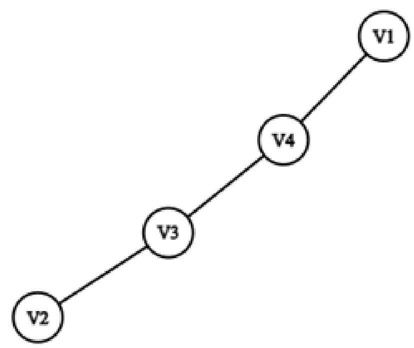

FIGURE 2. Structure of $\Gamma\left(\mathbb{Z}_{\rho^2 q}\right)$

In order to calculate Trinajstic index of $\Gamma\left(\mathbb{Z}_{\rho^2 q}\right)$ we must investigate these cases below:

Case 1: For the pair of sets V_1 and V_2 with $u \in V_1$ and $v \in V_2$, n_u and n_v are determined as follows:

$n_u = \mid V_4 \mid + \mid V_1 \mid$,
$n_v = \mid V_3 \mid + \mid V_2 \mid$.
Thus, $(n_u - n_v)^2 = (\mid V_4 \mid + \mid V_1 \mid - (\mid V_3 \mid + \mid V_2 \mid))^2$.

Case 2: For the pair of sets V_1 and V_3 with $u \in V_1$ and $v \in V_3$, n_u and n_v are as follows:

$n_u = 1$,
$n_v = \mid V_2 \mid + 1$.
Hence, $(n_u - n_v)^2 = (1 - |V_2| + 1)^2 = |V_2|^2$.

Case 3: For the pair of sets V_1 and V_4 with $u \in V_1$ and $v \in V_4$, n_u and n_v are described as follows:

$n_u = 1$,
$n_v = \mid V_1 \mid + \mid V_2 \mid + \mid V_3 \mid$.
Thus, $(n_u - n_v)^2 = (\mid V_1 \mid + \mid V_2 \mid + \mid V_3 \mid - 1)^2$.

Case 4: For the pair of sets V_2 and V_3 with $u \in V_2$ and $v \in V_3$, n_u and n_v values can be expressed as follows::

$n_u = \mid V_3 \mid$,
$n_v = \mid V_1 \mid + \mid V_2 \mid + \mid V_4 \mid$.

Therefore, $(n_u - n_v)^2 = (\mid V_1 \mid + \mid V_2 \mid + \mid V_4 \mid - \mid V_3 \mid)^2$.

Case 5: For the pair of sets V_2 and V_4 with $u \in V_2$ and $v \in V_4$, n_u and n_v are as follows:

$n_u = 1$,

$n_v = \mid V_1 \mid + \mid V_4 \mid$.

Hence, $(n_u - n_v)^2 = (\mid V_1 \mid + \mid V_4 \mid - 1)^2$.

Case 6: For the pair of sets V_3 and V_4 with $u \in V_3$ and $v \in V_4$, n_u and n_v are described as follows:

$n_u = 1 + \mid V_2 \mid$,

$n_v = \mid V_1 \mid + \mid V_3 \mid + 1 – 1 = \mid V_1 \mid + \mid V_3 \mid$.

Thus, $(n_u - n_v)^2 = (\mid V_1 \mid + \mid V_4 \mid - 1)^2$.

In this way, Trinajstic index of $\Gamma(\mathbb{Z}_(\rho^2 q))$ is

$$\begin{aligned} NT\left(\Gamma\left(\mathbb{Z}_{\rho^2 q}\right)\right) = & \sum_{\{u,v\} \in V\left(\Gamma\left(\mathbb{Z}_{\rho^2 q}\right)\right)} (n_u - n_v)^2 \\ = & \mid V_1 \mid\mid V_3 \mid\mid V_2 \mid^2 + \mid V_1 \mid\mid V_4 \mid (\mid V_1 \mid + \mid V_2 \mid + \mid V_3 \mid - 1)^2 + \\ & \mid V_2 \mid\mid V_3 \mid (\mid V_1 \mid + \mid V_2 \mid + \mid V_4 \mid - \mid V_3 \mid)^2 + \\ & \mid V_2 \mid\mid V_4 \mid (\mid V_1 \mid + \mid V_4 \mid - 1)^2 + \mid V_3 \mid\mid V_4 \mid (\mid V_1 \mid + \mid V_4 \mid - 1)^2 \end{aligned}$$

Then this equation will be

$$\begin{aligned} NT\left(\Gamma\left(\mathbb{Z}_{\rho^2 q}\right)\right) = & \rho(\rho - 1)\left(\rho^5 q - \rho^5 - 2\rho^4 q^2 + \rho^4 q + \rho^4 + \rho^3 q^3 + 13\rho^3 q^2 - 19\rho^3 q + 6\rho^3 - \right. \\ & \left. 3\rho^2 q^3 - 24\rho^2 q^2 + 36\rho^2 q - 14\rho^2 + 4\rho q^3 + 6\rho q^2 - 3\rho q + \rho - q^2 - 5q + 2\right). \end{aligned}$$

□

Table 5.3 provides the results for $\Gamma(\mathbb{Z}_{\rho^2 q})$ for primes $\rho < 20$ and $q < 20$.

Theorem 3.4. *Let ρ, q and r be distinct prime numbers and $n = \rho q r$. Trinajstic index of $\Gamma(\mathbb{Z}_{\rho q r})$ is as follows:*

$$\begin{aligned} NT(\Gamma(\mathbb{Z}_{\rho q r})) = & (\rho - 1)^2 (r - 1) (\rho q - 2q - \rho + qr + 1)^2 + \\ & (q - 1)^2 (r - 1) (\rho q - q - 2\rho + \rho r + 1)^2 + \\ & (\rho - 1)^2 (q - 1) (\rho r - 2r - \rho + qr + 1)^2 + \\ & (q - 1) (r - 1)^2 (\rho q - r - 2\rho + \rho r + 1)^2 + \\ & (\rho - 1) (q - 1)^2 (\rho r - 2r - q + qr + 1)^2 + \\ & (\rho - 1) (r - 1)^2 (\rho q - r - 2q + qr + 1)^2 + \\ & (\rho - 1)(q - 1)(r - 1) (\rho q - q - 3r - \rho + \rho r + qr + 2)^2 + \\ & (\rho - 1)(q - 1)(r - 1) (\rho q - 3q - r - \rho + \rho r + qr + 2)^2 + \\ & (\rho - 1)(q - 1)(r - 1) (\rho q - q - r - 3\rho + \rho r + qr + 2)^2 + \\ & r^2 (\rho - q)^2 (\rho - 1)(q - 1) + q^2 (\rho - r)^2 (\rho - 1)(r - 1) + \\ & \rho^2 (q - r)^2 (q - 1)(r - 1) + (\rho - r)^2 (\rho - 1) (q - 1)^2 (q - 2)^2 (r - 1) + \\ & (q - r)^2 (\rho - 1)^2 (\rho - 2)^2 (q - 1)(r - 1) + \\ & (\rho - q)^2 (\rho - 1)(q - 1) (r - 1)^2 (r - 2)^2. \end{aligned}$$

Proof. Since ρ, q, r, ρq, ρr and qr are divisors of $n = \rho qr$, vertex set of zero divisor graph could be partitioned as $V(\Gamma(\mathbb{Z}_n)) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$, $i \neq j$ and for $i, j \in 1, 2, ..., 6$. There are six subsets of $V(\Gamma(\mathbb{Z}_{\rho qr}))$ such that

$V_1 = \{\rho\alpha | \alpha = 1, 2, ..., qr - 1, q \nmid \alpha, r \nmid \alpha\}$,
$V_2 = \{q\alpha | \alpha = 1, 2, ..., \rho r - 1, \rho \nmid \alpha, r \nmid \alpha\}$,
$V_3 = \{r\alpha | \alpha = 1, 2, ..., \rho q - 1, \rho \nmid \alpha, q \nmid \alpha\}$,
$V_4 = \{\rho q\alpha | \alpha = 1, 2, ..., r - 1\}$,
$V_5 = \{\rho r\alpha | \alpha = 1, 2, ..., q - 1\}$,
$V_6 = \{qr\alpha | \alpha = 1, 2, ..., \rho - 1\}$.

Norm of each subsets are $|V_1| = (q-1)(r-1)$, $|V_2| = (q-1)(r-1)$, $|V_3| = (\rho-1)(rq-1)$, $|V_4| = (r-1)$, $|V_5| = (q-1)$ and $|V_6| = (\rho-1)$, respectively.

Graph was constructed by using V_1, V_2, V_3, V_4, V_5, and V_6 is below:

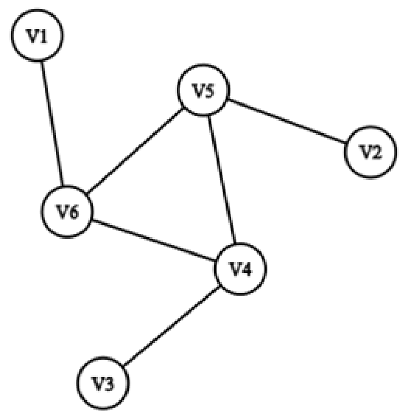

FIGURE 3. Structure of $\Gamma(\mathbb{Z}_{\rho qr})$

In order to calculate Trinajstic index of $\Gamma(\mathbb{Z}_{\rho qr})$ we must investigate these cases below:

Case 1: Considering the pair of sets V_1 and V_2, where $u \in V_1$ and $v \in V_2$, the values of n_u and n_v are described as follows:

$n_u = | V_6 | + | V_1 |$,
$n_v = | V_5 | + | V_2 |$.
Thus, $(n_u - n_v)^2 = (| V_5 | + | V_2 | - (| V_6 | + | V_1 |))^2$.

Case 2: Considering the pair of sets V_1 and V_3, where $u \in V_1$ and $v \in V_3$, the values of n_u and n_v are described as follows:

$n_u = | V_6 | + | V_1 |$,
$n_v = | V_4 | + | V_3 |$.
Hence, $(n_u - n_v)^2 = (| V_6 | + | V_1 | - (| V_4 | + | V_3 |))^2$.

Case 3: Considering the pair of sets V_1 and V_4, where $u \in V_1$ and $v \in V_4$, the values of n_u and n_v are described as follows:

$n_u = 0$,
$n_v = | V_3 | + | V_5 | + | V_2 |$.
Therefore, $(n_u - n_v)^2 = (| V_3 | + | V_5 | + | V_2 |)^2$.

Case 4: Considering the pair of sets V_1 and V_5, where $u \in V_1$ and $v \in V_5$, the values of n_u and n_v are described as follows:

$n_u = 0$,

$n_v = \mid V_2 \mid + \mid V_3 \mid + \mid V_4 \mid$.

Hence, $(n_u - n_v)^2 = (\mid V_2 \mid + \mid V_3 \mid + \mid V_4 \mid)^2$.

Case 5: Considering the pair of sets V_1 and V_6, where $u \in V_1$ and $v \in V_6$, the values of n_u and n_v are described as follows:

$n_u = \mid V_6 \mid$,

$n_v = \mid V_2 \mid + \mid V_3 \mid + \mid V_4 \mid + \mid V_5 \mid + \mid V_1 \mid$.

Thus $(n_u - n_v)^2 = (\mid V_2 \mid + \mid V_3 \mid + \mid V_4 \mid + \mid V_5 \mid + \mid V_1 \mid - \mid V_6 \mid)^2$.

Case 6: Considering the pair of sets V_2 and V_3, where $u \in V_2$ and $v \in V_3$, the values of n_u and n_v are described as follows:

$n_u = \mid V_5 \mid + \mid V_2 \mid$,

$n_v = \mid V_4 \mid + \mid V_3 \mid$.

Therefore, $(n_u - n_v)^2 = (\mid V_5 \mid + \mid V_2 \mid - (\mid V_4 \mid + \mid V_3 \mid))^2$.

Case 7: Considering the pair of sets V_2 and V_4, where $u \in V_2$ and $v \in V_4$, the values of n_u and n_v are described as follows:

$n_u = 0$,

$n_v = \mid V_3 \mid + \mid V_1 \mid + \mid V_6 \mid$.

Hence, $(n_u - n_v)^2 = (\mid V_3 \mid + \mid V_1 \mid + \mid V_6 \mid)^2$.

Case 8: Considering the pair of sets V_2 and V_5, where $u \in V_2$ and $v \in V_5$, the values of n_u and n_v are described as follows:

$n_u = \mid V_5 \mid$,

$n_v = \mid V_1 \mid + \mid V_2 \mid + \mid V_3 \mid + \mid V_4 \mid + \mid V_6 \mid$.

So, $(n_u - n_v)^2 = (\mid V_1 \mid + \mid V_2 \mid + \mid V_3 \mid + \mid V_4 \mid + \mid V_6 \mid - \mid V_5 \mid)^2$.

Case 9: Considering the pair of sets V_2 and V_6, where $u \in V_2$ and $v \in V_6$, the values of n_u and n_v are described as follows:

$n_u = 0$,

$n_v = \mid V_1 \mid + \mid V_3 \mid + \mid V_4 \mid$.

Accordingly, $(n_u - n_v)^2 = (\mid V_1 \mid + \mid V_3 \mid + \mid V_4 \mid)^2$.

Case 10: Considering the pair of sets V_3 and V_4, where $u \in V_3$ and $v \in V_4$, the values of n_u and n_v are described as follows:

$n_u = \mid V_4 \mid$,

$n_v = \mid V_1 \mid + \mid V_2 \mid + \mid V_3 \mid + \mid V_5 \mid + \mid V_6 \mid$.

Thus, $(n_u - n_v)^2 = (\mid V_1 \mid + \mid V_2 \mid + \mid V_3 \mid + \mid V_6 \mid + \mid V_6 \mid - \mid V_4 \mid)^2$.

Case 11: Considering the pair of sets V_3 and V_5, where $u \in V_3$ and $v \in V_5$, the values of n_u and n_v are described as follows:

$n_u = 0$,

$n_v = \mid V_1 \mid + \mid V_2 \mid + \mid V_6 \mid$.

Hence, $(n_u - n_v)^2 = (\mid V_1 \mid + \mid V_2 \mid + \mid V_6 \mid)^2$.

Case 12: Considering the pair of sets V_3 and V_6, where $u \in V_3$ and $v \in V_6$, the values of n_u and n_v are described as follows:

$n_u = 0$,

$n_v = \mid V_1 \mid + \mid V_2 \mid + \mid V_5 \mid$.

So, $(n_u - n_v)^2 = (\mid V_1 \mid + \mid V_2 \mid + \mid V_5 \mid)^2$.

Case 13: Considering the pair of sets V_4 and V_5, where $u \in V_4$ and $v \in V_5$, the values of n_u and n_v are described as follows:

$n_u = \mid V_3 \mid + \mid V_5 \mid$,

$n_v = \mid V_2 \mid + \mid V_4 \mid$.

Thus $(n_u - n_v)^2 = (\mid V_3 \mid + \mid V_5 \mid - (\mid V_2 \mid + \mid V_4 \mid))^2$.

Case 14: Considering the pair of sets V_4 and V_6, where $u \in V_4$ and $v \in V_6$, the values of n_u and n_v are described as follows:

$n_u = \mid V_3 \mid + \mid V_6 \mid$,

$n_v = \mid V_1 \mid + \mid V_4 \mid$.

Therefore, $(n_u - n_v)^2 = (\mid V_3 \mid + \mid V_6 \mid - (\mid V_1 \mid + \mid V_4 \mid))^2$.

Case 15: Considering the pair of sets V_5 and V_6, where $u \in V_5$ and $v \in V_6$, the values of n_u and n_v are described as follows:

$n_u = \mid V_2 \mid + \mid V_6 \mid$,

$n_v = \mid V_1 \mid + \mid V_5 \mid$.

Hence, $(n_u - n_v)^2 = (\mid V_2 \mid + \mid V_6 \mid - (\mid V_1 \mid + \mid V_5 \mid))^2$.

In this way, Trinajstic index of $\Gamma(Z_{\rho qr})$ is

$$\begin{aligned}
NT(\Gamma(\mathbb{Z}_{\rho qr})) = & \sum_{\{u,v\} \in \ V(\Gamma(Z_{\rho qr}))} (n_u - n_v)^2 \\
= & \mid V_1 \mid\mid V_2 \mid (\mid V_5 \mid + \mid V_2 \mid - (\mid V_6 \mid + \mid V_1 \mid))^2 + \\
& \mid V_1 \mid\mid V_3 \mid (\mid V_1 \mid + \mid V_6 \mid - (\mid V_3 \mid + \mid V_4 \mid))^2 + \\
& \mid V_1 \mid\mid V_4 \mid (\mid V_2 \mid + \mid V_3 \mid + \mid V_5 \mid)^2 + \mid V_1 \mid\mid V_5 \mid (\mid V_2 \mid + \mid V_3 \mid + \mid V_4 \mid)^2 + \\
& \mid V_1 \mid\mid V_6 \mid (\mid V_1 \mid + \mid V_2 \mid + \mid V_3 \mid + \mid V_4 \mid + \mid V_5 \mid - \mid V_6 \mid)^2 + \\
& \mid V_2 \mid\mid V_3 \mid (\mid V_2 \mid + \mid V_5 \mid - (\mid V_3 \mid + \mid V_4 \mid))^2 + \\
& \mid V_2 \mid\mid V_4 \mid (\mid V_1 \mid + \mid V_3 \mid + \mid V_6 \mid)^2 + \\
& \mid V_2 \mid\mid V_5 \mid (\mid V_1 \mid + \mid V_2 \mid + \mid V_3 \mid + \mid V_4 \mid + \mid V_6 \mid - \mid V_5 \mid)^2 + \\
& \mid V_2 \mid\mid V_6 \mid (\mid V_1 \mid + \mid V_3 \mid + \mid V_4 \mid)^2 + \\
& \mid V_3 \mid\mid V_4 \mid (\mid V_1 \mid + \mid V_2 \mid + \mid V_3 \mid + \mid V_5 \mid + \mid V_6 \mid - \mid V_4 \mid)^2 + \\
& \mid V_3 \mid\mid V_5 \mid (\mid V_1 \mid + \mid V_2 \mid + \mid V_6 \mid)^2 + \\
& \mid V_3 \mid\mid V_3 \mid (\mid V_1 \mid + \mid V_2 \mid + \mid V_5 \mid)^2 + \\
& \mid V_4 \mid\mid V_5 \mid (\mid V_2 \mid + \mid V_4 \mid - (\mid V_3 \mid + \mid V_5 \mid))^2 + \\
& \mid V_4 \mid\mid V_6 \mid (\mid V_1 \mid + \mid V_4 \mid - (\mid V_3 \mid + \mid V_6 \mid))^2 + \\
& \mid V_5 \mid\mid V_6 \mid (\mid V_1 \mid + \mid V_5 \mid - (\mid V_2 \mid + \mid V_6 \mid))^2
\end{aligned} \tag{3.5}$$

Then Equation 3.5 will be

$$
\begin{aligned}
NT\left(\Gamma\left(\mathbb{Z}_{\rho qr}\right)\right) = & (\rho-1)^2(r-1)(\rho q-2q-\rho+qr+1)^2+ \\
& (q-1)^2(r-1)(\rho q-q-2\rho+\rho r+1)^2+ \\
& (\rho-1)^2(q-1)(\rho r-2r-\rho+qr+1)^2+ \\
& (q-1)(r-1)^2(\rho q-r-2\rho+\rho r+1)^2+ \\
& (\rho-1)(q-1)^2(\rho r-2r-q+qr+1)^2+ \\
& (\rho-1)(r-1)^2(\rho q-r-2q+qr+1)^2+ \\
& (\rho-1)(q-1)(r-1)(\rho q-q-3r-\rho+\rho r+qr+2)^2+ \\
& (\rho-1)(q-1)(r-1)(\rho q-3q-r-\rho+\rho r+qr+2)^2+ \\
& (\rho-1)(q-1)(r-1)(\rho q-q-r-3\rho+\rho r+qr+2)^2+ \\
& r^2(\rho-q)^2(\rho-1)(q-1)+q^2(\rho-r)^2(\rho-1)(r-1)+ \\
& \rho^2(q-r)^2(q-1)(r-1)+(\rho-r)^2(\rho-1)(q-1)^2(q-2)^2(r-1)+ \\
& (q-r)^2(\rho-1)^2(\rho-2)^2(q-1)(r-1)+(\rho-q)^2(\rho-1)(q-1)(r-1)^2(r-2)^2
\end{aligned}
$$

□

Table 5.4 lists the results obtained for $\Gamma(\mathbb{Z}_{\rho qr})$ for primes $\rho < 10$, $q < 10$ and $r < 10$.

4. Conclusion

The Trinajstic index is a novel topological index that is one of the topological indexes to study on chemical graph theory, especially on chemical structure. The Trinajstic index could also be applicable on zero-divisor graphs except complete graph, star graph, path graph and cycle graph to improve theorems related to computer science and also graph theory too. As discussed in this paper, the Trinajstic index can be calculated analytically using prime numbers, without computational tools.

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5. Appendix

Table 5.2: Results of $\Gamma(\mathbb{Z}_{\rho q})$

ρ	q	$n=\rho q$	NT
2	2	4	0
2	3	6	2
2	5	10	36
2	7	14	150
2	11	22	810
2	13	26	1452
2	17	34	3600
2	19	38	5202

Table 5.2 – continued from previous page

ρ	q	$n=\rho q$	NT
2	23	46	9702
2	29	58	20412
2	31	62	25230
2	37	74	44100
2	41	82	60840
2	43	86	70602
2	47	94	93150
3	2	6	2
3	3	9	0
3	5	15	32
3	7	21	192
3	11	33	1280
3	13	39	2400
3	17	51	6272
3	19	57	9216
3	23	69	17600
3	29	87	37856
3	31	93	47040
3	37	111	83232
3	41	123	115520
3	43	129	134400
3	47	141	178112
5	2	10	36
5	3	15	32
5	5	25	0
5	7	35	96
5	11	55	1440
5	13	65	3072
5	17	85	9216
5	19	95	14112
5	23	115	28512
5	29	145	64512
5	31	155	81120
5	37	185	147456
5	41	205	207360
5	43	215	242592
5	47	235	324576
7	2	14	150
7	3	21	192
7	5	35	96

Table 5.2 – continued from previous page

ρ	q	$n = \rho q$	NT
7	7	49	0
7	11	77	960
7	13	91	2592
7	17	119	9600
7	19	133	15552
7	23	161	33792
7	29	203	81312
7	31	217	103680
7	37	259	194400
7	41	287	277440
7	43	301	326592
7	47	329	441600
11	2	22	810
11	3	33	1280
11	5	55	1440
11	7	77	960
11	11	121	0
11	13	143	480
11	17	187	5760
11	19	209	11520
11	23	253	31680
11	29	319	90720
11	31	341	120000
11	37	407	243360
11	41	451	360000
11	43	473	430080
11	47	517	596160
13	2	26	1452
13	3	39	2400
13	5	65	3072
13	7	91	2592
13	11	143	480
13	13	169	0
13	17	221	3072
13	19	247	7776
13	23	299	26400
13	29	377	86016
13	31	403	116640
13	37	481	248832
13	41	533	376320

Table 5.2 – continued from previous page

ρ	q	$n=\rho q$	NT
13	43	559	453600
13	47	611	638112
17	2	34	3600
17	3	51	6272
17	5	85	9216
17	7	119	9600
17	11	187	5760
17	13	221	3072
17	17	289	0
17	19	323	1152
17	23	391	12672
17	29	493	64512
17	31	527	94080
17	37	629	230400
17	41	697	368640
17	43	731	454272
17	47	799	662400
19	2	38	5202
19	3	57	9216
19	5	95	14112
19	7	133	15552
19	11	209	11520
19	13	247	7776
19	17	323	1152
19	19	361	0
19	23	437	6336
19	29	551	50400
19	31	589	77760
19	37	703	209952
19	41	779	348480
19	43	817	435456
19	47	893	649152
23	2	46	9702
23	3	69	17600
23	5	115	28512
23	7	161	33792
23	11	253	31680
23	13	299	26400
23	17	391	12672
23	19	437	6336

Table 5.2 – continued from previous page

ρ	q	$n = \rho q$	NT
23	23	529	0
23	29	667	22176
23	31	713	42240
23	37	851	155232
23	41	943	285120
23	43	989	369600
23	47	1081	582912
29	2	58	20412
29	3	87	37856
29	5	145	64512
29	7	203	81312
29	11	319	90720
29	13	377	86016
29	17	493	64512
29	19	551	50400
29	23	667	22176
29	29	841	0
29	31	899	3360
29	37	1073	64512
29	41	1189	161280
29	43	1247	230496
29	47	1363	417312
31	2	62	25230
31	3	93	47040
31	5	155	81120
31	7	217	103680
31	11	341	120000
31	13	403	116640
31	17	527	94080
31	19	589	77760
31	23	713	42240
31	29	899	3360
31	31	961	0
31	37	1147	38880
31	41	1271	120000
31	43	1333	181440
31	47	1457	353280
37	2	74	44100
37	3	111	83232
37	5	185	147456

Table 5.2 – continued from previous page

ρ	q	$n = \rho q$	NT
37	7	259	194400
37	11	407	243360
37	13	481	248832
37	17	629	230400
37	19	703	209952
37	23	851	155232
37	29	1073	64512
37	31	1147	38880
37	37	1369	0
37	41	1517	23040
37	43	1591	54432
37	47	1739	165600
41	2	82	60840
41	3	123	115520
41	5	205	207360
41	7	287	277440
41	11	451	360000
41	13	533	376320
41	17	697	368640
41	19	779	348480
41	23	943	285120
41	29	1189	161280
41	31	1271	120000
41	37	1517	23040
41	41	1681	0
41	43	1763	6720
41	47	1927	66240
43	2	86	70602
43	3	129	134400
43	5	215	242592
43	7	301	326592
43	11	473	430080
43	13	559	453600
43	17	731	454272
43	19	817	435456
43	23	989	369600
43	29	1247	230496
43	31	1333	181440
43	37	1591	54432
43	41	1763	6720

Table 5.2 – continued from previous page

ρ	q	$n = \rho q$	NT
43	43	1849	0
43	47	2021	30912
47	2	94	93150
47	3	141	178112
47	5	235	324576
47	7	329	441600
47	11	517	596160
47	13	611	638112
47	17	799	662400
47	19	893	649152
47	23	1081	582912
47	29	1363	417312
47	31	1457	353280
47	37	1739	165600
47	41	1927	66240
47	43	2021	30912
47	47	2209	0

Table 5.3: Results of $\Gamma(\mathbb{Z}_{\rho^2 q})$

ρ	q	ρ^2	$n = \rho^2 q$	NT
2	2	4	8	2
2	3	4	12	116
2	5	4	20	600
2	7	4	28	1836
2	11	4	44	8100
2	13	4	52	13896
2	17	4	68	32736
2	19	4	76	46548
3	2	9	18	1062
3	3	9	27	300
3	5	9	45	10404
3	7	9	63	26112
3	11	9	99	95832
3	13	9	117	156756
3	17	9	153	348012
3	19	9	171	485256
5	2	25	50	43860
5	3	25	75	95960
5	5	25	125	28880
5	7	25	175	561960
5	11	25	275	1843320
5	13	25	325	2957760
5	17	25	425	6470160
5	19	25	475	9002520
7	2	49	98	491946
7	3	49	147	966336
7	5	49	245	2107140
7	7	49	343	423612
7	11	49	539	11130000
7	13	49	637	17509716
7	17	49	833	38036796
7	19	49	931	53087328
11	2	121	242	12997710
11	3	121	363	23982200
11	5	121	605	42850500
11	7	121	847	62037360
11	11	121	1331	13069100
11	13	121	1573	174942900
11	17	121	2057	349767660

Table 5.3 – continued from previous page

ρ	q	ρ^2	$n = \rho^2 q$	NT
11	19	121	2299	483076440
13	2	169	338	43707612
13	3	169	507	80297256
13	5	169	845	139324224
13	7	169	1183	190351512
13	11	169	1859	320583432
13	13	169	2197	44974800
13	17	169	2873	782036112
13	19	169	3211	1058991336
17	2	289	578	305388816
17	3	289	867	563382176
17	5	289	1445	966976320
17	7	289	2023	1265474016
17	11	289	3179	1762029600
17	13	289	3757	2067512256
17	17	289	4913	319615232
17	19	289	5491	3857328288
19	2	361	722	682331382
19	3	361	1083	1263762504
19	5	361	1805	2177294436
19	7	361	2527	2838441312
19	11	361	3971	3787845624
19	13	361	4693	4268235924
19	17	361	6137	5720724972
19	19	361	6859	715825836

Table 5.4: Results of $\Gamma(\mathbb{Z}_{\rho qr})$

ρ	q	r	$n = \rho qr$	NT
2	2	2	8	2
2	2	3	12	116
2	2	5	20	600
2	2	7	28	1836
2	3	2	12	116
2	3	3	18	1062
2	3	5	30	14922
2	3	7	42	48980
2	5	2	20	600
2	5	3	30	14922
2	5	5	50	43860
2	5	7	70	273278
2	7	2	28	1836
2	7	3	42	48980
2	7	5	70	273278
2	7	7	98	491946
3	2	2	12	116
3	2	3	18	1062
3	2	5	30	14922
3	2	7	42	48980
3	3	2	18	1062
3	3	3	27	300
3	3	5	45	10404
3	3	7	63	26112
3	5	2	30	14922
3	5	3	45	10404
3	5	5	75	95960
3	5	7	105	734912
3	7	2	42	48980
3	7	3	63	26112
3	7	5	105	734912
3	7	7	147	966336
5	2	2	20	600
5	2	3	30	14922
5	2	5	50	43860
5	2	7	70	273278
5	3	2	30	14922
5	3	3	45	10404
5	3	5	75	95960

Table 5.4 – continued from previous page

ρ	q	r	$n = \rho qr$	NT
5	3	7	105	734912
5	5	2	50	43860
5	5	3	75	95960
5	5	5	125	28880
5	5	7	175	561960
5	7	2	70	273278
5	7	3	105	734912
5	7	5	175	561960
5	7	7	245	2107140
7	2	2	28	1836
7	2	3	42	48980
7	2	5	70	273278
7	2	7	98	491946
7	3	2	42	48980
7	3	3	63	26112
7	3	5	105	734912
7	3	7	147	966336
7	5	2	70	273278
7	5	3	105	734912
7	5	5	175	561960
7	5	7	245	2107140
7	7	2	98	491946
7	7	3	147	966336
7	7	5	245	2107140
7	7	7	343	423612

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A NOTE ON POINTWISE HEMI-SLANT CONFORMAL SUBMERSIONS

MOHAMMAD SHUAIB *

Abstract. The current study investigates the concept of pointwise hemi-slant conformal submersions from almost contact metric manifolds to Riemannian manifold. We investigate the geometrical implications of the horizontal and vertical vector fields ξ while studying the distribution integrability and total geodesicness of distribution leaves. Finally, we explore ϕ-pluriharmonicity from the almost contact metric manifold.

Keywords: Sasakian manifolds, Riemannian submersions, Pointwise hemi-slant conformal submersions, Conformal submersions.

2020 Mathematics Subject Classification: 53D15, 53C25.

1. Introduction

Immersions and submersions are well known to be special tools in both differential and Riemannian geometry. It is important in Riemannian geometry, particularly when the involved manifolds have a differentiable structures. Four decades ago, B. O'Neill [22] and A. Gray [12] separately formulated the cornerstone of the theory of Riemannian submersions, which has experienced major advances over the past two decades. In mathematics and physics, Riemannian submersions have been widely used, particularly in theories like Yang-Mills and Kaluza-Klein (see [8], [35], [20], [17]).

Riemannian submersions from almost Hermitian manifolds to Riemannian manifolds were studied in 1976 by B. Watson [34]. On the foundation of the findings from this study, B. Sahin [26] assessed the geometry and distinctive features of anti-invariant Riemannian submersions onto Riemannian manifolds. Afterwards, authors explored this field further, looking at slant submersions [10], [28], semi-slant submersions [16], [23], anti-invariant submersions [3], [26] and semi-invariant submersions [27] among other topics. Tastan, Sahin, and Yanan [33] defined and studied hemi-slant submersions from almost Hermitian manifolds as a generalized case of semi-invariant and semi-slant submersions.

J. W. Lee and B. Sahin [19] introduced pointwise slant submersions from almost Hermitian manifolds to Riemannian manifolds in this addition, thus expanding the concept of slant submersions even further. They established characterizations for pointwise slant submersions in

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addition to offering examples of this kind of submersion. As a generalization of Riemannian submersions, B. Fuglede [13] and T. Ishihara [18] presented the idea of conformal submersion and talked about some of its geometric characteristics. As a generalization of holomorphic submersions, conformal holomorphic submersions were studied by Gudmundsson and Wood [15]. They determined the necessary and sufficient conditions for harmonic morphisms of conformal holomorphic submersions. Later, conformal semi-invariant submersions [4], conformal slant submersions [2], conformal anti-invariant submersions [29], [24] and conformal semi-slant submersions [1] were studied and defined by Akyol and Sahin. Conformal hemi-slant submersions [31], [32], conformal bi-slant submersions [5] and quasi bi-slant conformal submersions [6] have all been discussed in the context of geometric studies recently, along with an assortment of decomposition theorems. Moreover, from almost Hermitian manifolds to almost contact metric manifolds, the concept of pluriharmonicity was extended.

We study the geometry of pointwise hemi-slant conformal submersions by considering both the horizontal and vertical aspects of the structural vector field ξ. The paper is organized as follows: Our investigation's goals can be achieved by introducing almost contact manifolds, such as the Sasakian manifold, which we discuss in Section 2. In the third part of this investigation, we define pointwise hemi-slant conformal submersions and describe some interesting results by considering the horizontal structure of the Reeb vector field ξ. A thorough examination of the total geodesic and the integrability of distributions is also given in Section 3. While in section 4, we consider vertical aspect of Reeb vector field ξ and study of pointwise hemi-slant conformal submersions. Additionally, as the nature of the Reeb vector field differs in Sections 3 and 4, we compared the findings of these two sections.

Note: We use the following abbreviations in this article.
Pointwise hemi-slant conformal submersion- $\mathcal{PWHSCS}$
Riemannian submersion- $\mathcal{RS}$
Riemannian Manifold-$\mathcal{RM}$
Horizontally Conformal Submersion-$\mathcal{HCS}$
Sasakian manifold-$\mathcal{SM}$
Almost contact metric manifold-$\mathcal{ACMM}$

2. Preliminaries

Authors find the study of $\mathcal{RS}$s to be an extremely intriguing topic. We now begin with a discussion of a few significant points and some useful results that are highly beneficial to our research.

Definition 2.1. [34] *Let (Ξ_1, g_1) and (Ξ_2, g_2) be two $\mathcal{RM}$s and $\bar{\alpha}$ be a smooth map between (Ξ_1, g_1) and (Ξ_2, g_2) where, m_1 and m_2 are the dimensions of Ξ_1 and Ξ_2 respectively. Then $\bar{\alpha}$ is called horizontally weakly conformal or semi conformal at $x \in \Xi_1$ if either*

(i) $\bar{\alpha}_{*x} = 0$, *or*

(ii) $\bar{\alpha}_{*x}$ *maps horizontal space* $\aleph_x = (\ker(\bar{\alpha}_{*x}))^{\perp}$ *conformally onto* $T_{\bar{\alpha}*}(\Xi_2)$ *i.e.,* $\bar{\alpha}_{*x}$ *is surjective and there exits a number* $\Lambda(x) \neq 0$ *such that*

$$g_2(\bar{\alpha}_{*x}\beta_1, \bar{\alpha}_{*x}\beta_2) = \Lambda(x) g(\beta_1, \beta_2),$$

for any $\beta_1, \beta_2 \in \aleph_x$.

Above equation can be reduce as:

$$(\bar{\alpha}_* g_2)_x \mid_{\aleph_x \times \aleph_x} = \Lambda(x) g(x) \mid_{\aleph_x \times \aleph_x}.$$

A point x is a critical point of $\bar{\alpha}$ if it fulfills (i) in the definition above. A point that satisfies (ii) is known as a regular point. At a critical point, $\bar{\alpha}_{*x}$ has rank 0, whereas at a regular point, it has rank n and defines a submersion. The number $\lambda(x)$ is called the square dilation and its square root $\lambda(x) = \sqrt{\Lambda(x)}$ is called the dilation of $\bar{\alpha}$ at x which is necessarily non-negative. A map $\bar{\alpha}$ is considered horizontally weakly conformal or semi-conformal on Ξ_1 if it is weakly conformal at all points of Ξ_1. If $\bar{\alpha}$ has no critical points, it is considered a (horizontally) conformal submersion.

Definition 2.2. [7] *Let $\bar{\alpha}$ be a $\mathcal{RS}$ between two $\mathcal{RM}s$. Then $\bar{\alpha}$ is called a horizontally conformal submersion, if there is a positive function λ such that*

$$g_1(\omega_1, \hat{\omega_2}) = \frac{1}{\lambda^2} g_2(\bar{\alpha}_* \omega_1, \bar{\alpha}_* \omega_2), \tag{2.1}$$

for any $\omega_1, \omega_2 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$. It is obvious that every $\mathcal{RS}s$ is particularly a horizontally conformal submersion with $\lambda = 1$.

Let be a $\mathcal{RS}$ $\bar{\alpha} : (\Xi_1, g_1) \to (\Xi_2, g_2)$. If $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$, then a vector field β_1 on Ξ_1 is referred to as a basic vector field and $\bar{\alpha}$-related to Ξ_2 using a vector field β_1 i.e $\bar{\alpha}_*(\beta_1(q)) = \beta_1 \bar{\alpha}(q)$ for $q \in \Xi_1$.

According to O'Neill, the two equations of the $(1, 2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ are:

$$\mathcal{A}_{E_1} F_1 = \aleph \nabla_{\aleph E_1} v F_1 + v \nabla_{\aleph E_1} \aleph F_1, \tag{2.2}$$

$$\mathcal{T}_{E_1} F_1 = \aleph \nabla_{v E_1} v F_1 + v \nabla_{v E_1} \aleph F_1, \tag{2.3}$$

for any $E_1, F_1 \in \Gamma(T\Xi_1)$ and ∇ is a Levi-Civita connection of g_1. From above two equations of O'Neill, we can deduce that

$$\nabla_{\omega_1} \omega_2 = \mathcal{T}_{\omega_1} \omega_2 + v \nabla_{\omega_1} \omega_2 \tag{2.4}$$

$$\nabla_{\omega_1} \beta_1 = \mathcal{T}_{\omega_1} \beta_1 + \aleph \nabla_{\omega_1} \beta_1 \tag{2.5}$$

$$\nabla_{\beta_1} \omega_1 = \mathcal{A}_{\beta_1} \omega_1 + v_1 \nabla_{\beta_1} \omega_1 \tag{2.6}$$

$$\nabla_{\beta_1} \beta_2 = \aleph \nabla_{\beta_1} \beta_2 + \mathcal{A}_{\beta_1} \beta_2 \tag{2.7}$$

for any vector fields $\omega_1, \omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$ [11].

We note that

$$g(\mathcal{A}_{\beta_1} E_1, F_1) = -g(E_1, \mathcal{A}_{\beta_1} F_1), \;\; g(\mathcal{T}_{\omega_2} E_1, F_1) = -g(E_1, \mathcal{T}_{\omega_2} F_1),$$

for any vector fields $E_1, F_1 \in \Gamma(T\Xi_1)$. In the unique scenario where $\bar{\alpha}$ represents a $\mathcal{HCS}$, we possess:

Proposition 2.1. [14] *Let $\bar{\alpha} : (\Xi_1, g_1) \to (\Xi_2, g_2)$ be a horizontally conformal submersion with dilation λ and β_1, β_2 be the horizontal vectors, then*

$$A_{\beta_1} \beta_2 = \frac{1}{2} \{ v[\beta_1, \beta_2] - \lambda^2 g(\beta_1, \beta_2) grad_v (\frac{1}{\lambda^2}) \} \tag{2.8}$$

measures the obstruction integrability of the horizontal distribution.

The second fundamental form of smooth map $\bar{\alpha}$ is provided by the formula

$$(\nabla\bar{\alpha}_*)(\omega_1,\omega_2)=\nabla^{\bar{\alpha}}_{\omega_1}\bar{\alpha}_*\omega_2-\bar{\alpha}_*\nabla_{\omega_1}\omega_2, \tag{2.9}$$

and the map be totally geodesic if $(\nabla\bar{\alpha}_*)(\omega_1,\omega_2)=0$ for all $\omega_1,\omega_2\in\Gamma(T\Xi_1)$ where ∇ and $\nabla^{\bar{\alpha}}$ are Levi-Civita and pullback connections.

Lemma 2.1. *Let $\bar{\alpha}:\Xi_1\to\Xi_2$ be a $\mathcal{HCS}$. Then, we have*

(i) $(\nabla\bar{\alpha}_*)(\beta_1,\beta_2)=\beta_1(\ln\lambda)\bar{\alpha}_*(\beta_2)+\beta_2(ln\lambda)\bar{\alpha}_*(\beta_1)-g_1(\beta_1,\beta_2)\bar{\alpha}_*(\text{grad } ln\lambda)$,

(ii) $(\nabla\bar{\alpha}_*)(\omega_1,\omega_2)=-\bar{\alpha}_*(\mathcal{T}_{\omega_1}\omega_2)$,

(iii) $(\nabla\bar{\alpha}_*)(\beta_1,\omega_1)=-\bar{\alpha}_*(\nabla_{\beta_1}\omega_1)=-\bar{\alpha}_*(\mathcal{A}_{\beta_1}\omega_1)$

for any $\beta_1,\beta_2\in\Gamma(ker\bar{\alpha}_*)^{\perp}$ and $\omega_1,\omega_2\in\Gamma(ker\bar{\alpha}_*)$ [34].

Let M be a $(2n+1)$-dimensional almost contact manifold with almost contact structures (ϕ,ξ,η), where a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2=-I+\eta\otimes\xi,\ \phi\xi=0,\ \eta\circ\phi=0,\ \eta(\xi)=1, \tag{2.10}$$

where I is the identity tensor and there exists a Riemannian metric g in such a way that

$$g(\phi\omega_1,\phi\omega_2)=g(\omega_1,\omega_2)-\eta(\omega_1)\eta(\omega_2), \tag{2.11}$$

which can be noticed as follows:

$$\eta(\omega_1)=g(\omega_1,\xi), \tag{2.12}$$

for any $\omega_1,\ \omega_2\in\Gamma(TM)$. Then (ϕ,ξ,η,g)-structure is called an almost contact metric structure. A normal contact metric structure is called a Sasakian structure, which satisfies

$$(\nabla_{\omega_1}\phi)\omega_2=g(\omega_1,\omega_2)\xi-\eta(\omega_2)\omega_1 \tag{2.13}$$

where ∇ is the Levi-Civita connection of g. For a $\mathcal{SM}$, we can deduce that

$$\nabla_{\omega_1}\xi=-\phi\omega_1. \tag{2.14}$$

The covariant derivative of ϕ is defined by

$$(\nabla_{\beta_1}\phi)\beta_2=\nabla_{\beta_1}\phi\beta_2-\phi\nabla_{\beta_1}\beta_2, \tag{2.15}$$

for all vector fields β_1,β_2 in M. S. A. Sepet and M. Ergut [30] defined pointwise slant submersion as:

Definition 2.3. *Let $(\bar{\Xi}_1,\phi,\xi,\eta,g_1)$ be an $\mathcal{ACMM}$ and $(\bar{\Xi}_1,\phi,\xi,\eta,g_1)$ be a $\mathcal{RM}$.Let $\bar{\alpha}$ be a $\mathcal{RS}$ from Ξ_1 to Ξ_2. If the wirtinger angle $\theta(\beta_1)$ between $\phi\beta_1$ and the space* ker$\bar{\alpha}_*$ *is independent of the choice of the non-zero vector field $\beta_1\in\Gamma(\ker\bar{\alpha}_*)-<\xi>$ at each given point $q\in\Xi_1$, then $\bar{\alpha}$ is a pointwise slant submersion. The angle θ represents a function on $\bar{\Xi}_1$, known as the slant function of the pointwise slant submersion.*

Now, we extended the concept of ϕ-pluriharmonicity from almost contact metric manifolds to $(\Xi_1,\phi,\xi,\eta,g_1)$ which was once studied and defined by Y. Ohnita [21]. Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from almost contact metric manifolds to $(\Xi_1,\phi,\xi,\eta,g_1)$ onto a Riemannian manifold (Ξ_2,g_2). Then $\mathcal{PWHSCS}$ is $\mathfrak{D}^{\perp}$-ϕ-pluriharmonic, $\mathfrak{D}^{\theta}$-ϕ-pluriharmonic, $(\mathfrak{D}^{\perp}-\mathfrak{D}^{\theta})$-$\phi$ pluriharmonic, ker$\bar{\alpha}_*$-ϕ-pluriharmonic, $(\ker\bar{\alpha}_*)^{\perp}$-$\phi$-pluriharmonic and $((\ker\bar{\alpha}_*)^{\perp}-\ker\bar{\alpha}_*)$-$\phi$-pluriharmonic if

$$(\nabla\bar{\alpha}_*)(\beta_1,\beta_2)+(\nabla\bar{\alpha}_*)(\phi\beta_1,\phi\beta_2)=0, \tag{2.16}$$

for any $\beta_1, \beta_2 \in \Gamma(\mathfrak{D}^{\perp})$, for any $\beta_1, \beta_2 \in \Gamma(\mathfrak{D}^{\theta})$, for any $\beta_1 \in \Gamma(\mathfrak{D}^{\perp}), \beta_2 \in \Gamma(\mathfrak{D}^{\theta})$, for any $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)$, for any $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$ and for any $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)$.

3. Pointwise hemi-slant $\xi^{\perp}$-conformal submersions

In this section, we will revisit the idea of $\mathcal{PWHSCS}$ with $\xi \in \Gamma(ker\bar{\alpha})^{\perp}$.

Definition 3.1. *Let $\bar{\alpha} : (\Xi_1, \phi, \xi, \eta, g_1) \to (\Xi_2, g_2)$ be a $\mathcal{HCS}$ where $(\Xi_1, \phi, \xi, \eta, g_1)$ is an $\mathcal{ACMM}$ and (Ξ_2, g_2) is a $\mathcal{RM}$. A $\mathcal{HCS}$ $\bar{\alpha}$ is called a pointwise hemi-slant conformal submersion with $\xi \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$ if there exists two distributions $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$ such that $\ker\bar{\alpha}_* = \mathfrak{D}^{\theta} \oplus \mathfrak{D}^{\perp}$, $\phi(\mathfrak{D}^{\perp}) \subseteq \Gamma(\ker\bar{\alpha}_*)^{\perp}$ and for any given point $q \in \Xi_1$ and $\beta_1 \in (\mathfrak{D}^{\theta})_q$, the angle $\theta = \theta(\beta_1)$ between $\phi\beta_1$ and space $(\mathfrak{D}^{\theta})_q$ is independent of choice of non-zero vector $\beta_1 \in (\mathfrak{D}^{\theta})_q$, where $\mathfrak{D}^{\theta}$ is the orthogonal complement of $\mathfrak{D}^{\perp}$ in $\ker\bar{\alpha}_*$. In this case, the angle θ can be regarded as a slant function and called pointwise hemi-slant function of submersion.*

Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2). Then, for any $\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$, we have

$$\omega_2 = \mathfrak{P}\omega_2 + \mathfrak{Q}\omega_2 \tag{3.17}$$

where $\mathfrak{P}$ and $\mathfrak{Q}$ are the projections morphism onto $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$. Now, for any $\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$, we have

$$\phi\omega_2 = \bar{\delta}\omega_2 + \bar{\mathcal{F}}\omega_2 \tag{3.18}$$

where $\bar{\delta}\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\bar{\mathcal{F}}\omega_2 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$. From (3.17) and (3.18), we have

$$\begin{aligned}\phi\zeta_1 =& \phi(\mathfrak{P}\zeta_1) + \phi(\mathfrak{Q}\zeta_1)\\ =& \bar{\delta}(\mathfrak{P}\zeta_1) + \bar{\mathcal{F}}(\mathfrak{P}\zeta_1) + \bar{\delta}(\mathfrak{Q}\zeta_1) + \bar{\mathcal{F}}(\mathfrak{Q}\zeta_1),\end{aligned}$$

for any $\zeta_1 \in \Gamma(\ker\bar{\alpha}_*)$. Since $\phi\mathfrak{D}^{\perp} \subseteq \Gamma(\ker\bar{\alpha}_*)^{\perp}$, we have $\bar{\delta}(\mathfrak{P}\zeta_1) = 0$, we have

$$\phi\zeta_1 = \bar{\mathcal{F}}(\mathfrak{P}\zeta_1) + \bar{\delta}(\mathfrak{Q}\zeta_1) + \bar{\mathcal{F}}(\mathfrak{Q}\zeta_1).$$

Now, we have the following decomposition

$$(\ker\bar{\alpha}_*)^{\perp} = \bar{\mathcal{F}}\mathfrak{D}^{\theta} \oplus \bar{\mathcal{F}}\mathfrak{D}^{\perp} \oplus \nu, \tag{3.19}$$

where ν is the orthogonal complement to $\bar{\mathcal{F}}\mathfrak{D}^{\theta} \oplus \bar{\mathcal{F}}\mathfrak{D}^{\perp}$ in $(\ker\bar{\alpha}_*)^{\perp}$ such that ν is invariant with respect to ϕ. Now, for any $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$, we have

$$\phi\beta_1 = \mathcal{J}\beta_1 + \mathcal{N}\beta_1 \tag{3.20}$$

where $\mathcal{J}\beta_1 \in \Gamma(\ker\bar{\alpha}_*)$ and $\mathcal{N}\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$.

Lemma 3.1. *Let $(\Xi_1, \phi, \xi, \eta, g_1)$ be an $\mathcal{ACMM}$ and (Ξ_2, g_2) be a $\mathcal{RM}$. If $\bar{\alpha} : \Xi \to \Xi_2$ is a $\mathcal{PWHSCS}$, then we have*

$$-\omega_1 = -\bar{\delta}^2\omega_1 + \mathcal{J}\bar{\mathcal{F}}\omega_1,\ \bar{\mathcal{F}}\bar{\delta}\omega_1 + \mathcal{N}\bar{\mathcal{F}}\omega_1 = 0,\ -\beta_1 = \bar{\mathcal{F}}\mathcal{J}\beta_1 + \mathcal{N}^2\beta_1,\ \eta(\beta_1)\xi = \bar{\delta}\mathcal{J}\beta_1 + \mathcal{J}\mathcal{N}\beta_1,$$

for any vector field $\omega_1 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$.

Proof. By considering the (3.18) and (3.20), the proof of Lemma exists. □

Let us now provide some helpful outcomes, which will be applied throughout the research paper.

Lemma 3.2. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2), then we have*

$$\bar{\delta}^2\zeta_2 = (-\cos^2\theta)\zeta_2, \tag{3.21}$$

for any vector fields $\zeta_2 \in \Gamma(\mathfrak{D}^\theta)$.

Lemma 3.3. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2), then we have*

(i) $g_1(\bar{\delta}\zeta_1, \bar{\delta}\zeta_2) = \cos^2\theta\, g_1(\zeta_1, \zeta_2)$,
(ii) $g_1(\bar{F}\zeta_1, \bar{F}\zeta_2) = \sin^2\theta g_1(\zeta_1, \zeta_2)$,

for any vector fields $\zeta_1, \zeta_2 \in \Gamma(\mathfrak{D}^\theta)$.

Proof. The proof of the preceding Lemmas is identical to the proof of Theorem (2.2) of [9]. As a result, we omit the proofs. □

Lemma 3.4. *Let $\bar{\alpha} : \Xi_1 \to \Xi_2$ be a $\mathcal{PWHSCS}$ with hemi-slant function θ where, $(\Xi_1, \phi, \xi, \eta, g_1)$ a $\mathcal{SM}$ and (Ξ_2, g_2) a $\mathcal{RM}$, then we have*

(i) $\mathcal{A}_{\beta_1}\mathcal{N}\beta_2 + v\nabla_{\beta_1}\mathcal{J}\beta_2 = \mathcal{J}\aleph\nabla_{\beta_1}\beta_2 + \bar{\delta}\mathcal{A}_{\beta_1}\beta_2$
(ii) $\aleph\nabla_{\beta_1}\mathcal{N}\beta_2 + \mathcal{A}_{\beta_1}\mathcal{J}\beta_2 = \mathcal{N}\aleph\nabla_{\beta_1}\beta_2 + \bar{F}\mathcal{A}_{\beta_1}\beta_2 + g_1(\beta_1, \beta_2)\xi - \eta(\beta_2)\beta_1$
(iii) $v\nabla_{\beta_1}\bar{\delta}\omega_2 + \mathcal{A}_{\beta_1}\bar{F}\omega_2 = \mathcal{J}\mathcal{A}_{\beta_1}\omega_2 + \bar{\delta}v\nabla_{\beta_1}\omega_2$
(iv) $\mathcal{A}_{\beta_1}\bar{\delta}\omega_2 + \aleph\nabla_{\beta_1}\bar{F}\omega_2 = \mathcal{N}\mathcal{A}_{\beta_1}\omega_2 + \bar{F}v\nabla_{\beta_1}\omega_2$
(v) $v\nabla_{\omega_2}\mathcal{J}\beta_1 + \mathcal{T}_{\omega_2}\mathcal{N}\beta_1 = \bar{\delta}\mathcal{T}_{\omega_2}\beta_1 + \mathcal{J}\aleph\nabla_{\omega_2}\beta_1 + \eta(\beta_1)\omega_2$
(vi) $\mathcal{T}_{\omega_2}\mathcal{J}\beta_1 + \aleph\nabla_{\omega_2}\mathcal{N}\beta_1 = \bar{F}\mathcal{T}_{\omega_2}\beta_1 + \mathcal{N}\aleph\nabla_{\omega_2}\beta_1$
(vii) $v\nabla_{\omega_1}\bar{\delta}\omega_2 + \mathcal{T}_{\omega_1}\bar{F}\omega_2 = \bar{\delta}v\nabla_{\omega_1}\omega_2 + \mathcal{J}\mathcal{T}_{\omega_1}\omega_2$
(viii) $\mathcal{T}_{\omega_1}\bar{\delta}\omega_2 + \aleph\nabla_{\omega_1}\bar{F}\omega_2 = \mathcal{N}\mathcal{T}_{\omega_1}\omega_2 + \bar{F}v\nabla_{\omega_1}\omega_2 - g_1(\omega_1, \omega_2)\xi$,

for any vector fields $\omega_1, \omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. Using (2.13), (2.15), and (2.7) (3.20), we obtain the first two relations (i) and (ii). Equations (2.13), (2.15) (2.7), (2.4)-(2.7), and (3.18) (3.20 yield the expected results. □

To investigate the geometry of $\mathcal{PWHSCS}$ $\bar{\alpha} : \Xi_1 \to \Xi_2$, we will now review several key findings. Direct calculations might lead to the following conclusions:

(a) $(\nabla_{\omega_1}\bar{\delta})\omega_2 = v\nabla_{\omega_1}\bar{\delta}\omega_2 - \bar{\delta}v\nabla_{\omega_1}\omega_2$
(b) $(\nabla_{\omega_1}\bar{F})\omega_2 = \aleph\nabla_{\omega_1}\bar{F}\omega_2 - \bar{F}v\nabla_{\omega_1}\omega_2$
(c) $(\nabla_{\beta_1}\mathcal{J})\beta_2 = v\nabla_{\beta_1}\mathcal{J}\beta_2 - \mathcal{J}\aleph\nabla_{\beta_1}\beta_2$
(d) $(\nabla_{\beta_1}\mathcal{N})\beta_2 = \aleph\nabla_{\beta_1}\mathcal{N}\beta_2 - \mathcal{N}\aleph\nabla_{\beta_1}\beta_2$,

for any vector fields $\omega_1, \omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Lemma 3.5. *Let $\bar{\alpha} : \Xi_1 \to \Xi_2$ be a $\mathcal{PWHSCS}$ with hemi-slant function θ from a $\mathcal{SM}$ onto a $\mathcal{RM}$, then we have*

(i) $(\nabla_{\omega_1}\bar{\delta})\omega_2 = \mathcal{J}\mathcal{T}_{\omega_1}\omega_2 - \mathcal{T}_{\omega_1}\bar{F}\omega_2$
(ii) $(\nabla_{\omega_1}\bar{F})\omega_2 = \mathcal{N}\mathcal{T}_{\omega_1}\omega_2 - \mathcal{T}_{\omega_1}\bar{\delta}\omega_2 + g_1(\omega_1, \omega_2)\xi$
(iii) $(\nabla_{\beta_1}\mathcal{J})\beta_2 = \bar{\delta}\mathcal{A}_{\beta_1}\beta_2 - \mathcal{A}_{\beta_1}\mathcal{N}\beta_2$
(iv) $(\nabla_{\beta_1}\mathcal{N})\beta_2 = \bar{F}\mathcal{A}_{\beta_1}\beta_2 - \mathcal{A}_{\beta_1}\mathcal{J}\beta_2 - \eta(\beta_2)\beta_1 + g_1(\beta_1, \beta_2)\xi$,

for all vector fields $\omega_1, \omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. The results may be obtained by using the above-mentioned formulae $(a)-(d)$, as well as (2.15), (2.4)-(2.7). □

The tensor fields $\bar{\delta}$ and $\bar{F}$, if they are parallel with regard to the Levi-Civita connection ∇ of Ξ_1, then we obtain

$$\mathcal{J}\mathcal{T}_{\omega_1}\omega_2 = \mathcal{T}_{\omega_1}\bar{F}\omega_2,\ \mathcal{N}\mathcal{T}_{\omega_1}\omega_2 - g_1(\omega_1,\omega_2)\xi = \mathcal{T}_{\omega_1}\bar{\delta}\omega_2$$

for any vector fields $\omega_1, \omega_2 \in \Gamma(T\Xi_1)$.

We are going to discuss about the anti-invariant distribution $\mathfrak{D}^\perp$ and the slant distribution $\mathfrak{D}^\theta$, as well as their integrability and total geodesic.

Theorem 3.1. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from a $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ onto a $\mathcal{RM}$ (Ξ_2,g_2) such that the structure vector field ξ is horizontal. Then the following are equivalent*

(i) *The anti-invariant distribution $\mathfrak{D}^\perp$ is integrable.*

(ii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_2}\bar{\alpha}_*(\bar{F}\zeta_1) - \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2), \bar{\alpha}_*(\bar{F}\omega_1))$
$= g(\nabla_{\zeta_1}\bar{F}\bar{\delta}\zeta_2 + \nabla_{\zeta_2}\bar{F}\bar{\delta}\zeta_1, \omega_1) - g(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2 + \mathcal{T}_{\zeta_2}\bar{F}\zeta_1, \bar{\delta}\omega_1)$

for any $\zeta_1,\zeta_2 \in \Gamma(D^\perp)$ and $\omega_1 \in \mathfrak{D}^\theta$.

Proof. By using (2.11), (2.13),(3.18), (2.4) and (2.5), we can write

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = -g_1(\phi\nabla_{\zeta_1}\bar{\delta}\zeta_2,\omega_1) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) + g_1(\aleph\nabla_{\zeta_1}\bar{F}\zeta_2,\bar{F}\omega_1),$$

for any $\zeta_1,\zeta_2 \in \Gamma(D)^\perp$ and $\omega_1 \in \Gamma(D)^\theta$. Taking account the fact from (3.17), (2.13), (2.5), and (3.18), we have

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = -g_1(\nabla_{\zeta_1}\bar{\delta}^2\zeta_2,\omega_1) - g_1(\nabla_{\zeta_1}\bar{F}\bar{\delta}\zeta_2,\omega_1) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) + g_1(\aleph\nabla_{\zeta_1}\bar{F}\zeta_2,\bar{F}\omega_1).$$

By using the horizontal conformality of $\bar{\alpha}$ and changing the role of ζ_1,ζ_2 and from (2.9) with Lemma 2.1, Lemma 3.2, we finally get

$$\begin{aligned}\sin^2\theta g_1([\zeta_1,\zeta_2],\omega_1) =& g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) + g_1(\mathcal{T}_{\zeta_2}\bar{F}\zeta_1,\bar{\delta}\omega_1) - g_1(\nabla_{\zeta_1}\bar{F}\bar{\delta}\zeta_2,\omega_1) - g_1(\nabla_{\zeta_2}\bar{F}\bar{\delta}\zeta_1,\omega_1)\\ &+ \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_2}(\bar{F}\zeta_1),\bar{\alpha}_*(\bar{F}\omega_1)) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}(\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)).\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $\bar{\alpha}:\Xi_1\to\Xi_2$ be a $\mathcal{PWHSCS}$ from a $\mathcal{SM}$ Ξ_1 onto a $\mathcal{RM}$ Ξ_2 such that structure vector field ξ is horizontal. Then the following are equivalent.*

(i) *Slant distribution $\mathfrak{D}^\theta$ is integrable.*

(ii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\omega_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1)) + g_1([\omega_1,\zeta_1],\omega_2)$
$= \sin2\theta g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2) - g_1(\mathcal{T}_{\omega_2}\bar{F}\omega_1,\phi\zeta_1),$

for any $\omega_1,\omega_2 \in \Gamma(\mathfrak{D}^\theta)$ and $\zeta_1 \in \Gamma(\mathfrak{D}^\perp)$.

Proof. For any $\omega_1,\omega_2 \in \Gamma(D)^\theta$ and $\zeta_1 \in \Gamma(D)^\perp$ with taking account the fact from (2.11), (2.13) and (2.15), we get

$$g_1([\omega_1,\omega_2],\zeta_1) = -g_1([\omega_1,\zeta_1],\omega_2) - g_1(\nabla_{\zeta_1}\phi\omega_1,\phi\omega_2) - g_1(\nabla_{\omega_2}\phi\omega_1,\phi\zeta_1).$$

By using (2.4), (2.5) and (2.13), we can write

$$\begin{aligned}g_1([\omega_1,\omega_2],\zeta_1) =& -g_1([\omega_1,\zeta_1],\omega_2) + g_1(\nabla_{\zeta_1}\bar{\delta}^2\omega_1,\omega_2) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2)\\ &- g_1(\aleph\nabla_{\zeta_1}\bar{F}\omega_1,\bar{F}\omega_2) - g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\phi\zeta_1) - g_1(\aleph\nabla_{\omega_2}\bar{F}\omega_1,\phi\zeta_1).\end{aligned}$$

In the light of (2.1) with Lemma 3.2, we get

$$\begin{aligned} g_1([\omega_1,\omega_2],\zeta_1) =& -g_1([\omega_1,\zeta_1],\omega_2)+\sin 2\theta\zeta_1(\theta)g_1(\omega_1,\omega_2)-\cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2)\\ &-\frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\zeta_1}\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))-\frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\omega_2}\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1))\\ &-g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\phi\zeta_1)-g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2). \end{aligned}$$

By using the horizontal conformality of $\bar{\alpha}$ with Lemma 2.1 and (2.5), we finally have

$$\begin{aligned} g_1([\omega_1,\omega_2],\zeta_1) =& -g_1([\omega_1,\zeta_1],\omega_2)+\sin^2\theta\zeta_1(\theta)g_1(\omega_1,\omega_2)-\cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2)\\ &-g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2)-\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))\\ &-\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\omega_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1))+\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_2,\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1))\\ &-g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\phi\zeta_1)+\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)). \end{aligned}$$

□

This is the required proof of theorem.

Theorem 3.3. *Let $\bar{\alpha}:(\Xi_1,\phi,\xi,\eta,g_1)\to(\Xi_2,g_2)$ be a $\mathcal{PWHSCS}$ from a $\mathcal{SM}$ Ξ_1 onto a $\mathcal{RM}$ Ξ_2 such that structure vector field ξ is horizontal. Then the following are equivalent.*

(i) *Vertical distribution $(\ker\bar{\alpha}_*)$ is integrable.*
(ii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\mathfrak{P}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))+\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\mathfrak{Q}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))$
$= \cos^2\theta g_1(\nabla_{\beta_1}\mathfrak{Q}\omega_1,\omega_2)-\sin 2\theta\beta_1(\theta)g_1(\mathfrak{Q}\omega_1,\omega_2)+g_1([\omega_1,\beta_1],\omega_2)-g_1(\mathcal{A}_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{N}\beta_1)$
$-g_1(v\nabla_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{J}\beta_1)+g_1(\mathcal{A}_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\bar{\delta}\omega_2)-g_1(\nabla_{\beta_1}\bar{F}\bar{\delta}\mathfrak{Q}\omega_1,\omega_2)+g_1(\mathcal{A}_{\beta_1}\bar{F}\mathfrak{Q}\omega_1,\bar{\delta}\omega_2)$
$+g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{N}\beta_1)+g_1(v\nabla_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1)+g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1)+g_1(\aleph\nabla_{\omega_2}\bar{F}\omega_1,\mathcal{N}\beta_1)$
$+g_1(\beta_1,\bar{F}\omega_2)g_1(\bar{F}\mathfrak{P}\omega_1,\operatorname{grad}\ln\lambda)+g_1(\bar{F}\mathfrak{P}\omega_1,\bar{F}\omega_2)g_1(\beta_1,\operatorname{grad}\ln\lambda)$
$-g_1(\beta_1,\bar{F}\mathfrak{P}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda)+g_1(\beta_1,\bar{F}\omega_2)g_1(\bar{F}\mathfrak{Q}\omega_1,\operatorname{grad}\ln\lambda)$
$+g_1(\bar{F}\mathfrak{Q}\omega_1,\bar{F}\omega_2)g_1(\beta_1,\operatorname{grad}\ln\lambda)-g_1(\beta_1,\bar{F}\mathfrak{Q}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda),$

for any $\omega_1,\omega_2\in\Gamma(\ker\bar{\alpha}_*)$ and $\beta_1\in\Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. By using (2.11), (2.13), (2.15) and (3.17), we have

$$g_1([\omega_1,\omega_2],\beta_1)=-g_1([\omega_1,\beta_1],\omega_2)-g_1(\nabla_{\beta_1}\phi\omega_1,\phi\omega_2)+g_1(\nabla_{\omega_2}\phi\omega_1,\phi\beta_1),$$

for any $\omega_1,\omega_2\in\Gamma(\ker\bar{\alpha}_*)$ and $\beta_1\in\Gamma(\ker\bar{\alpha}_*)^\perp$. In the light of (2.4)-(2.7) and (2.15), we can write

$$\begin{aligned} g_1([\omega_1,\omega_2],\beta_1) =& -g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{N}\beta_1)-g_1(v\nabla_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1)-g_1(\mathcal{T}_{\omega_2}\bar{F}\omega_1,\mathcal{J}\beta_1)\\ &-g_1([\omega_1,\beta_1],\omega_2)+g_1(\mathcal{A}_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{N}\beta_1)+g_1(v\nabla_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{J}\beta_1)\\ &-g_1(\mathcal{A}_{\beta_1}\bar{F}\mathfrak{P}\omega_1,\bar{\delta}\omega_2)-g_1(\aleph\nabla_{\beta_1}\bar{F}\mathfrak{P}\omega_1,\bar{F}\omega_2)+g_1(\nabla_{\beta_1}\phi\bar{\delta}\mathfrak{Q}\omega_1,\omega_2)\\ &-g_1((\nabla_{\beta_1}\phi)\bar{\delta}\mathfrak{Q}\omega_1,\omega_2)-g_1(\mathcal{A}_{\beta_1}\bar{F}\mathfrak{Q}\omega_1,\bar{\delta}\omega_2)-g_1(\aleph\nabla_{\beta_1}\bar{F}\mathfrak{Q}\omega_1,\bar{F}\omega_2)\\ &-g_1(\aleph\nabla_{\omega_2}\bar{F}\omega_1,\mathcal{N}\beta_1). \end{aligned}$$

By using Lemma 3.2 with (2.1), we get

$$\begin{aligned} g_1([\omega_1,\omega_2],\beta_1) =& -g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{N}\beta_1) - g_1(v\nabla_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1)\\ &+ \sin 2\theta\beta_1(\theta)g_1(\mathfrak{Q}\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\beta_1}\mathfrak{Q}\omega_1,\omega_2) + g_1(\nabla_{\beta_1}\bar{F}\bar{\delta}\mathfrak{Q}\omega_1,\omega_2)\\ &- g_1([\omega_1,\beta_1],\omega_2) + g_1(\mathcal{A}_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{N}\beta_1) + g_1(v\nabla_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{J}\beta_1)\\ &- \frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\beta_1}\bar{F}\mathfrak{P}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) - \frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\beta_1}\bar{F}\mathfrak{Q}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))\\ &- g_1(\mathcal{A}_{\beta_1}\bar{F}\mathfrak{Q}\omega_1,\bar{\delta}\omega_2) - g_1(\aleph\nabla_{\omega_2}\bar{F}\omega_1,\mathcal{N}\beta_1) - g_1(\mathfrak{A}_{\beta_1}\bar{F}\mathfrak{P}\omega_1,\bar{\delta}\omega_2). \end{aligned}$$

□

By using the horizontal conformality of $\bar{\alpha}$ from Lemma 2.1 and with (2.9), we finally deduce that

$$\begin{aligned} g_1([\omega_1,\omega_2],\beta_1) =& -g_1([\omega_1,\beta_1],\omega_2) + g_1(\mathcal{A}_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{N}\beta_1) + g_1(v\nabla_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{J}\beta_1)\\ &+ \sin 2\theta\beta_1(\theta)g_1(\mathfrak{Q}\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\beta_1}\mathfrak{Q}\omega_1,\omega_2) + g_1(\nabla_{\beta_1}\bar{F}\bar{\delta}\mathfrak{Q}\omega_1,\omega_2)\\ &- g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{N}\beta_1) - g_1(v\nabla_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1)\\ &- g_1(\beta_1,\bar{F}\omega_2)g_1(\bar{F}\mathfrak{P}\omega_1,\operatorname{grad}\ln\lambda) - g_1(\bar{F}\mathfrak{P}\omega_1,\bar{F}\omega_2)g_1(\beta_1,\operatorname{grad}\ln\lambda)\\ &- g_1(\beta_1,\bar{F}\mathfrak{P}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\mathfrak{P}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))\\ &- g_1(\beta_1,\bar{F}\omega_2)g_1(\bar{F}\mathfrak{Q}\omega_1,\operatorname{grad}\ln\lambda) - g_1(\bar{F}\mathfrak{Q}\omega_1,\bar{F}\omega_2)g_1(\beta_1,\operatorname{grad}\ln\lambda)\\ &- g_1(\beta_1,\bar{F}\mathfrak{Q}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\mathfrak{Q}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))\\ &- g_1(\mathfrak{A}_{\beta_1}\bar{F}\mathfrak{P}\omega_1,\bar{\delta}\omega_2) - g_1(\mathcal{A}_{\beta_1}\bar{F}\mathfrak{Q}\omega_1,\bar{\delta}\omega_2) - g_1(\aleph\nabla_{\omega_2}\bar{F}\omega_1,\mathcal{N}\beta_1). \end{aligned}$$

This completes the proof of theorem.

Theorem 3.4. *Let $\bar{\alpha}:(\Xi_1,\phi,\xi,\eta,g_1)\to(\Xi_2,g_2)$ be a $\mathcal{PWHSCS}$ where $(\Xi_1,\phi,\xi,\eta,g_1)$ is a $\mathcal{SM}$ and (Ξ_2,g_2) is a $\mathcal{RM}$ such that structure vector field ξ is horizontal. Then the anti-invariant distribution $\mathfrak{D}^\perp$ defines totally geodesic foliation if and only if*

$$\begin{aligned} \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)) =& \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)) - g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{F})\\ &- g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) =& \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) - g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{N}\beta_1)\\ &- g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\mathcal{J}\beta_1) + g_1(\bar{\delta}\zeta_1,\zeta_2)\eta(\beta_1), \end{aligned} \tag{3.23}$$

for any $\zeta_1,\zeta_2\in\Gamma(\mathfrak{D}^\perp),\omega_1\in\Gamma(\mathfrak{D}^\theta)$ and $\beta_1\in\Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. By using (2.11), (2.13) and (2.15), we get

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = g_1(\nabla_{\zeta_1}\bar{\delta}\zeta_2,\phi\omega_1) + g_1(\nabla_{\zeta_1}\bar{F}\zeta_2,\phi\omega_1).$$

In the light of (2.1), (2.4), and (2.5), we can write

$$\begin{aligned} g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) =& g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) + \frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\zeta_1}\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1))\\ &+ g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{F}\omega_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1). \end{aligned}$$

By using the horizontal conformality of $\bar{\alpha}$ with Lemma 3.2 and (2.1), (2.5), we finally get

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{F}\omega_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) - \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)).$$

On the other hand, for any $\zeta_1,\zeta_2 \in \Gamma(\mathfrak{D}^{\perp})$, $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$ and by using (2.11), (2.13), (2.15), we have

$$g_1(\nabla_{\zeta_1}\zeta_2,\beta_1) = g_1(\nabla_{\zeta_1}\phi\zeta_2,\phi\beta_1) + g_1(\bar{\delta}\zeta_1,\zeta_2)\eta(\beta_1).$$

Finally, in the light of (2.4), (2.5), (2.1) and (2.9), we get

$$g_1(\nabla_{\zeta_1}\zeta_2,\beta_1) = \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) - \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) + g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{N}\beta_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{J}\beta_1) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\mathcal{J}\beta_1) + g_1(\bar{\delta}\zeta_1,\zeta_2)\eta(\beta_1).$$

This is the required proof of theorem. □

Theorem 3.5. *Let $\bar{\alpha} : (\Xi_1,\phi,\xi,\eta,g_1) \to (\Xi_2,g_2)$ be a $\mathcal{PWHSCS}$ from a $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ onto a $\mathcal{RM}$ (Ξ_2,g_2) such that the structure vector field ξ is horizontal. Then the slant distribution defines totally geodesic foliation if and only if*

$$\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) - \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2) = g_1([\omega_1,\zeta_1],\omega_2) - \sin 2\theta\zeta_1(\theta)g_1(\omega_1,\omega_2) + \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\bar{\delta}W,\omega_2)$$

and

$$\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_2,\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) = g_1([\omega_1,\beta_2],\omega_2) - \sin 2\theta\beta_2(\theta)g_1(\omega_1,\omega_2) + \cos^2\theta g_1(\nabla_{\beta_2}\omega_1,\omega_2) - g_1(\mathfrak{A}_{\beta_2}\bar{F}\bar{\delta}\omega_1,\omega_2) + g_1(\mathfrak{A}_{\beta_2}\bar{F}\omega_1,\bar{\delta}\omega_2) + g_1(\beta_2,\bar{F}\omega_2)g_1(\bar{F}\omega_1,\operatorname{grad}\ln\lambda) + g_1(\bar{F}\omega_1,\bar{F}\omega_2)g_1(\beta_2,\operatorname{grad}\ln\lambda) - g_1(\beta_2,\bar{F}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda),$$

for any $\omega_1,\omega_2 \in \Gamma(\mathfrak{D}^{\theta}), \zeta_1 \in \Gamma(\mathfrak{D}^{\perp})$ and $\beta_2 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$.

Proof. For any $\omega_1,\omega_2 \in \Gamma(\mathfrak{D}^{\theta})$ and $\zeta_1 \in \Gamma(\mathfrak{D}^{\perp})$ with using (2.10) , (2.11), (2.12) and (2.15), we have

$$g_1(\nabla_{\omega_1}\omega_2,\zeta_1) = g_1([\omega_1,\zeta_1],\omega_2) + g_1(\nabla_{\zeta_1}\bar{\delta}^2\omega_1,\omega_2) + g_1(\nabla_{\zeta_1}\bar{F}\bar{\delta}\omega_1,\omega_2) - g_1(\nabla_{\zeta_1}\bar{F}\omega_1,\phi\omega_2).$$

In the light of (2.5), (2.10) and (2.9) with Lemma 3.2, we can write

$$g(\nabla_{\omega_1}\omega_2,\zeta_1) = -g_1([\omega_1,\zeta_1],\omega_2) + \sin 2\theta\zeta_1(\theta)g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2) - \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)).$$

On the other hand, for any $\omega_1,\omega_2 \in \Gamma(\mathfrak{D}^{\theta})$ and $\beta_2 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$ with using (2.10), (2.11), (2.12) and (2.15), we get

$$g_1(\nabla_{\omega_1}\omega_2,\beta_2) = -g_1([\omega_1,\beta_2],\omega_2) - g_1(\nabla_{\beta_2}\bar{\delta}^2\omega_1,\omega_2) + g_1(\nabla_{\beta_2}\bar{F}\bar{\delta}\omega_1,\omega_2) - g_1(\nabla_{\beta_2}\bar{F}\omega_1,\phi\omega_2).$$

From (2.7) and with lemma 3.2, we have

$$g_1(\nabla_{\omega_1}\omega_2,\beta_2) = -\, g_1([\omega_1,\beta_2],\omega_2) + \sin 2\theta\beta_2(\theta)g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\beta_2}\omega_1,\omega_2) + g_1(\mathcal{A}_{\beta_2}\bar{F}\bar{\delta}\omega_1,\omega_2) - g_1(\mathcal{A}_{\beta_2}\bar{F}\omega_1,\bar{\delta}\omega_2) - g_1(\aleph\nabla_{\beta_2}\bar{F}\omega_1,\bar{F}\omega_2).$$

Finally by using the horizontal conformality of $\bar{\alpha}$ with (2.1), (2.9) and Lemma 2.1, we can deduce that

$$\begin{aligned} g_1(\nabla_{\omega_1}\omega_2,\beta_2) = &-\, g_1([\omega_1,\beta_2],\omega_2) + \sin 2\theta\beta_2(\theta)g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\beta_2}\omega_1,\omega_2) \\ &+ g_1(\mathcal{A}_{\beta_2}\bar{F}\bar{\delta}\omega_1,\omega_2) - g_1(\mathcal{A}_{\beta_2}\bar{F}\omega_1,\bar{\delta}\omega_2) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) \\ &- g_1(\beta_2,\bar{F}\omega_2)g_1(\bar{F}\omega_1,\operatorname{grad}\ln\lambda) - g_1(\bar{F}\omega_1,\bar{F}\omega_2)g_1(\beta_2,\operatorname{grad}\ln\lambda) \\ &+ g_1(\beta_2,\bar{F}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda). \end{aligned}$$

□

This completes the proof of theorem.

Theorem 3.6. *Let $\bar{\alpha}:\Xi_1\to\Xi_2$ be a $\mathcal{PWHSCS}$ from $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ onto $\mathcal{RM}$ (Ξ_2,g_2) with structure vector field ξ is horizontal. Then $(\ker\bar{\alpha}_*)^\perp$ defines totally geodesic foliation if and only if*

$$\begin{aligned} &\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\mathcal{N}\beta_2),\bar{\alpha}_*(\bar{F}\zeta_1)) \\ &= g_1(\beta_1,\bar{F}\zeta_1)g_1(\mathcal{N}\beta_2,\operatorname{grad}\ln\lambda) + g_1(\mathcal{N}\beta_2,\bar{F}\zeta_1)g_1(\beta_1,\operatorname{grad}\ln\lambda) \\ &\;- g_1(\beta_1,\mathcal{N}\beta_2)g_1(\bar{F}\zeta_1,\operatorname{grad}\ln\lambda) - g_1(\mathfrak{A}_{\beta_1}\mathcal{J}\beta_2,\bar{F}\zeta_1) \\ &\;- g_1(v\nabla_{\beta_1}\mathcal{J}\beta_2,\bar{\delta}\zeta_1) - g_1(\mathfrak{A}_{\beta_1}\mathcal{N}\beta_2,\bar{\delta}\zeta_1) - \eta(\beta_2)g_1(\beta_1,\bar{F}\zeta_1), \end{aligned}$$

for any $\beta_1,\beta_2\in\Gamma(\ker\bar{\alpha}_*)^\perp$ and $\zeta_1\in\Gamma(\ker\bar{\alpha}_*)$.

Proof. For any $\beta_1,\beta_2\in\Gamma(\ker\bar{\alpha}_*)^\perp$ and $\zeta_1\in\Gamma(\ker\bar{\alpha}_*)$ with using (2.10), (2.11), (2.12), (2.13), (2.10) and (2.7), we can write

$$\begin{aligned} g_1(\nabla_{\beta_1}\beta_2,\zeta_1) =& g_1(\mathcal{A}_{\beta_1}\mathcal{J}\beta_2,\bar{F}\zeta_1) + g_1(v\nabla_{\beta_1}\mathcal{J}\beta_2,\bar{\delta}\zeta_1) + g_1(\mathcal{A}_{\beta_1}\mathcal{N}\beta_2,\bar{\delta}\zeta_1) \\ &+ g_1(\mathcal{H}\nabla_{\beta_1}\mathcal{N}\beta_2,\bar{F}\zeta_1) + \eta(\beta_2)g_1(\beta_1,\bar{F}\zeta_1). \end{aligned}$$

By using the horizontal conformality of $\bar{\alpha}$ with (2.1), (2.9) and Lemma 2.1, we finally get

$$\begin{aligned} g_1(\nabla_{\beta_1}\beta_2,\zeta_1) =& g_1(\mathcal{A}_{\beta_1}\mathcal{J}\beta_2,\bar{F}\zeta_1) + g_1(v\nabla_{\beta_1}\mathcal{J}\beta_2,\bar{\delta}\zeta_1) + g_1(\mathcal{A}_{\beta_1}\mathcal{N}\beta_2,\bar{\delta}\zeta_1) \\ &- g_1(\beta_1,\bar{F}\zeta_1)g_1(\mathcal{N}\beta_2,\operatorname{grad}\ln\lambda) - g_1(\mathcal{N}\beta_2,\bar{F}\zeta_1)g_1(\beta_1,\operatorname{grad}\ln\lambda) \\ &+ g_1(\beta_1,\mathcal{N}\beta_2)g_1(\bar{F}\zeta_1,\operatorname{grad}\ln\lambda) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\mathcal{N}\beta_2),\bar{\alpha}_*(\bar{F}\zeta_1)) \\ &+ \eta(\beta_2)g_1(\beta_1,\bar{F}\zeta_1), \end{aligned}$$

which is the required proof of the theorem. □

Theorem 3.7. *Let $\bar{\alpha}:(\Xi_1,\phi,\xi,\eta,g_1)\to(\Xi_2,g_2)$ be a $\mathcal{PWHSCS}$ where, $(\Xi_1,\phi,\xi,\eta,g_1)$ a $\mathcal{SM}$ and (Ξ_2,g_2) a $\mathcal{RM}$ with structure vector field ξ is horizontal. Then the following are equivalent.*

(i) $(\ker\bar{\alpha}_*)$ *defines a totally geodesic foliation.*
(ii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_2}\bar{\alpha}_*(\bar{F}Q\beta_1),\bar{\alpha}_*(\bar{F}\beta_2))+g_1([\beta_1,\bar{F}],\beta_2)+g_1(\mathcal{A}_{\zeta_2}\bar{\delta}P\beta_1,\bar{F}\beta_2)+g_1(\mathcal{A}_{\zeta_2}\bar{F}P\beta_1,\bar{\delta}\beta_2)$ $= \sin 2\theta\zeta_2(\theta)g_1(Q\beta_1,\beta_2) - \cos^2\theta g_1(\nabla_{\zeta_2}Q\beta_1,\beta_2) - g_1(v\nabla_{\zeta_2}\bar{\delta}P\beta_1,\bar{\delta}\zeta_2)$

$$- g_1(\mathcal{H}\nabla_{\zeta_2}\bar{F}P\beta_1, \bar{F}\beta_2) - g_1(\mathcal{A}_{\zeta_2}\bar{F}Q\beta_1, \bar{\delta}\beta_2) - \eta(\beta_1)g_1(\zeta_2, \bar{F}\beta_2) + \eta(\beta_2)g_1(\bar{\delta}\zeta_2, \beta_1)$$
$$+ g_1(\zeta_2, \bar{F}\beta_2)g_1(\bar{F}Q\beta_1, \operatorname{grad}\ln\lambda) + g_1(\bar{F}Q\beta_1, \bar{F}\beta_2)g_1(\zeta_2, \operatorname{grad}\ln\lambda)$$
$$- g_1(\zeta_2, \bar{F}Q\beta_1)g_1(\bar{F}\beta_2, \operatorname{grad}\ln\lambda),$$

for any $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)$ *and* $\zeta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. Considering the fact

$$g_1(\nabla_{\beta_1}\beta_2, \zeta_2) = -g_1([\beta_1, \zeta_2], \beta_2) - g_1(\nabla_{\zeta_2}\phi\beta_1, \phi\beta_2) - \eta(\beta_1)g_1(\zeta_2, \bar{F}\beta_2) + \eta(\beta_2)g_1(\bar{\delta}\zeta_2, \beta_1),$$

by using (2.11), (2.15), (2.10) and (2.13) for any $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\zeta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$. In the light of (3.17), (3.18) and (3.20), we can write

$$\begin{aligned} g_1(\nabla_{\beta_1}\beta_2, \zeta_2) = &- g_1([\beta_1, \zeta_2], \beta_2) - g_1(\nabla_{\zeta_2}\bar{\delta}P\beta_1, \phi\beta_2) - g_1(\nabla_{\zeta_2}\bar{F}P\beta_1, \phi\beta_2) \\ &- g_1(\nabla_{\zeta_2}\bar{F}Q\beta_1, \phi\beta_2) - \eta(\beta_1)g_1(\zeta_2, \bar{F}\beta_2) + \eta(\beta_2)g_1(\bar{\delta}\zeta_2, \beta_1) \\ &- g_1(\nabla_{\zeta_2}\bar{\delta}Q\beta_1, \phi\beta_2). \end{aligned}$$

From (2.10), (2.7), (2.15) and (2.13), we have

$$\begin{aligned} g_1(\nabla_{\beta_1}\beta_2, \zeta_2) = &- g_1([\beta_1, \zeta_2], \beta_2) - g_1(\mathcal{A}_{\zeta_2}\bar{\delta}P\beta_1, \bar{F}\beta_2) - g_1(v\nabla_{\zeta_2}\bar{\delta}P\beta_1, \bar{F}\beta_2) \\ &- g_1(\mathcal{H}\nabla_{\zeta_2}\bar{F}P\beta_1, \bar{F}\beta_2) + g_1(\nabla_{\zeta_2}\bar{\delta}^2Q\beta_1, \beta_2) - g_1(\mathcal{A}_{\zeta_2}\bar{F}Q\beta_1, \bar{\delta}\beta_2) \\ &- g_1(\mathcal{H}\nabla_{\zeta_2}\bar{F}Q\beta_1, \bar{F}\beta_2) - \eta(\beta_1)g_1(\zeta_2, \bar{F}\beta_2) + \eta(\beta_2)g_1(\bar{\delta}\zeta_2, \beta_1) \\ &- g_1(\mathcal{A}_{\zeta_2}\bar{F}P\beta_1, \bar{\delta}\beta_2). \end{aligned}$$

By using (2.1), (2.9) and from the fact that $\bar{\alpha}$ is a $\mathcal{PWHSCS}$, we finally get

$$\begin{aligned} g_1(\nabla_{\beta_1}\beta_2, W) = &- g_1([\beta_1, \zeta_2], \beta_2) - g_1(\mathcal{A}_{\zeta_2}\bar{\delta}P\beta_1, \bar{F}\beta_2) - g_1(v\nabla_{\zeta_2}\bar{\delta}P\beta_1, \bar{F}\beta_2) \\ &- g_1(\mathcal{H}\nabla_{\zeta_2}\bar{F}P\beta_1, \bar{F}\beta_2) + \sin 2\theta\zeta_2(\theta)g_1(Q\beta_1, \beta_2) - \cos^2\theta g_1(\nabla_{\zeta_2}Q\beta_1, \beta_2) \\ &- g_1(\mathcal{A}_{\zeta_2}\bar{F}Q\beta_1, \bar{\delta}\beta_2) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_2}\bar{\alpha}_*(\bar{F}Q\beta_1), \bar{\alpha}_*(\bar{F}\beta_2)) \\ &+ g_1(\zeta_2, \bar{F}\beta_2)g_1(\bar{F}Q\beta_1, \operatorname{grad}\ln\lambda) + g_1(\bar{F}Q\beta_1, , \bar{F}\beta_2)g_1(\zeta_2, \operatorname{grad}\ln\lambda) \\ &- g_1(\zeta_2, \bar{F}Q\beta_1)g_1(\bar{F}\beta_2, \operatorname{grad}\ln\lambda) - \eta(\beta_1)g_1(\zeta_2, \bar{F}\beta_2) + \eta(\beta_2)g_1(\bar{\delta}\zeta_2, X) \\ &- g_1(\mathcal{A}_{\zeta_2}\bar{F}P\beta_1, \bar{\delta}\beta_2). \end{aligned}$$

This is complete proof of the theorem. □

Theorem 3.8. *Let* $\bar{\alpha} : (\Xi_1, \phi, \xi, \eta, g_1) \to (\Xi_1, g_2)$ *be a* $\mathcal{PWHSCS}$ *from a* $\mathcal{SM}$ Ξ_1 *onto a* $\mathcal{RM}$ Ξ_2 *such that the structure vector field* ξ *is horizontal. Then* $\bar{\alpha}$ *is totally geodesic map if and only if*

(i) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_2}\bar{\alpha}_*(\bar{F}\omega_1), \bar{\alpha}_*(\bar{F}\zeta_1)) = \sin 2\theta\beta_2(\theta)g_1(\omega_1, \zeta_1) - \cos^2\theta g_1(\nabla_{\beta_2}\omega_1, \zeta_1)$
$- g_1(\mathcal{A}_{\beta_2}\bar{F}\bar{\delta}\omega_1, \zeta_1) - g_1(\mathcal{A}_{\beta_2}\bar{F}\omega_1, \bar{\delta}\zeta_1) + g_1(\beta_2, \bar{F}\zeta_1)g_1(\bar{F}\omega_1, \operatorname{grad}\ln\lambda)$
$+ g_1(\bar{F}\omega_1, \bar{F}\zeta_1)g_1(\beta_2, \operatorname{grad}\ln\lambda) - g_1(\beta_2, \bar{F}\omega_1)g_1(\bar{F}\zeta_1, \operatorname{grad}\ln\lambda) - g_1([\beta_2, \omega_1], \zeta_1)$

(ii) $\mathcal{T}_{\zeta_2}\mathcal{J}\bar{F}\omega_1 + \nabla_{\zeta_2}\mathcal{N}\bar{F}\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$

(iii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\bar{\delta}\omega_1), \bar{\alpha}_*(\beta_2)) = \cos^2\theta g_1(\nabla_{\beta_1}\omega_1, \beta_2) - g_1(\mathfrak{A}_{\beta_1}\bar{F}\omega_1, \mathcal{J}\beta_2)$
$+ \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_1, \bar{F}\bar{\delta}\omega_1), \bar{\alpha}_*(\beta_2)) + \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_1, \bar{F}\omega_1), \bar{\alpha}_*(\mathcal{N}\beta_2))$
$- \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\omega_1), \bar{\alpha}_*(\mathcal{N}\beta_2)) - g_1(\mathcal{J}\beta_1, \omega_1)\eta(\beta_2),$

for any $\omega_1, \zeta_1 \in \Gamma(\mathfrak{D}^\theta), \zeta_2, \omega_2 \in \Gamma(\mathfrak{D}^\perp)$ *and* $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. By using (2.1), (2.15), (2.9) and (2.13), we can write

$$\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2)) = -g([\beta_2,\omega_1],\zeta_1) - g_1(\nabla_{\beta_2}\phi\omega_1,\phi\zeta_1),$$

for any $\omega_1,\zeta_1\in\Gamma(\mathfrak{D}^\theta)$ and $\beta_2\in\Gamma(\ker\bar{\alpha}_*)^\perp$. In the light of (3.18), (2.13) and Lemma 3.2, we get

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2)) =& -g([\beta_2,\omega_1],\zeta_1) + \sin 2\theta\beta_2(\theta)g_1(\omega_1,\zeta_1) - \cos^2\theta g_1(\nabla_{\beta_2}\omega_1,\zeta_1)\\ &- g_1(\mathcal{A}_{\beta_2}\bar{F}\omega_1,\bar{\delta}\zeta_1) - g_1(\mathcal{H}\nabla_{\beta_2}\bar{F}\omega_1,\bar{F}\zeta_1).\end{aligned}$$

From (2.9) and by using the horizontal conformality from Lemma 2.1, we have

$$\begin{aligned}&\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2))\\ &= -g([\beta_2,\omega_1],\zeta_1) + \sin 2\theta\beta_2(\theta)g_1(\omega_1,\zeta_1) - \cos^2\theta g_1(\nabla_{\beta_2}\omega_1,\zeta_1)\\ &- g_1(\mathcal{A}_{\beta_2}\bar{F}\omega_1,\bar{\delta}\zeta_1) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\zeta_1))\\ &+ g_1(\beta_2,\bar{F}\zeta_1)g_1(\bar{F}\omega_1,\operatorname{grad}\ln\lambda) + g_1(\bar{F}\omega_1,\bar{F}\zeta_1)g_1(\beta_2,\operatorname{grad}\ln\lambda)\\ &- g_1(\beta_2,\bar{F}\omega_1)g_1(\bar{F}\zeta_1,\operatorname{grad}\ln\lambda),\end{aligned}$$

which is the proof of (i). On the other hand, from (2.1), (2.9), (2.13) and (2.15), we get

$$\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2)) = -g_1(\nabla_{\zeta_2}\phi\omega_2,\phi\beta_1),$$

for $\zeta_2,\omega_2\in\Gamma(\mathfrak{D}^\perp)$ and $\beta_1\in\Gamma(\ker\bar{\alpha}_*)^\perp$. By using the fact $\phi\omega_2\in\Gamma(\ker\bar{\alpha}_*)^\perp,\omega_2\in\Gamma(\mathfrak{D}^\perp)$ and from (2.4), (2.5), we have

$$\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2)) = g_1(\mathcal{A}_{\zeta_2}\mathcal{J}\bar{F}\omega_2,\beta_1) + g_1(\mathcal{H}\nabla_{\zeta_2}\mathcal{N}\bar{F}\omega_2,\beta_1),$$

which proves the (ii) part. For part (iii), by using (2.1), (2.9), (2.11) and (2.13), we can write

$$\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_1,\omega_1),\bar{\alpha}_*(\beta_2)) = -g_1(\nabla_{\beta_1}\phi\omega_1,\phi\beta_2) - g_1(\mathcal{J}\beta_1,\omega_1)\eta(\beta_2),$$

for any $\omega_1\in\Gamma(\mathfrak{D}^\theta)$ and $\beta_1,\beta_2\in\Gamma(\bar{\alpha}_*)^\perp$. In the light of (3.18), (2.11), (2.13), (2.7) and Lemma 3.2, we get

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2)) =& -\cos^2\theta g_1(\nabla_{\beta_1}\omega_1,\beta_2) - g_1(\mathcal{H}\nabla_{\beta_1}\bar{F}\bar{\delta}\omega_1,\beta_2) - g_1(\mathcal{A}_{\beta_1}\bar{F}\omega_1,\mathcal{J}\beta_2)\\ &- g_1(\mathcal{H}\nabla_{\beta_1}\bar{F}\omega_1,\mathcal{N}\beta_2) - g_1(\mathcal{J}\beta_1,\omega_1)\eta(\beta_2).\end{aligned}$$

By using the horizontal conformality of $\bar{\alpha}$ with (2.9) and Lemma 2.1, we finally have

$$\begin{aligned}&\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2))\\ &= -\cos^2\theta g_1(\nabla_{\beta_1}\omega_1,\beta_2) - g_1(\mathcal{A}_{\beta_1}\bar{F}\omega_1,\mathcal{J}\beta_2) - g_1(\mathcal{J}\beta_1,\omega_1)\eta(\beta_2)\\ &+ \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_1,\bar{F}\bar{\delta}\omega_1),\bar{\alpha}_*(\beta_2)) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\bar{\delta}\omega_1),\bar{\alpha}_*(\beta_2))\\ &+ \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_1,\bar{F}\omega_1),\bar{\alpha}_*(\mathcal{N}\beta_2)) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\mathcal{N}\beta_2)).\end{aligned}$$

This completes the proof of theorem. □

Theorem 3.9. *Let $\bar{\alpha} : (\Xi_1, \phi, \xi, \eta, g_1) \to (\Xi_2, g_2)$ be a $\mathcal{PWHSCS}$ where, $(\Xi_1, \phi, \xi, \eta, g_1)$ a $\mathcal{SM}$ and (Ξ_2, g_2) a $\mathcal{RM}$ with structure vector field ξ is horizontal. Suppose $\bar{\alpha}$ is $\mathfrak{D}^\theta$-ϕ-pluriharmonic. Then the following are equivalent.*

(i) *$\mathfrak{D}^\theta$ defines a totally geodesic foliation.*

(ii) $$\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\zeta_2) + \nabla^{\bar{\alpha}}_{\phi\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_1) - \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2) = \cos^2\theta\bar{\alpha}_*(\mathcal{NH}\nabla_{\bar{\delta}\zeta_1}\zeta_2 + \bar{F}\mathcal{T}_{\bar{\delta}\zeta_1}\zeta_2 + \bar{F}v\nabla_{\bar{F}\zeta_1}\zeta_2) - \sin 2\theta(\bar{\delta}\zeta_1(\theta)\bar{\alpha}_*(\bar{F}\zeta_2) + \bar{F}\zeta_1(\theta)\bar{\alpha}_*(\bar{F}\zeta_2)) - \eta(\nabla_{\bar{F}\zeta_1}\bar{\delta}\zeta_2)\bar{\alpha}_*\xi - \bar{\alpha}_*(\mathcal{H}\nabla_{\bar{\delta}\zeta_1}\zeta_2 + \bar{F}\mathcal{A}_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2 + \mathcal{NH}\nabla_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2) - \bar{F}\zeta_1(\ln\lambda)\bar{\alpha}_*(\bar{F}\zeta_2) - \bar{F}\zeta_2(\ln\lambda)\bar{\alpha}_*(\bar{F}\zeta_1) + g_1(\bar{F}\zeta_1, \bar{F}\zeta_2)\bar{\alpha}_*(\operatorname{grad}\ln\lambda) - \bar{\alpha}_*(\mathcal{H}_{\bar{\delta}\zeta_1}\bar{F}\zeta_2) + \mathcal{NA}_{\bar{F}\zeta_1}\zeta_2,$$

for any $\zeta_1, \zeta_2 \in \Gamma(\mathfrak{D}^\theta)$.

Proof. By using the concept of ϕ-pluriharmonicity with (3.18) and (2.9), we have

$$\bar{\alpha}_*\nabla_{\zeta_1}\zeta_2 = \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\zeta_2) + \nabla^{\bar{\alpha}}_{\bar{\delta}\zeta_1}\bar{\alpha}_*(\phi\zeta_2) - \bar{\alpha}_*\nabla_{\bar{\delta}\zeta_1}\bar{\delta}\zeta_2 - \bar{\alpha}_*\nabla_{\bar{\delta}\zeta_1}\bar{F}\zeta_2 - \bar{\alpha}_*\nabla_{\bar{F}\zeta_1}\bar{\delta}\zeta_2 - \bar{\alpha}_*\nabla_{\bar{F}\zeta_1}\bar{F}\zeta_2,$$

for any $\zeta_1, \zeta_2 \in \Gamma(\mathfrak{D}^\theta)$. In the light of (2.5), (2.13) and (2.12), we can write

$$\begin{aligned}
&\bar{\alpha}_*\nabla_{\zeta_1}\zeta_2 \\
&= \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\zeta_2) + \nabla^{\bar{\alpha}}_{\phi\zeta_1}\bar{\alpha}_*(\phi\zeta_2) + \bar{\alpha}_*(\phi\nabla_{\bar{\delta}\zeta_1}\phi\bar{\delta}\zeta_2 + \eta(\nabla_{\bar{\delta}\zeta_1}\bar{\delta}\zeta_2)\xi) \\
&\quad + \bar{\alpha}_*(\phi\nabla_{\bar{F}\zeta_1}\phi\bar{\delta}\zeta_2 + \eta(\nabla_{\bar{F}\zeta_1}\bar{\delta}\zeta_2)\xi) - \bar{\alpha}_*\nabla_{\bar{F}\zeta_1}\bar{F}\zeta_2 - \bar{\alpha}_*\nabla_{\bar{\delta}\zeta_1}\bar{F}\zeta_2.
\end{aligned}$$

By using (2.4), (2.7) and Lemma 3.2, we get

$$\begin{aligned}
&\bar{\alpha}_*\nabla_{\zeta_1}\zeta_2 \\
&= \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\zeta_2) + \nabla^{\bar{\alpha}}_{\phi\zeta_1}\bar{\alpha}_*(\phi\zeta_2) + \sin 2\theta\bar{\delta}\zeta_1(\theta)\bar{\alpha}_*(\phi\zeta_2) - \cos^2\theta\bar{\alpha}_*(\phi\nabla_{\bar{\delta}\zeta_1}\zeta_2) \\
&\quad + \sin 2\theta\bar{F}\zeta_1(\theta)\bar{\alpha}_*(\phi\zeta_2) - \cos^2\theta\bar{\alpha}_*(\phi\nabla_{\bar{F}\zeta_1}\zeta_2) + \bar{\alpha}_*\{\phi(\mathcal{T}_{\bar{\delta}\zeta_1}\zeta_2 + \mathcal{H}\nabla_{\bar{\delta}\zeta_1}\zeta_2)\} \\
&\quad + \eta(\nabla_{\bar{\delta}\zeta_1}\bar{\delta}\zeta_2)\bar{\alpha}_*\xi + \bar{\alpha}_*\{\phi(\mathcal{A}_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2 + \mathcal{H}\nabla_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2)\} + \eta(\nabla_{\bar{F}\zeta_1}\bar{\delta}\zeta_2)\bar{\alpha}_*\xi \\
&\quad - \nabla^{\bar{\alpha}}_{\bar{F}\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2) + (\nabla\bar{\alpha}_*)(\bar{F}\zeta_1, \bar{F}\zeta_2) - \bar{\alpha}_*(\mathcal{T}_{\bar{\delta}\zeta_1}\bar{F}\zeta_2 + \mathcal{H}_{\bar{\delta}\zeta_1}\bar{F}\zeta_2).
\end{aligned}$$

Finally, by using the horizontal conformality of $\bar{\alpha}$ from Lemma 2.1 and with (2.7), (2.1), we have

$$\begin{aligned}
\bar{\alpha}_*\nabla_{\zeta_1}\zeta_2 =& \nabla^{\bar{\alpha}}_{\phi\zeta_1}\bar{\alpha}_*(\phi\zeta_2) + \sin 2\theta\bar{\delta}\zeta_1(\theta)\bar{\alpha}_*(\bar{F}\zeta_2) - \cos^2\theta\bar{\alpha}_*(\mathcal{NH}\nabla_{\bar{\delta}\zeta_1}\zeta_2 + \bar{F}\mathcal{T}_{\bar{\delta}\zeta_1}\zeta_2) \\
&+ \sin 2\theta\bar{F}\zeta_1(\theta)\bar{\alpha}_*(\bar{F}\zeta_2) - \cos^2\theta\bar{\alpha}_*(\mathcal{NA}_{\bar{F}\zeta_1}\zeta_2 + \bar{F}v\nabla_{\bar{F}\zeta_1}\zeta_2) + \bar{\alpha}_*(\mathcal{H}\nabla_{\bar{\delta}\zeta_1}\zeta_2) \\
&+ \eta(\nabla_{\bar{\delta}\zeta_1}\bar{\delta}\zeta_2)\bar{\alpha}_*\xi + \bar{\alpha}_*\{\bar{F}(\mathcal{A}_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2 + \mathcal{NH}\nabla_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2)\} + \eta(\nabla_{\bar{F}\zeta_1}\bar{\delta}\zeta_2)\bar{\alpha}_*\xi \\
&+ \bar{F}\zeta_1(\ln\lambda)\bar{\alpha}_*(\bar{F}\zeta_2) + \bar{F}\zeta_2(\ln\lambda)\bar{\alpha}_*(\bar{F}\zeta_1) - g_1(\bar{F}\zeta_1, \bar{F}\zeta_2)\bar{\alpha}_*(\operatorname{grad}\ln\lambda) \\
&- \bar{\alpha}_*(\mathcal{H}_{\bar{\delta}\zeta_1}\bar{F}\zeta_2) + \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\zeta_2) - \nabla^{\bar{\alpha}}_{\bar{F}\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2).
\end{aligned}$$

□

This completes the proof of the theorem.

4. Pointwise hemi-slant conformal submersions with vertical Reeb vector field-ξ

This section will go over the definitions and results that will help us understand and investigate the concept of pointwise hemi-slant conformal submersions from $\mathcal{ACMMs}$ by considering the Reeb vector filed ξ vertical.

Definition 4.1. *Let $\bar{\alpha} : (\Xi_1, \phi, \xi, \eta, g_1) \to (\Xi_2, g_2)$ be a $\mathcal{HCS}$ where $(\Xi_1, \phi, \xi, \eta, g_1)$ is an $\mathcal{ACMM}$ and (Ξ_2, g_2) is a $\mathcal{RM}$. A $\mathcal{HCS}$ $\bar{\alpha}$ is called a pointwise hemi-slant conformal submersion if there exists distributions $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$ such that $\ker\bar{\alpha}_* = \mathfrak{D}^\theta \oplus \mathfrak{D}^\perp \oplus <\xi>$, $\phi(\mathfrak{D}^\perp) \subseteq \Gamma(\ker\bar{\alpha})^\perp$ and for any given point $q \in \Xi_1$ and $\beta_1 \in (\mathfrak{D}^\theta)_q$, the angle $\theta = \theta(\beta_1)$ between $\phi\beta_1$ and space $(\mathfrak{D}^\theta)_q$ is independent of choice of non-zero vector $\beta_1 \in (\mathfrak{D}^\theta)_q$, where $\mathfrak{D}^\theta$ is the orthogonal complement of $\mathfrak{D}^\perp$ in $\ker\bar{\alpha}_*$ and $<\xi>$ is 1-dimensional distribution. The angle θ is a slant function, often known as the pointwise hemi-slant function of submersion.*

Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2) with vertical Reeb vector field ξ. Then, for any $Y \in (\ker\bar{\alpha}_*)$, we have

$$Y = \mathfrak{P}\beta_2 + \mathfrak{Q}\beta_2 + \eta(\beta_2)\xi \tag{4.24}$$

where $\mathfrak{P}$ and $\mathfrak{Q}$ are the projections morphism onto $\mathfrak{D}^\perp$ and $\mathfrak{D}^\theta$.

Lemma 4.1. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2), then we have*

$$\bar{\delta}^2\omega_2 = -\cos^2\theta(I - \eta\otimes\xi)\omega_2, \tag{4.25}$$

for any vector field $\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$.

Lemma 4.2. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ with vertical ξ, from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2), then we have*

(i) $g_1(\bar{\delta}\zeta_1, \bar{\delta}\zeta_2) = \cos^2\theta\{g_1(\zeta_1, \zeta_2) - \eta(\zeta_1)\eta(\zeta_2)\}$,
(ii) $g_1(\bar{F}\zeta_1, \bar{F}\zeta_2) = \sin^2\theta\{g_1(\zeta_1, \zeta_2) - \eta(\zeta_1)\eta(\zeta_2)\}$,

for any vector fields $\zeta_1, \zeta_2 \in \Gamma(\ker\bar{\alpha}_*)$.

Moving further, we shall talk about the integrability of slant and anti-invariant distributions $\mathfrak{D}^\theta$ and $\mathfrak{D}^\perp$ respectively.

Theorem 4.1. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from $\mathcal{SM}$ onto a $\mathcal{RM}$ with vertical ξ and θ is a hemi-slant function, tfae*

(i) *The anti-invariant distribution $\mathfrak{D}^\perp$ is integrable.*
(ii) $\frac{1}{\lambda^2} g_2(\nabla^{\bar{\alpha}}_{\zeta_2}\bar{\alpha}_*(\bar{F}\zeta_1) - \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2), \bar{\alpha}_*(\bar{F}\omega_1))$
$= g(\nabla_{\zeta_1}\bar{F}\bar{\delta}\zeta_2 + \nabla_{\zeta_2}\bar{F}\bar{\delta}\zeta_1, \omega_1) - g(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2 + \mathcal{T}_{\zeta_2}\bar{F}\zeta_1, \bar{\delta}\omega_1)$,

for any $\zeta_1, \zeta_2 \in \Gamma(D^\perp)$ and $\omega_1 \in \mathfrak{D}^\theta$.

By comparing the preceding conclusion with Theorem 3.1, it is inescapable that there is no influence of the Reeb vector field ξ, whether horizontal or vertical. For slant distribution, we have

Lemma 4.3. *Let $\bar{\alpha} : (\Xi_1, \phi, \xi, \eta, g_1) \to (\Xi_2, g_2)$ be a $\mathcal{PWHSCS}$ with $\xi \in \Gamma(ker\bar{\alpha}_*)$ where, $(\Xi_1, \phi, \xi, \eta, g_1)$ a $\mathcal{SM}$ and (Ξ_2, g_2) a $\mathcal{RM}$. Then the slant distribution is not integrable.*

Since the slant distribution is not integrable, now we will discuss about distribution $\mathfrak{D}^\theta \oplus <\xi>$.

Theorem 4.2. *Let $\bar{\alpha} : \Xi_1 \to \Xi_2$ be a $\mathcal{PWHSCS}$ from a $\mathcal{SM}$ Ξ_1 onto a $\mathcal{RM}$ Ξ_2 such that ξ is vertical. Then the following are equivalent.*

(i) *Slant distribution* $\mathfrak{D}^{\theta}\oplus<\xi>$ *is integrable.*

(ii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))+\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\omega_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1))+g_1([\omega_1,\zeta_1],\omega_2)$
$=\sin 2\theta g_1(\omega_1,\omega_2)-\cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2)-g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2)-g_1(\mathcal{T}_{\omega_2}\bar{F}\omega_1,\phi\zeta_1),$

for any $\omega_1,\omega_2\in\Gamma(\mathfrak{D}^{\theta}\oplus<\xi>)$ *and* $\zeta_1\in\Gamma(\mathfrak{D}^{\perp})$.

Proof. For any $\omega_1,\omega_2\in\Gamma(\mathfrak{D}^{\theta}\oplus<\xi>)$ and $\zeta_1\in\Gamma(D)^{\perp}$ with taking account the fact from (2.11), (2.13), (2.15), (2.4), (2.5), (2.1) with Lemma 3.2, we get

$$\begin{aligned}&g_1([\omega_1,\omega_2],\zeta_1)\\&=-g_1([\omega_1,\zeta_1],\omega_2)+\sin 2\theta\zeta_1(\theta)g_1(\omega_1,\omega_2)-\cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2)-g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\phi\zeta_1)\\&-g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2)-\frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\zeta_1}\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))-\frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\omega_2}\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1)).\end{aligned}$$

By using the horizontal conformality of $\bar{\alpha}$ with Lemma 2.1 and (2.5), we finally have

$$\begin{aligned}&g_1([\omega_1,\omega_2],\zeta_1)\\&=-g_1([\omega_1,\zeta_1],\omega_2)+\sin^2\theta\zeta_1(\theta)g_1(\omega_1,\omega_2)-\cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2)-g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\phi\zeta_1)\\&-g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2)-\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))+\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))\\&-\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\omega_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1))+\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_2,\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1)).\end{aligned}$$

□

Although the nature of ξ differs, the proofs of Theorem 3.2 and the previous result are identical as well.

Corollary 4.1. *Let* $\bar{\alpha}:(\Xi_1,\phi,\xi,\eta,g_1)\rightarrow(\Xi_2,g_2)$ *be a* $\mathcal{PWHSCS}$ *from* $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ *onto a* $\mathcal{RM}$ (Ξ_2,g_2) *with hemi-slant function* θ*. The following conditions holds.*

Let $\bar{\alpha}:(\Xi_1,\phi,\xi,\eta,g_1)\rightarrow(\Xi_2,g_2)$ *be a* $\mathcal{PWHSCS}$ *from* $\mathcal{SM}$ *onto a* $\mathcal{RM}$ *with hemi slant function* θ*. Then*	*(i)* $\mathfrak{D}^{\theta}\oplus<\xi>$ *is integrable with* $\xi\in\Gamma(ker\bar{\alpha}_*)$ *if and only if*	*(ii)* $\mathfrak{D}^{\theta}$ *is integrable with* $\xi\in\Gamma(ker\bar{\alpha}_*)^{\perp}$ *if and only if*
	$\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))$ $+\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\omega_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1))$ $+g_1([\omega_1,\zeta_1],\omega_2)=\sin 2\theta$ $g_1(\omega_1,\omega_2)-g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2)$ $-\cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2)$ $-g_1(\mathcal{T}_{\omega_2}\bar{F}\omega_1,\phi\zeta_1)$	$\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))$ $+\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\omega_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1))$ $+g_1([\omega_1,\zeta_1],\omega_2)=\sin 2\theta$ $g_1(\omega_1,\omega_2)-g_1(\mathcal{T}_{\omega_2}\bar{F}\omega_1,\phi\zeta_1)$ $-\cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2)$ $-g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2)$

Then, for (i), $\omega_1,\omega_2\in\Gamma(\mathfrak{D}^{\theta}\oplus<\xi>)$ and $\zeta_1\in\Gamma(\mathfrak{D}^{\perp})$, for (ii), $\omega_1,\omega_2\in\Gamma(\mathfrak{D}^{\theta})$ and $\zeta_1\in\Gamma(\mathfrak{D}^{\perp})$.

For totally geodesicness of anti-invariant distribution $\mathfrak{D}^{\perp}$ considering ξ to be vertical specially $\mathfrak{D}^{\perp}\oplus<\xi>$, we have

Theorem 4.3. *Let* $\bar{\alpha}$ *be a* $\mathcal{PWHSCS}$ *from* $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ *onto a* $\mathcal{RM}$ (Ξ_2,g_2) *with hemi-slant function* θ *and vertical Reeb vector field* ξ*. Then the anti-invariant distribution* $\mathfrak{D}^{\perp}\oplus<\xi>$ *defines a totally geodesic foliation if and only if*

$$\begin{aligned}\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1))=&\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1))-g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{F})\\&-g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1)-g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1)\end{aligned}$$

and

$$\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) = \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) - g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{N}\beta_1)$$
$$- g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\mathcal{J}\beta_1) + g_1(\zeta_1,\mathfrak{B}\beta_1)\eta(\zeta_2),$$

for any $\zeta_1,\zeta_2 \in \Gamma(\mathfrak{D}^{\perp}\oplus <\xi>), \omega_1 \in \Gamma(\mathfrak{D}^{\theta})$ *and* $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$.

Proof. From (2.11), (2.13, (2.1), (2.4), (2.5) and (2.15), we obtain

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) + \frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\zeta_1}\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1))$$
$$+ g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{F}\omega_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1).$$

Lemma 3.2, (2.1), and the horizontal conformality of $\bar{\alpha}$ allow us to ultimately obtain

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{F}\omega_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1)$$
$$- \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)).$$

However, using (2.11), (2.13), (2.4), (2.5), (2.1), (2.9) and (2.15), for any $\zeta_1,\zeta_2 \in \Gamma(\mathfrak{D}^{\perp})$ and $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$, we have

$$g_1(\nabla_{\zeta_1}\zeta_2,\beta_1) = \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) - \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1))$$
$$+ g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{N}\beta_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{J}\beta_1) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\mathcal{J}\beta_1)$$
$$+ g_1(\zeta_1,\mathfrak{B}\beta_1)\eta(\zeta_2).$$

This is the required proof of theorem. □

Theorem 4.4. *Let* $\bar{\alpha}$ *be a* $\mathcal{PWHSCS}$ *from* $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ *onto a* $\mathcal{RM}$ (Ξ_2,g_2) *with hemi-slant function* θ *and vertical Reeb vector field* ξ. *Then the slant distribution* $\mathfrak{D}^{\perp}$ *not defines totally geodesic foliation.*

Since the slant distribution is not defines totally geodesic foliation, we can discuss the total geodesicness of $\mathfrak{D}^{\theta}\oplus <\xi>$ as follows :

Corollary 4.2. *Let* $\bar{\alpha}$ *be a* $\mathcal{PWHSCS}$ *from* $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ *onto a* $\mathcal{RM}$ (Ξ_2,g_2) *with hemi-slant function* θ *and vertical Reeb vector field* ξ. *Then the slant distribution* $\mathfrak{D}^{\perp}\oplus <\xi>$ *defines a totally geodesic foliation if and only if*

$$\begin{aligned}\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)) = &\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)) - g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{F})\\ &- g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1)\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}&\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1))\\ &= \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) - g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{N}\beta_1)\\ &- g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\mathcal{J}\beta_1) - g_1(\bar{\delta}\bar{F}\zeta_1,\beta_1)\eta(\zeta_2),\end{aligned} \tag{4.27}$$

for any $\zeta_1,\zeta_2 \in \Gamma(\mathfrak{D}^{\perp}), \omega_1 \in \Gamma(\mathfrak{D}^{\theta})$ *and* $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$.

Theorem 3.4 provides an easy way to prove the above conclusion by taking the vertical character of ξ-. When we compare the proof of both results, there is no change in equations

(3.22) and (4.26), but in comparison (3.23) and (4.27), single term $g_1(\bar{\delta}\zeta_1, \zeta_2)\eta(\beta_1)$ is substituted by $-g_1(\bar{\delta}\bar{F}\zeta_1, \beta_1)\eta(\zeta_2)$.

5. Conclusion

This research article examined the effect of a vector field ξ- with dual nature (vertical and horizontal) on pointwise hemi-slant conformal submersions from Sasakian manifolds. The conditions of distribution integrabiities and their leaves' total geodesicness are also examined.

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GEOMETRIC STUDY OF RICCI SOLITONS IN PERFECT FLUID SPACETIMES WITH LORENTZIAN CONCIRCULAR STRUCTURE MANIFOLDS

GURUPADAVVA INGALAHALLI *

Abstract. In this article, we examine the interaction between Ricci solitons and the properties of the geometric structure in a perfect fluid spacetimes that admits a Lorentzian Concircular structure manifold with a Concircular curvature tensor. We investigate the conditions under which a Ricci soliton exists within such a framework and analyze its implications on the curvature properties of the spacetime. The study focuses on the influence of the soliton potential on the energy-momentum tensor of the perfect fluid and examines the interplay between the Ricci curvature and the Concircular structure. Further, we establish key geometric conditions that characterize the nature of the Ricci soliton in this setting and derive significant constraints on the manifolds topology. Our findings contribute to the broader understanding of the role of Ricci solitons in relativistic fluids and their impact on spacetime geometry.

Keywords: Ricci soliton, Einstein, perfect fluid, Lorentz space.

2020 Mathematics Subject Classification: 53B50, 53C44, 53C50, 83C02.

1. Introduction

Geometric flows have emerged as a crucial tool in the study of Riemannian and semi-Riemannian manifolds, as well as in the theory of general relativity. In his foundational work, Hamilton [12] identified the Ricci flow as an effective method for refining the structure of a manifold. This process modifies the metric of a Riemannian manifold M over time, helping to smooth out its irregularities. The Ricci flow is defined by:

$$\frac{\partial g}{\partial t} = -2Ric \tag{1.1}$$

where g represents the components of the metric tensor, Ric is the Ricci curvature tensor, and t denotes the time parameter. Ricci solitons represent self-similar solutions to the Ricci flow. They have attracted considerable attention in both differential geometry and general relativity due to their strong connection with the Ricci flow and their role as a generalization of Einstein metrics. On a Riemannian manifold (M, g), a Ricci soliton is a particular type of

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solution to the Ricci flow equation. It can be viewed as a natural extension of an Einstein metric and is characterized by a triple (g, V, a), where the following condition holds:

$$\mathfrak{L}_V g + 2Ric + 2ag = 0, \tag{1.2}$$

where Ric is the Ricci curvature tensor of the metric g, $\mathfrak{L}_V g$ denotes the Lie derivative of g along the vector field V on M, and a is a scalar constant. A Ricci soliton (g, V, a) on a manifold M is classified as shrinking, steady, or expanding depending on the sign of the constant a. Specifically, the soliton is: shrinking if $a < 0$: the manifold contracts over time, steady if $a = 0$: the metric evolves trivially under the flow, and expanding if $a > 0$: the manifold expands over time.

Ricci solitons are fundamental in the analysis of geometric flows and general relativity, acting as self-similar solutions to the Ricci flow equation. Their importance has led to extensive investigations across various spacetime geometries, as they offer a deeper understanding of how geometric structures evolve over time. Notably, exploring Ricci solitons within perfect fluid spacetimes offers meaningful contributions to the fields of relativistic hydrodynamics and cosmological modeling. A perfect fluid spacetime is an idealized model in which the energy-momentum tensor represents a fluid with no viscosity or heat conduction, making it a fundamental framework in general relativity. When such a spacetime admits a Lorentzian Concircular structure, it imposes additional geometric constraints that influence the curvature properties and behavior of Ricci solitons. The Lorentzian Concircular structure, characterized by a Concircular vector field, plays a significant role in studying the conformal geometry of spacetime and its interaction with fluid dynamics.

As a natural extension of the Lorentzian para-Sasakian manifold (commonly referred to as the LP-Sasakian manifold, introduced by Matsumoto) A.A. Shaikh [21] developed the concept of Lorentzian Concircular structure manifolds, exploring their existence and significance in both cosmology and the general theory of relativity. These manifolds, denoted as $(LCS)_n$-manifolds, form a notable subclass within the broader category of semi-Riemannian manifolds and play a crucial role in the analysis of spacetime geometry, especially in four-dimensional equipped with a Lorentzian metric g with signature $(-, +, +, +)$. The Lorentzian metric, which stands out among indefinite metrics, introduces a distinct geometric framework where not all directions are equivalent. This leads to a classification of vectors into timelike, lightlike (null), and spacelike, depending on how they interact with the metric. The foundation of Lorentzian geometry lies in understanding the causal character of these vectors. This causal structure is what makes Lorentzian manifolds particularly well-suited for modeling spacetime in the context of Einsteins theory of general relativity [14].

In recent years, Ricci solitons have been extensively explored by several geometers across a wide range of geometric frameworks in [1], [4], [5], [7], [9], [10], [11], [16], [17], [19], [20]. Furthermore, a number of researchers have investigated perfect fluid spacetimes from the perspective of Ricci soliton geometry, highlighting their structural and curvature properties in [3], [6], [8], [18], [24].

Based on the above this research investigates the relationship between Ricci solitons and the geometric framework of a perfect fluid spacetime that possesses a Lorentzian Concircular structure. By examining the essential characteristics of these spacetimes, the study aims to understand the role Ricci solitons play in shaping their geometric progression and stability. Additionally, the work seeks to identify the criteria necessary for the existence of such solitons, thereby offering valuable insights into their significance within the realms of differential geometry and theoretical physics.

Relativistic fluid models play a vital role across various areas of physics, including astrophysics, plasma physics, and nuclear physics. In the context of general relativity, perfect fluids serve as simplified yet powerful models for describing matter distributions, such as those found within stars or in an isotropic universe. Einsteins field equations can be employed to analyze the dynamics of a perfect fluid enclosed in a spherical body, while the FLRW equations are instrumental in modeling the large-scale evolution of the universe. Within general relativity, the energy-momentum tensor acts as the source of spacetime curvature. A perfect fluid is fully described by its mass density in the rest frame and its isotropic pressure. It lacks shear stresses, viscosity, and heat conduction, and its energy-momentum tensor takes the following form:

$$T(U,V) = pg(U,V) + (\sigma + p)\eta(U)\eta(V). \tag{1.3}$$

For any vector fields $U, V \in \chi(M)$, where p denotes the isotropic pressure, σ represents the energy density, and g is the Minkowski metric tensor, the velocity vector of the fluid is given by $\xi := \sharp(\eta)$, satisfying the normalization condition $g(\xi,\xi) = -1$. When the relation $\sigma = -p$ holds, the energy-momentum tensor becomes Lorentz-invariant, expressed as $T = -\sigma g$, corresponding to the vacuum state. Alternatively, when $\sigma = 3p$, the matter content is identified as a radiation fluid.

The motion of a perfect fluid is governed by Einstein's field equations, which describe the interaction between matter and the curvature of spacetime:

$$Ric(U,V) + (\lambda - \frac{r}{2})g(U,V) = kT(U,V). \tag{1.4}$$

For any vector fields $U, V \in \chi(M)$, where λ denotes the cosmological constant, k represents the gravitational constant (often taken as $8\pi G$ in geometric units with G the universal gravitational constant), Ric is the Ricci curvature tensor, and r is the scalar curvature associated with the metric g. These modified field equations arise from Einsteins original formulation, where the cosmological constant was introduced in an attempt to model a static universe. In contemporary cosmology, however, λ is interpreted as a potential form of dark energy responsible for the observed accelerated expansion of the universe. Substituting the expression for T from equation (1.3) into (1.4) we obtain:

$$Ric(U,V) = -\left[\lambda - \frac{r}{2} - kp\right] g(U,V) + (\sigma k + pk)\eta(U)\eta(V), \tag{1.5}$$

for any $U, V \in \chi(M)$. Recall that a manifold exhibits a particular geometric property when its Ricci tensor Ric can be written as a functional combination of g and $\eta \otimes \eta$, for η the g dual 1-form of a unitary vector field, is called quasi-Einstein.

By contracting equation (1.5) and considering that $g(\xi,\xi) = -1$, we obtain:

$$r = 4\lambda + k\sigma - 3kp. \tag{1.6}$$

Therefore, the resulting expression simplifies to:

$$Ric(U,V) = \left[\frac{2\lambda + k\sigma - pk}{2}\right] g(U,V) + (k\sigma + kp)\eta(U)\eta(V). \tag{1.7}$$

This relation holds for all vector fields $U, V \in \chi(M)$.

$$QU = \left[\frac{2\lambda + k\sigma - pk}{2}\right] U + (k\sigma + kp)\eta(U)\xi. \tag{1.8}$$

Here, Ric denotes the Ricci tensor associated with the Ricci operator Q, defined by the relation $Ric(U,V) = g(QU,V)$.

2. Basic Concepts of Lorentzian Concircular structure manifold

In this section, we explore key concepts of Lorentzian Concircular structure manifolds: A Lorentzian Concircular structure manifold is a smooth manifold M equipped with both a Lorentzian metric g and a Concircular structure.

An n-dimensional smooth, connected, and paracontact Hausdorff manifold M, equipped with a Lorentzian metric g, is referred to as a Lorentzian manifold. This implies that M possesses a smooth, symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the bilinear form $g_p : T_pM \times T_pM \longrightarrow R$ defines a non-degenerate inner product with signature $(-,+,\ldots,+)$. Here, T_pM denotes the tangent vector space to M at p, and R represents the field of real numbers. A non-zero vector $v \in T_pM$ is classified as timelike if $g_p(v,v) < 0$, non-spacelike if $g_p(v,v) \leq 0$, null if $g_p(v,v) = 0$, or spacelike $g_p(v,v) > 0$, [15]. The causal character of a vector refers to the category it falls into based on this classification.

A Lorentzian manifold M admits a unit timelike concircular vector field ξ, referred to as the characteristic vector field of the manifold. The manifold satisfies the following fundamental conditions:

$$g(\mathcal{U},\xi) = \eta(\mathcal{U}), \quad g(\xi,\xi) = -1, \quad (\nabla_{\mathcal{U}}\eta)(\mathcal{V}) = \alpha\{g(\mathcal{U},\mathcal{V}) + \eta(\mathcal{U})\eta(\mathcal{V})\}, \tag{2.9}$$

where g denotes the Lorentzian metric, ξ is the unit timelike concircular vector field, η is the associated 1-form, ∇ represents the Levi-Civita connection, and α is a smooth scalar function on M. From equation (2.9), we obtain:

$$\nabla_{\mathcal{U}}\xi = \alpha\{\mathcal{U} + \eta(\mathcal{U})\xi\}. \tag{2.10}$$

For any vector fields $\mathcal{U}, \mathcal{V}$ on M and ∇ represents the covariant derivative operator associated with the Lorentzian metric g, and α is a non-zero scalar function that satisfies:

$$\nabla_{\mathcal{U}}\alpha = (\mathcal{U}\alpha) = d\alpha(\mathcal{U}) = \rho\eta(\mathcal{U}). \tag{2.11}$$

Let ρ denote a scalar function defined by $\rho = -(\xi\alpha)$. By setting:

$$\nabla_{\mathcal{U}}\xi = \alpha\phi\mathcal{U}. \tag{2.12}$$

Then, using equations (2.10) and (2.12), we get:

$$\phi\mathcal{U} = \mathcal{U} + \eta(\mathcal{U})\xi. \tag{2.13}$$

Here, ϕ is a $(1,1)$-type tensor field, referred to as the structure tensor of M. A Lorentzian manifold M, equipped with a unit timelike concircular vector field ξ, its associated 1-form

η, and structure tensor field ϕ, is known as a Lorentzian Concircular structure manifold, abbreviated as an $(LCS)_n$-manifold [21]. In such a manifold, the following fundamental relations are satisfied:

$$\phi^2\mathcal{V} = \mathcal{V}+\eta(\mathcal{V})\xi, \quad \eta(\phi\mathcal{V})=0, \quad \eta(\xi)=-1, \quad \phi\cdot\xi=0, \tag{2.14}$$

$$g(\phi\mathcal{U},\phi\mathcal{V}) = g(\mathcal{U},\mathcal{V})+\eta(\mathcal{U})\eta(\mathcal{V}), \tag{2.15}$$

$$\eta(R(\mathcal{U},\mathcal{V})Z) = (\alpha^2-\rho)[g(\mathcal{V},Z)\eta(\mathcal{U})-g(\mathcal{U},Z)\eta(\mathcal{V})], \tag{2.16}$$

$$R(\mathcal{U},\mathcal{V})\xi = (\alpha^2-\rho)[\eta(\mathcal{V})\mathcal{U}-\eta(\mathcal{U})\mathcal{V}], \tag{2.17}$$

$$Ric(\mathcal{U},\xi) = (n-1)(\alpha^2-\rho)\eta(\mathcal{U}). \tag{2.18}$$

For all vector fields $\mathcal{U},\mathcal{V},Z$ on M, let R denote the Riemann curvature tensor associated with the Lorentzian metric g, and let Ric represent the Ricci tensor corresponding to the Ricci operator Q, defined by $Ric(\mathcal{U},\mathcal{V})=g(Q\mathcal{U},\mathcal{V})$.

The Concircular curvature tensor Ç [3] is defined by the expression:

$$\text{Ç}(\mathcal{U},\mathcal{V})Z = R(\mathcal{U},\mathcal{V})Z-\frac{r}{n(n-1)}[g(\mathcal{V},Z)\mathcal{U}-g(\mathcal{U},Z)\mathcal{V}], \tag{2.19}$$

where R is the Riemann curvature tensor, r denotes the scalar curvature, and Ric represent the Ricci tensor associated with operator Q, that is, $Ric(\mathcal{U},\mathcal{V})=g(Q\mathcal{U},\mathcal{V})$.

3. Certain Geometric Properties of a Perfect Fluid Spacetime Admitting Lorentzian Concircular structure manifolds

In this portion of the paper, we investigate the geometric characteristics of a perfect fluid spacetime modeled on a Lorentzian Concircular structure manifold (denoted as $(LCS)_n$-manifold). Our aim is to explore how the intrinsic geometry of such a manifold interacts with the energy-momentum distribution of a perfect fluid. We begin by recalling the fundamental definitions and structure tensors associated with a Lorentzian Concircular structure manifold and the energy-momentum tensor for a perfect fluid, followed by derivations of curvature conditions, symmetry properties, and physical interpretations relevant to relativistic fluid dynamics.

The Concircular curvature tensor Ç [3] in perfect fluid spacetime endowed with a 4-dimensional Lorentzian Concircular structure manifold is defined as follows:

$$\text{Ç}(\mathcal{U},\mathcal{V})Z = R(\mathcal{U},\mathcal{V})Z-\frac{r}{12}[g(\mathcal{V},Z)\mathcal{U}-g(\mathcal{U},Z)\mathcal{V}]. \tag{3.20}$$

After Covariantly differentiating equation (3.20) and contracting, we derive

$$(div\text{Ç})(\mathcal{U},\mathcal{V})Z = (divR)(\mathcal{U},\mathcal{V})Z-\frac{1}{12}[g(\mathcal{V},Z)(\mathcal{U}r)-g(\mathcal{U},Z)(\mathcal{V}r)]. \tag{3.21}$$

Assuming $div\text{Ç}=\nabla\cdot\text{Ç}=0$, where div represents the divergence, we derive the following from equation (3.21):

$$(\nabla_{\mathcal{U}}Ric)(\mathcal{V},Z)-(\nabla_{\mathcal{V}}Ric)(\mathcal{U},Z)=\frac{1}{12}[g(\mathcal{V},Z)(\mathcal{U}r)-g(\mathcal{U},Z)(\mathcal{V}r)]. \tag{3.22}$$

Since r is a constant scalar curvature, the preceding relation indicates that

$$g((\nabla_{\mathcal{U}}Q)\mathcal{V}-(\nabla_{\mathcal{V}}Q)\mathcal{U},Z)=0 \Longrightarrow (\nabla_{\mathcal{U}}Q)\mathcal{V}-(\nabla_{\mathcal{V}}Q)\mathcal{U}=0. \tag{3.23}$$

By substituting equation (1.8) into (3.23), we obtain:

$$k(\sigma+p)[g(\mathcal{V},\nabla_{\mathcal{U}}\xi)\xi+\eta(\mathcal{V})\nabla_{\mathcal{U}}\xi-g(\mathcal{U},\nabla_{\mathcal{V}}\xi)\xi-\eta(\mathcal{U})\nabla_{\mathcal{V}}\xi]=0. \tag{3.24}$$

Inserting (2.12), (2.13) in (3.24) and on simplification, we get:

$$k(\sigma+p)\alpha[\eta(\mathcal{V})\mathcal{U}-\eta(\mathcal{U})\mathcal{V}]=0. \tag{3.25}$$

Here, k denotes the gravitational constant defined as $k=8\pi G$ in geometrized units, where G is the universal gravitational constant. The preceding equation leads to the condition $p=-\sigma$, under the assumption that $\alpha\neq 0$. This leads us to the following conclusion:

Theorem 3.1. *Let (M,g) be a general relativistic perfect fluid spacetime endowed with a Lorentzian Concircular structure manifold satisfying (1.7). If the divergence of the Concircular curvature tensor vanishes, that is, div$\c{C}=\nabla\cdot\c{C}=0$, then the pressure and energy density satisfy $p=-\sigma$ provided that $\alpha\neq 0$.*

If the Concircular curvature tensor is Concircularly flat, that is, $\c{C}(\mathcal{U},\mathcal{V})Z=0$, then by equation (1.6), it follows that:

$$R(\mathcal{U},\mathcal{V})Z=\frac{4\lambda+k(\sigma-3p)}{12}[g(\mathcal{V},Z)\mathcal{U}-g(\mathcal{U},Z)\mathcal{V}]. \tag{3.26}$$

If the condition $p=-\sigma$ is imposed in equation (3.26), then it results in the following expression:

$$R(\mathcal{U},\mathcal{V})Z=\frac{\lambda+k\sigma}{3}[g(\mathcal{V},Z)\mathcal{U}-g(\mathcal{U},Z)\mathcal{V}]. \tag{3.27}$$

Based on this, we propose the following statement:

Theorem 3.2. *Let (M,g) be a general relativistic perfect fluid spacetime that satisfies equation (1.6) and possesses a Lorentzian Concircular structure. If the Concircular curvature tensor $\c{C}=0$ vanishes, implying that the manifold is Concircularly flat, then the spacetime has constant curvature given by $\frac{\lambda+k\sigma}{3}$.*

In this case, we investigate the curvature properties under the assumption that the spacetime is 4-dimensional ξ-Concircularly flat that is:
Using the definition of the Concircular curvature tensor, we have:

$$\c{C}(\mathcal{U},\mathcal{V})\xi \;=\; R(\mathcal{U},\mathcal{V})\xi-\frac{r}{12}[g(\mathcal{V},\xi)\mathcal{U}-g(\mathcal{U},\xi)\mathcal{V}]. \tag{3.28}$$

If the manifold is ξ-Concircularly flat, then $\c{C}(\mathcal{U},\mathcal{V})\xi=0$, which implies:

$$R(\mathcal{U},\mathcal{V})\xi=\frac{r}{12}[g(\mathcal{V},\xi)\mathcal{U}-g(\mathcal{U},\xi)\mathcal{V}]. \tag{3.29}$$

By applying equations (1.6) and (2.17) in (3.29), the following result is derived:

$$\left[\frac{-[4\lambda+k(\sigma-3p)]+12(\alpha^2-\rho)}{12}\right][\eta(\mathcal{V})\mathcal{U}-\eta(\mathcal{U})\mathcal{V}]=0. \tag{3.30}$$

From (3.30), we get:

$$p=\frac{4\lambda+k\sigma-12(\alpha^2-\rho)}{3k}. \tag{3.31}$$

As a result, we conclude the following:

Theorem 3.3. *Let (M, g) be a general relativistic perfect fluid spacetime satisfying equation (1.7) and admitting a Lorentzian Concircular structure. If the Concircular curvature tensor satisfies $\c{C}(\mathcal{U}, \mathcal{V})\xi = 0$, then the pressure p is equal to $\frac{4\lambda + k\sigma - 12(\alpha^2 - \rho)}{3k}$.*

Definition 3.1. *A second-order tensor ℓ is referred to as a parallel tensor if its covariant derivative vanishes, i.e., $\nabla \ell = 0$, where ∇ represents the covariant derivative operator associated with the metric tensor g.*

Consider ℓ as a symmetric second-order tensor. It is said to be parallel if it satisfies $\nabla \ell = 0$. Employing the Ricci commutation identity, we proceed as follows:

$$\nabla^2_{\mathcal{U},\mathcal{V}} \ell(\mathfrak{Z}, \mathfrak{W}) - \nabla^2_{\mathcal{U},\mathcal{V}} \ell(\mathfrak{W}, \mathfrak{Z}) = 0. \tag{3.32}$$

Accordingly, the resulting expression is:

$$\ell(R(\mathcal{U}, \mathcal{V})\mathfrak{Z}, \mathfrak{W}) + \ell(\mathfrak{Z}, R(\mathcal{U}, \mathcal{V})\mathfrak{W}) = 0. \tag{3.33}$$

For arbitrary vector fields $\mathcal{U}, \mathcal{V}, \mathfrak{Z}, \mathfrak{W}$ on M, by assigning $\mathfrak{Z} = \mathfrak{W} = \xi$ in equation (3.33), and applying equation (3.27) along with the symmetry property of ℓ, we obtain:

$$\frac{2(\lambda + \sigma k)}{3}[\eta(\mathcal{V})\ell(\mathcal{U}, \xi) - \eta(\mathcal{U})\ell(\mathcal{V}, \xi)] = 0. \tag{3.34}$$

From equation(3.34), it follows that either $\lambda = -\sigma k$, which is equivalent to $\sigma = \frac{-\lambda}{k}$ or

$$\eta(\mathcal{V})\ell(\mathcal{U}, \xi) - \eta(\mathcal{U})\ell(\mathcal{V}, \xi) = 0. \tag{3.35}$$

By inserting $\mathcal{U} = \xi$ in (3.35) and on simplification, we obtain:

$$\ell(\mathcal{V}, \xi) = -\eta(\mathcal{V})\ell(\xi, \xi). \tag{3.36}$$

The fact that ℓ is parallel, combined with equation (3.36), leads to the conclusion that $\ell(\xi, \xi)$ is constant:

$$\begin{aligned}&(\nabla_{\mathfrak{U}}\ell)(V, \xi) + \ell(\nabla_{\mathfrak{U}}V, \xi) + \ell(V, \nabla_{\mathfrak{U}}\xi)\\&= -\{[g(\nabla_{\mathfrak{U}}V, \xi) + g(V, \nabla_{\mathfrak{U}}\xi)]\ell(\xi, \xi) + \eta(V)[(\nabla_{\mathfrak{U}}\ell)(\xi, \xi) + 2\ell(\nabla_{\mathfrak{U}}\xi, \xi)]\}.\end{aligned} \tag{3.37}$$

By considering $\nabla \ell = 0$ and by virtue of (3.36) in (3.37), we obtain:

$$\ell(\nabla_{\mathfrak{U}}V, \xi) + \alpha\ell(V, \phi\mathfrak{U}) = -\{[\eta(\nabla_{\mathfrak{U}}V) + \alpha g(V, \phi\mathfrak{U})]\ell(\xi, \xi) + 2\alpha\eta(V)\ell(\phi\mathfrak{U}, \xi)\}. \tag{3.38}$$

By using (3.36) in (3.38) and by virtue of (2.13) and on simplification, we have:

$$\ell(\mathfrak{U}, V) = -g(\mathfrak{U}, V)\ell(\xi, \xi). \tag{3.39}$$

For any arbitrary vector fields $\mathfrak{U}, V$ on M and assuming ℓ is parallel, it follows that $\ell(\xi, \xi)$ remains constant. Therefore, we conclude the following:

Theorem 3.4. *In a perfect fluid spacetime that is Concircularly flat and equipped with a Lorentzian Concircular structure manifolds, the presence of a symmetric parallel tensor of second order implies that either the condition $\lambda = -\sigma k$ is satisfied, or the tensor must be a constant scalar multiple of the metric tensor g.*

If $\lambda + \sigma k \neq 0$ in equation (3.34), then Concircularly flat perfect fluid spacetime endowed with a Lorentzian Concircular structure manifolds that admits a second-order symmetric parallel tensor is a regular spacetime. Accordingly, we present the following corollary:

Corollary 3.1. *In a Concircularly flat, regular perfect fluid spacetime endowed with a Lorentzian Concircular structure, a second-order symmetric parallel tensor possesses the same directional and symmetry characteristics as the metric tensor and is uniformly scaled by a constant factor across the manifold. Hence, it is a constant multiple of the metric tensor* g.

4. Perfect fluid spacetime satisfying $(\xi, \cdot)_R \cdot Ric = 0$ and Lorentzian Concircular structure manifold

This section focuses on the analysis of a perfect fluid spacetime in Lorentzian Concircular structure manifolds that fulfills the curvature condition $(\xi, \cdot)_R \cdot Ric = 0$, which is equivalent to the following [3]:

$$\begin{aligned} ((\xi, U)_R \cdot Ric)(V, Z) &= ((\xi \wedge_R U) \cdot Ric)(V, Z) \\ &= Ric((\xi \wedge_R U)V, Z) + Ric(V, (\xi \wedge_R U)Z), \end{aligned} \tag{4.40}$$

where the curvature operator $(U \wedge_R V)$ is defined by its action on a vector field Z as $(U \wedge_R V)Z = R(U, V)Z$. Utilizing this definition, equation (4.40) leads to:

$$Ric(R(\xi, U)V, Z) + Ric(V, R(\xi, U)Z) = 0. \tag{4.41}$$

Inserting (1.7) in (4.41), we get

$$\begin{aligned} &\left(\frac{2\lambda + k(\sigma - p)}{2}\right)[g(R(\xi, U)V, Z) + g(V, R(\xi, U)Z)] \\ &+(k\sigma + kp)[\eta(R(\xi, U)V)\eta(Z) + \eta(V)\eta(R(\xi, U)Z)] = 0. \end{aligned} \tag{4.42}$$

By substituting equations (2.16) and (2.17) into (4.42), we obtain the following expression:

$$(k\sigma + kp)(\alpha^2 - \rho)[g(U, V)\eta(Z) + g(U, Z)\eta(V) + 2\eta(U)\eta(V)\eta(Z)] = 0. \tag{4.43}$$

By replacing Z with ξ in equation (4.43), we obtain:

$$(\alpha^2 - \rho)(k\sigma + kp)[g(U, V) + \eta(U)\eta(V)] = 0, \tag{4.44}$$

we obtain $\sigma = -p$ and $(\alpha^2 - \rho) \neq 0$. Therefore, we can conclude the following:

Theorem 4.1. *Let* (M, g) *be a perfect fluid spacetime in general relativity that admits a Lorentzian Concircular structure and satisfies equation (1.7). If the curvature condition* $(\xi, \cdot)_R \cdot Ric = 0$ *holds, then the pressure* p *and energy density* σ *satisfy* $\sigma = -p$.

5. Perfect fluid spacetime satisfying $(\xi, \cdot)_{Ric} \cdot R = 0$ and Lorentzian Concircular structure manifold

The current section is concerned with the perfect fluid spacetimes in Lorentzian Concircular structure manifolds that satisfy the curvature condition $(\xi, \cdot)_{Ric} \cdot R = 0$. This condition is equivalent to the following expression:

$$((\xi, U)_{Ric} \cdot R)(V, Z)W = (\xi \wedge_{Ric} U)R(V, Z)W + R((\xi \wedge_{Ric} U)V, Z)W + R(V, (\xi \wedge_{Ric} U)Z)W + R(V, Z)(\xi \wedge_{Ric} U)W, \quad (5.45)$$

where $(U \wedge_{Ric} V)Z = Ric(V, Z)U - Ric(U, Z)V$. Rewriting the preceding relation, we obtain:

$$Ric(U, R(V, Z)W)\xi - Ric(\xi, R(V, Z)W)U + Ric(U, V)R(\xi, Z)W - Ric(\xi, V)R(U, Z)W + Ric(U, Z)R(V, \xi)W - Ric(\xi, Z)R(V, U)W + Ric(U, W)R(V, Z)\xi - Ric(\xi, W)R(V, Z)U = 0. \quad (5.46)$$

Applying the inner product with the vector field ξ in equation (5.46), we obtain:

$$-Ric(U, R(V, Z)W) - Ric(\xi, R(V, Z)W)\eta(U) + Ric(U, V)\eta(R(\xi, Z)W) - Ric(\xi, V)\eta(R(U, Z)W) + Ric(U, Z)\eta(R(V, \xi)W) - Ric(\xi, Z)\eta(R(V, U)W) + Ric(U, W)\eta(R(V, Z)\xi) - Ric(\xi, W)\eta(R(V, Z)U) = 0. \quad (5.47)$$

Inserting (1.7) in (5.47), we get

$$\left(\frac{2\lambda + k(\sigma - p)}{2}\right)[-g(U, R(V, Z)W) - \eta(R(V, Z)W)\eta(U) + g(U, V)\eta(R(\xi, Z)W) - \eta(V)\eta(R(U, Z)W) + g(U, Z)\eta(R(V, \xi)W) - \eta(Z)\eta(R(V, U)W) + g(U, W)\eta(R(V, Z)\xi) - \eta(W)\eta(R(V, Z)U)] + k(\sigma + p)[\eta(U)\eta(V)\eta(R(\xi, Z)W) + \eta(V)\eta(R(U, Z)W) + \eta(U)\eta(Z)\eta(R(V, \xi)W) + \eta(Z)\eta(R(V, U)W) + \eta(U)\eta(W)\eta(R(V, Z)\xi) + \eta(W)\eta(R(V, Z)U)] = 0. \quad (5.48)$$

By using (2.16), (2.17) in (5.48), we arrive at

$$\left(\frac{2\lambda + k(\sigma - p)}{2}\right)[-g(U, R(V, Z)W) + (\alpha^2 - \rho)\{2g(V, W)\eta(U)\eta(Z) - 2g(Z, W)\eta(U)\eta(V) - g(U, V)g(Z, W) + g(U, Z)g(V, W)\}] + k(\sigma + p)(\alpha^2 - \rho)[g(U, Z)\eta(W)\eta(V) - g(U, V)\eta(W)\eta(Z)] = 0. \quad (5.49)$$

By inserting $Z = W = \xi$ in (5.49) and on simplification, we have

$$(\alpha^2 - \rho)[2\lambda - 2kp][g(U, V) + \eta(U)\eta(V)] = 0. \quad (5.50)$$

From the above equation, it follows that $p = \frac{\lambda}{k}$ and $(\alpha^2 - \rho) \neq 0$. Consequently, we present the following result:

Theorem 5.1. *Let (M, g) be a general relativistic perfect fluid spacetime satisfying equation (1.7) and admitting a Lorentzian Concircular structure manifold. If the curvature condition $(\xi, \cdot)_{Ric} \cdot R = 0$ holds, then the pressure satisfies $p = \frac{\lambda}{k}$, provided that $(\alpha^2 - \rho) \neq 0$.*

6. Perfect fluid spacetime satisfying $(\xi, \cdot)_{Ç} \cdot Ric = 0$ and Lorentzian Concircular structure manifold

In this section, we investigate perfect fluid spacetimes admitting a Lorentzian Concircular structure manifold that satisfy the condition $(\xi, \cdot)_{Ç} \cdot Ric = 0$. This condition is equivalent to the following relation:

$$\begin{aligned}((\xi, U)_{Ç} \cdot Ric)(V, Z) &= ((\xi \wedge_{Ç} U) \cdot Ric)(V, Z) \\ &= Ric((\xi \wedge_{Ç} U)V, Z) + Ric(V, (\xi \wedge_{Ç} U)Z), \end{aligned} \tag{6.51}$$

where the operator $(U \wedge_{Ç} V)$ acts on a vector field Z as $(U \wedge_{Ç} V)Z = Ç(U, V)Z$. Reformulating the preceding expression using this definition, we get:

$$Ric(Ç(\xi, U)V, Z) + Ric(V, Ç(\xi, U)Z) = 0, \tag{6.52}$$

Inserting (1.7) in (6.52), we obtain:

$$\begin{aligned}&\left(\frac{2\lambda + k\sigma - kp)}{2}\right)[g(Ç(\xi, U)V, Z) + g(V, Ç(\xi, U)Z)] \\ &+(k\sigma + kp)[\eta(Ç(\xi, U)V)\eta(Z) + \eta(V)\eta(Ç(\xi, U)Z)] = 0. \end{aligned} \tag{6.53}$$

By using (2.16), (2.17), and (2.19) in (6.53), we get:

$$\begin{aligned}&(k\sigma + kp)\left\{\frac{12(\alpha^2 - \rho) - [4\lambda + k(\sigma - 3p)]}{12}\right\}[g(U, V)\eta(Z) + g(U, Z)\eta(V) \\ &+2\eta(U)\eta(V)\eta(Z)] = 0. \end{aligned} \tag{6.54}$$

Replacing Z by ξ in equation (6.54), gives the following result:

$$(k\sigma + kp)\left\{\frac{12(\alpha^2 - \rho) - [4\lambda + k(\sigma - 3p)]}{12}\right\}[g(U, V) + \eta(U)\eta(V)] = 0. \tag{6.55}$$

Therefore, we arrive at two possibilities: either $\sigma = -p$ or $p = \frac{4\lambda + k\sigma - 12(\alpha^2 - \rho)}{3k}$. As a consequence, we establish the following result:

Theorem 6.1. *Let (M, g) be a general relativistic perfect fluid spacetime endowed with a Lorentzian Concircular structure manifolds satisfying equation (1.7). If the condition $(\xi, \cdot)_{Ç} \cdot Ric = 0$ holds, then either $\sigma = -p$ or $p = \frac{4\lambda + k\sigma - 12(\alpha^2 - \rho)}{3k}$.*

7. Perfect fluid spacetime satisfying $(\xi, \cdot)_{Ric} \cdot Ç = 0$ and Lorentzian Concircular structure manifolds

In this section, we examine perfect fluid spacetimes in Lorentzian Concircular structure manifolds satisfying the condition $(\xi, \cdot)_{Ric} \cdot Ç = 0$. This condition is equivalent to the following:

$$\begin{aligned}((\xi, U)_{Ric} \cdot Ç)(V, Z)W &= (\xi \wedge_{Ric} U)Ç(V, Z)W + Ç((\xi \wedge_{Ric} U)V, Z)W \\ &\quad +Ç(V, (\xi \wedge_{Ric} U)Z)W + Ç(V, Z)(\xi \wedge_{Ric} U)W \end{aligned} \tag{7.56}$$

where $(X \wedge_{Ric} Y)Z = Ric(Y, Z)X - Ric(X, Z)Y$.

The preceding equation can be expressed as:

$$\begin{aligned}&Ric(U, Ç(V, Z)W)\xi - Ric(\xi, Ç(V, Z)W)U + Ric(U, V)Ç(\xi, Z)W \\ &-Ric(\xi, V)Ç(U, Z)W + Ric(U, Z)Ç(V, \xi)W - Ric(\xi, Z)Ç(V, U)W \\ &+Ric(U, W)Ç(V, Z)\xi - Ric(\xi, W)Ç(V, Z)U = 0. \end{aligned} \tag{7.57}$$

Taking the inner product of equation (7.57) with ξ, we obtain:

$$\begin{aligned}
&-Ric(U,Ç(V,Z)W) - Ric(\xi,Ç(V,Z)W)\eta(U) + Ric(U,V)\eta(Ç(\xi,Z)W) \\
&-Ric(\xi,V)\eta(Ç(U,Z)W) + Ric(U,Z)\eta(Ç(V,\xi)W) - Ric(\xi,Z)\eta(Ç(V,U)W) \\
&+Ric(U,W)\eta(Ç(V,Z)\xi) - Ric(\xi,W)\eta(Ç(V,Z)U) = 0. \qquad (7.58)
\end{aligned}$$

Inserting equation (1.7) into (7.58) gives:

$$\begin{aligned}
&\left(\frac{2\lambda + k(\sigma - p)}{2}\right)[-g(U,Ç(V,Z)W) - \eta(Ç(V,Z)W)\eta(U) + g(U,V)\eta(Ç(\xi,Z)W) \\
&-\eta(V)\eta(Ç(U,Z)W) + g(U,Z)\eta(Ç(V,\xi)W) - \eta(Z)\eta(Ç(V,U)W) + g(U,W)\eta(Ç(V,Z)\xi) \\
&-\eta(W)\eta(Ç(V,Z)U)] + k(\sigma+p)[\eta(U)\eta(V)\eta(Ç(\xi,Z)W) + \eta(V)\eta(Ç(U,Z)W) \\
&+\eta(U)\eta(Z)\eta(Ç(V,\xi)W) + \eta(Z)\eta(Ç(V,U)W) + \eta(U)\eta(W)\eta(Ç(V,Z)\xi) \\
&+\eta(W)\eta(Ç(V,Z)U)] = 0. \qquad (7.59)
\end{aligned}$$

By substituting equations (2.16), (2.17) and (2.19) into (7.59), we obtain:

$$\begin{aligned}
&-\left(\frac{2\lambda + k(\sigma - p)}{2}\right) g(U,Ç(V,Z)W) \\
&+\left(\frac{2\lambda + k(\sigma - p)}{2}\right)\left\{\frac{12(\alpha^2-\rho) - [4\lambda + k(\sigma - 3p)]}{12}\right\}[2g(V,W)\eta(U)\eta(Z) \\
&-2g(Z,W)\eta(U)\eta(V) - g(U,V)g(Z,W) + g(U,Z)g(V,W)] \\
&+(k\sigma + kp)\left\{\frac{12(\alpha^2-\rho) - [4\lambda + k(\sigma - 3p)]}{12}\right\}[g(U,Z)\eta(W)\eta(V) \\
&-g(U,V)\eta(W)\eta(Z)] = 0. \qquad (7.60)
\end{aligned}$$

By substituting $Z = W = \xi$ in (7.60) and on simplification, we have

$$[2\lambda - 2kp]\left\{\frac{12(\alpha^2-\rho) - [4\lambda + k(\sigma - 3p)]}{12}\right\}[g(U,V) + \eta(U)\eta(V)] = 0. \qquad (7.61)$$

Therefore, either $p = \frac{\lambda}{k}$ or $p = \frac{4\lambda + k\sigma - 12(\alpha^2-\rho)}{3k}$. Hence, we present the following:

Theorem 7.1. *Let (M,g) represent a general relativistic perfect fluid spacetime satisfying (1.7), endowed with a Lorentzian Concircular structure manifold. If the condition $(\xi,\cdot)_{Ric}\cdot Ç = 0$ holds, then either $p = \frac{\lambda}{k}$ or $p = \frac{4\lambda + k\sigma - 12(\alpha^2-\rho)}{3k}$.*

8. Ricci solitons within perfect fluid spacetimes possessing a Lorentzian Concircular structure manifold

This section explores Ricci solitons within the framework of a perfect fluid spacetime endowed with a Lorentzian Concircular structure.

Consider the Ricci solitons equation:

$$\mathfrak{L}_\xi g + 2Ric + 2ag = 0. \qquad (8.62)$$

Let g be a pseudo-Riemannian metric, Ric the Ricci tensor, ξ a vector field, and a a real constant. A triple (g,ξ,a) that satisfies equation (8.62) is called a Ricci soliton on M. The soliton is said to be shrinking, steady, or expanding depending on whether a is negative, zero, or positive, respectively.

From the Lie derivative, it follows that:

$$(\mathfrak{L}_\xi g)(U,V) = g(\nabla_U \xi, V) + g(U, \nabla_V \xi). \tag{8.63}$$

By using (8.63) in (8.62), we obtain

$$Ric(U,V) = -ag(U,V) - \frac{1}{2}[g(\nabla_U \xi, V) + g(U, \nabla_V \xi)]. \tag{8.64}$$

By contracting both sides of (8.64), we arrive at:

$$r = -a \cdot dim(M) - div(\xi), \tag{8.65}$$

where $dim(M)$ denotes the dimension of the manifold M, and $div(\xi)$ is the divergence of the vector field ξ.

Let (M, g) be a spacetime representing a general relativistic perfect fluid with a Lorentzian Concircular structure, and let (g, ξ, a) define a Ricci soliton on M. From equations (1.7) and (8.64), we obtain:

$$\left[a + \lambda + \frac{k(\sigma - p)}{2}\right] g(U,V) + k(\sigma + p)\eta(U)\eta(V)$$
$$+\frac{1}{2}[g(\nabla_U \xi, V) + g(U, \nabla_V \xi)] = 0. \tag{8.66}$$

By an orthonormal frame field e_i substituting $U = V = e_i$ in (8.66) and by incorporating the fluid parameters and the divergence of the vector field ξ, the soliton constant a takes the form:

$$a = -\lambda - \frac{k(\sigma - 3p)}{4} - \frac{div(\xi)}{4}. \tag{8.67}$$

Therefore, we present the following statement:

Theorem 8.1. *A Ricci soliton (g, ξ, a), where $a = -\lambda - \frac{k(\sigma-3p)}{4} - \frac{div(\xi)}{4}$, is classified as steady when $p = \frac{4\lambda}{3k} + \frac{\sigma}{3} + \frac{div\xi}{3k}$; it is expanding if $p > \frac{4\lambda}{3k} + \frac{\sigma}{3} + \frac{div\xi}{3k}$, and shrinking if $p < \frac{4\lambda}{3k} + \frac{\sigma}{3} + \frac{div\xi}{3k}$.*

9. Conclusion

This investigation highlights the fundamental role of Ricci solitons in perfect fluid spacetimes characterized by a Lorentzian Concircular structure and a Concircular curvature tensor. Through a detailed analysis of the relationship between the soliton potential, energy-momentum tensor, and the geometric properties of the manifold, we have identified essential conditions for the existence and nature of Ricci solitons. The results impose meaningful constraints on the curvature and topology of the spacetime, offering new perspectives on the geometric behavior of relativistic fluids. These contributions not only strengthen the theoretical understanding of Ricci solitons but also expand their relevance within the framework of general relativity.

Looking ahead, this study can be extended by examining Ricci solitons in more generalized geometric environments, such as manifolds with torsion or non-metric connections. Further research could also address the stability, evolution, and physical significance of Ricci solitons in dynamic spacetimes. Incorporating numerical approaches and analyzing specific solutions to Einsteins field equations may offer practical insights, enhancing the applicability of the theoretical findings to problems in cosmology and gravitational physics.

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TAUBERIAN THEOREMS IN NEUTROSOPHIC N-NORMED LINEAR SPACES VIA STATISTICAL CESARO SUMMABILITY

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Abstract. In this study, the connection between statistical Cesaro summability as well as sequence of statistical convergence within neutrosophic n-normed linear space ($\mathfrak{NnNLS}$) is investigated. Although Cesaro summability along with its statistical variant within classical normed spaces, fuzzy, intuitionistic fuzzy, and neutrosophic are covered in the literature, this study is notable for both its methodology and its thorough approach, which covers a wider range among spaces in addition explains the process beginning with the statistical Cesaro summability concepts towards statistical convergence. The Tauberian theorems in $\mathfrak{NnNLS}$ will follow from these findings.

Keywords: Neutrosophic n-normed linear space; Cauchy sequence; Tauberian theorem; Cesaro summability

2020 Mathematics Subject Classification: 40G05; 03E72; 40E05;

1. Introduction

In 1965, Zadeh[23] initially presented the theory among fuzzy sets. He developed this theory to deal with the idea of partial truth, in which truth values fall somewhere between being entirely true and being entirely untrue. This strategy was especially helpful for handling ambiguous or imprecise data, which conventional binary logic was ill-equipped to handle. Atanassov[2], [3] introduced intuitionistic fuzzy set(IFS) theory in 1986. This theory adds a degree among membership as well as a degree among non-membership to the usual fuzzy set theory. Florentin Smarandache[18][19] introduced the concept of neutrosophic sets as to extend of the IFS. The degree of indeterminacy and the neutrosophic set were established as distinct components in his 1995 manuscript, which was published in 1998. Compared with traditional fuzzy sets, this enables a representation of imprecision and uncertainty, which makes it especially helpful in situations where judgments must take into account ambiguous or incomplete data. Gunawan and Mashadi[9], Kim and Cho[13], Malceski[14], and other researchers have looked at n-normed linear spaces. Vijayabalaji and Narayanan[21] defined

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a fuzzy n-normed linear space. Saadati and Park[17] introduced the concept of intuitionistic fuzzy normed space. Many more authors have conducted research on generalised difference sequence spaces. Jeyaraman et al.[10],[11] established the concepts of Logarithmic summability and Cesaro summability in neutrosophic n- normed linear spaces. Praveena et al.[16] generalized the concept of Cesaro summability method in Neutrosophic Normed spaces using the Tauberian conditions.
Our aim in this research is to introduce the idea of statistical summability theory in a neutrosophic n- normed linear spaces $\mathfrak{NnNLS}$. In the context of $\mathfrak{NnN}$, this work will assist us in establishing Tauberian conditions which enable the shift beginning with the statistical Cesaro summability towards statistical convergence among sequences. In order to accomplish this, we provide the ideas among Cesaro as well as statistical Cesaro summability. Future research into related Tauberian theorems in a $\mathfrak{NnNLS}$ environment is made possible by these ideas.

2. Preliminaries

This phase contains some of the basic definitions in addition to the notation required for the next section.

Definition 2.1. [10] *The following axioms define a continuous t-norm as a binary operation* $*$*:*$[0,1]\times[0,1]\to[0,1]$
(i) $*$ *is continuous, commutative and associative,*
(ii) $\mathfrak{p}*1=\mathfrak{p}$ *forevery* $\mathfrak{p}\in[0,1]$*,*
(iii) If $\mathfrak{p}\leq\mathfrak{r}$ *and* $\mathfrak{q}\leq\mathfrak{s}$ *then* $\mathfrak{p}*\mathfrak{q}\leq\mathfrak{r}*\mathfrak{s}$*, for each* $\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s}\in[0,1]$

Definition 2.2. [10] *The seven-tuple* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$ *is recognized as a* $\mathfrak{NnNLS}$*, where* $\mathfrak{U}$ *represents the space of vectors among dimensions that vary* $d\geq n$ *on the domain* $\mathbb{R}$*,* $*$ *indicates a continuous t-norm,* $\diamond$*, and* $\circ$ *represent a continuous t-conorms, and* $\hat{\mu}$*,* $\breve{\nu}$*, and* $\tilde{\omega}$ *are fuzzy sets described on* $\mathfrak{U}^n\times(0,\infty)$*. In this context,* $\hat{\mu}$ *denotes the membership degree,* $\breve{\nu}$ *denotes the non-membership degree and* $\tilde{\omega}$ *indicates the degree of indeterminacy for elements* $(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda})\in\mathfrak{U}^n\times(0,\infty)$*. The following requirements are met for each* $(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\mathrm{n})\in\mathfrak{U}^n$ *and* $s,\hat{\lambda}>0$*:*
(i) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda})+\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda})+\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda})\leq 3$*;*
(ii) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda})=1$*,* $\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda})=0$ *and* $\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda})=0$ *for every positive* $\hat{\lambda}$ *iff* $\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n$ *are linearly dependent;*
(iii) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda})$*,* $\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda})$ *and* $\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda})$ *are not influenced by any particular arrangement of* $\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n$*;*
(iv) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{c}\mathfrak{f}_n,\hat{\lambda})=\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\frac{\hat{\lambda}}{|\mathfrak{c}|}\right)$*,* $\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{c}\mathfrak{f}_n,\hat{\lambda})=\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\frac{\hat{\lambda}}{|\mathfrak{c}|}\right)$ *and* $\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{c}\mathfrak{f}_n,\hat{\lambda})=\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\frac{\hat{\lambda}}{|\mathfrak{c}|}\right)$ *if* $\mathfrak{c}\neq 0,\mathfrak{c}\in F$*;*
(v) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,s)*\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}'_n,\hat{\lambda})\leq\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n+\mathfrak{f}'_n,s+\hat{\lambda})$*;*
(vi) $\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,s)\diamond\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}'_n,\hat{\lambda})\geq\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n+\mathfrak{f}'_n,s+\hat{\lambda})$*;*
(vii) $\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,s)\circ\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}'_n,\hat{\lambda})\geq\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n+\mathfrak{f}'_n,s+\hat{\lambda})$*;*
(vii) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda}):(0,\infty)\to[0,1]$*,* $\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda}):(0,\infty)\to[0,1]$ *and* $\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_n,\hat{\lambda}):(0,\infty)\to[0,1]$ *are always continuous in* $\hat{\lambda}$*;*

(viii) $\lim_{\hat{\lambda}\to\infty} \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 1$ *and* $\lim_{\hat{\lambda}\to 0} \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 0$ *;*
(ix) $\lim_{\hat{\lambda}\to\infty} \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 0$ *and* $\lim_{\hat{\lambda}\to 0} \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 1$.
(ix) $\lim_{\hat{\lambda}\to\infty} \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 0$ *and* $\lim_{\hat{\lambda}\to 0} \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 1$.

Definition 2.3. [22] *Let* $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$ *be* $\mathfrak{NnNLS}$.
(i) The $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_k$ *a sequence in* $\mathfrak{U}$ *is considered to converge with* $\tilde{\mathfrak{L}} \in \mathfrak{U}$ *under* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$*, if* $\forall\ \breve{\varrho} \in (0,1), \hat{\lambda} > 0$*, and also* $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1} \in \mathfrak{U}$*,* $\exists$ *a natural number* k_0 *in a way*
$\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{h}}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) > 1 - \breve{\varrho}$*,* $\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{h}}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) < \breve{\varrho}$ *and*
$\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{h}}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) < \breve{\varrho}\ \forall\ k \geq k_0$.
In order to indicate this convergence, $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n - \lim \hat{\mathfrak{h}} = \tilde{\mathfrak{L}}$ *or* $\hat{\mathfrak{h}}_k \overset{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}{\to} \tilde{\mathfrak{L}}$ *as* $k \to \infty$.
(ii) The $\hat{\mathfrak{h}} = \hat{\mathfrak{h}}_k$ *a sequence within* $\mathfrak{U}$ *is defined to be Cauchy in relation to* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$*, if* $\forall\ \breve{\varrho} \in (0,1), \hat{\lambda} > 0$ *and also* $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1} \in \mathfrak{U}$*,* $\exists$ *a natural number* k_0 *in a way that*
$\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{h}}_k - \hat{\mathfrak{h}}_m, \hat{\lambda}) > 1 - \breve{\varrho}$*,* $\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{h}}_k - \hat{\mathfrak{h}}_m, \hat{\lambda}) < \breve{\varrho}$ *and*
$\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{h}}_k - \hat{\mathfrak{h}}_m, \hat{\lambda}) < \breve{\varrho}$ *for any* $k, m \geq k_0$.
(iii) If all Cauchy sequences in $\mathfrak{U}$ *converge, then a* $\mathfrak{NnNLS}$ $\mathfrak{U}$ *is complete with regard to* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$.

Definition 2.4. *Let* $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$ *represent a* $\mathfrak{NnNLS}$ *as well as* $\mathfrak{V}$ *indicate any subset of* $\mathfrak{U}$*. The set* $\mathfrak{V}$ *is considered bound if* $\exists\ \breve{\varrho} > 0$ *and* $\hat{\lambda}_0 > 0$ *are such that*
$\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}_0) > 1 - \breve{\varrho}$*,* $\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}_0) < \breve{\varrho}$ *and* $\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}_0) < \breve{\varrho}$
for all $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n \in \mathfrak{V}$*. We tell that the set* $\mathfrak{V}$ *is p-bounded if* $\lim_{\hat{\lambda}\to\infty} \Phi_{\mathfrak{V}}(\hat{\lambda}) = 1$ *and* $\lim_{\hat{\lambda}\to\infty} \Psi_{\mathfrak{V}}(\hat{\lambda}) = 0$ *and* $\lim_{\hat{\lambda}\to\infty} \varphi_{\mathfrak{V}}(\hat{\lambda}) = 0$ *where*
$\Phi_{\mathfrak{V}}(r) = \inf\{\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\mathfrak{r}}) : \mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n \in \mathfrak{V}\}$*;*
$\Psi_{\mathfrak{V}}(r) = \sup\{\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\mathfrak{r}}) : \mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n \in \mathfrak{V}\}$
$\varphi_{\mathfrak{V}}(r) = \sup\{\breve{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\mathfrak{r}}) : \mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n \in \mathfrak{V}\}$.

Definition 2.5. *Let* $\mathfrak{V}$ *be subset of* $\mathbb{N}$. $\delta(\mathfrak{V}) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in \mathfrak{V}\}|$*, where* $|\mathfrak{V}|$ *indicates the cardinality of the set* $\mathfrak{V}$ *and determines the natural density of* $\mathfrak{V}$ *whenever the limit exists.*

Definition 2.6. *A sequence* $\mathfrak{f} = \{\mathfrak{f}_k\}$ *among numbers is assumed statistically(*$\mathfrak{st}$*)-convergent to* $\tilde{\mathfrak{L}}$*, when* $\forall\ \breve{\varrho} > 0$, $\breve{\delta}(\{k \in \mathbb{N} : |\mathfrak{f}_k - \tilde{\mathfrak{L}}| \geq \breve{\varrho}\}) = 0$*. That a case, we represent* $\mathfrak{st} - \lim \mathfrak{f} = \tilde{\mathfrak{L}}$.

Definition 2.7. *([20]). Let* $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$ *be* $\mathfrak{NnNLS}$*. The* $\mathfrak{f} = \{\mathfrak{f}_k\}$ *a sequence within* $\mathfrak{U}$ *is assumed* $\mathfrak{st}$*- convergent towards* $\tilde{\mathfrak{L}} \in \mathfrak{U}$ *in relation with* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$*, when for all* $\breve{\varrho} \in (0,1), \hat{\lambda} > 0$ *along with* $\mathrm{h}_1, \mathrm{h}_2, \ldots, \mathrm{h}_{n-1} \in \mathfrak{U}$*,*
$\breve{\delta}(\{k \in \mathbb{N} : \hat{\mu}(\mathrm{h}_1, \mathrm{h}_2, \ldots, \mathrm{h}_{n-1}, \mathfrak{f}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) \leq 1 - \breve{\varrho}$*,*
$\breve{\nu}(\mathrm{h}_1, \mathrm{h}_2, \ldots, \mathrm{h}_{n-1}, \mathfrak{f}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) \geq \breve{\varrho}, \tilde{\omega}(\mathrm{h}_1, \mathrm{h}_2, \ldots, \mathrm{h}_{n-1}, \mathfrak{f}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) \geq \breve{\varrho}\}) = 0$*. This is represented by* $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim \mathfrak{f} = \tilde{\mathfrak{L}}$.

3. Statistical Cesaro summability in $\mathfrak{NnNLS}$

We begin by introducing the concept of Cesaro summability.

Definition 3.1. *([7]). Let* $\{\mathrm{a}_n\}$ *indicate a sequence within* $\mathfrak{NnNLS}$ $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$*. The equation* $\breve{\mathcal{X}}_n = \frac{1}{n+1} \sum_{k=0}^{n} \mathrm{a}_k$ *describes the arithmetic means (AM)* $\breve{\mathcal{X}}_n$ *among* a_n*.* $\{\mathrm{a}_n\}$ *is* *referred to be Cesaro summable towards* $\mathrm{a} \in \mathfrak{U}$ *when* $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n - \lim_{m\to\infty} \breve{\mathcal{X}}_m = \mathrm{a}$. *Further,* $\{\mathrm{a}_n\}$ *is indicated as a* $\mathfrak{st}$ *Cesaro summable towards* $\mathrm{a} \in \mathfrak{U}$ *when* $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{m\to\infty} \breve{\mathcal{X}}_m = \mathrm{a}$.

In a $\mathfrak{NnNLS}$ under p-boundedness of sequence, the $\mathfrak{st}$ Cesaro summability method is regular, as demonstrate by the following theorem.

Theorem 3.1. *Let* $\{\mathrm{a}_n\}$ *indicate a* p*-bounded sequence within a* $\mathfrak{NnNLS}$ $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$. *If* $\{\mathrm{a}_n\}$ *converges statistically to* $\mathrm{a} \in \mathfrak{U}$, *then* $\{\mathrm{a}_n\}$ *serves as a* $\mathfrak{st}$ *Cesaro summable to* $\mathfrak{U}$ *in relation to* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$.

Proof. Let $\{\mathrm{a}_n\}$ $\mathfrak{st}$ converges towards $\mathrm{a} \in \mathfrak{U}$ and also assume that it is p-bounded. Put $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1} \in \mathfrak{U}$. If $\breve{\varrho} > 0$, then there is $\mathrm{T}, \mathrm{T}' > 0$ which means
$\inf_{n\in\mathbb{N}} \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \hat{\mathfrak{t}}) > 1 - \breve{\varrho}$, $\sup_{n\in\mathbb{N}} \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \hat{\mathfrak{t}}) < \breve{\varrho}$, and
$\sup_{n\in\mathbb{N}} \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \hat{\mathfrak{t}}) < \breve{\varrho}$, for every $\hat{\mathfrak{t}} > 2\mathrm{T}$.
Therefore, $\inf_{n\in\mathbb{N}} \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{t}}}{2}\right) > 1 - \breve{\varrho}$, $\sup_{n\in\mathbb{N}} \breve{\nu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{t}}}{2}\right) < \breve{\varrho}$ and $\sup_{n\in\mathbb{N}} \tilde{\omega}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{t}}}{2}\right) < \breve{\varrho}$ for every $\hat{\mathfrak{t}} > 2\mathrm{T}'$.
Thus, the following inequalities are implied:

$$\inf_{n\in\mathbb{N}} \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{t}})$$
$$\geq \min\left\{\inf_{n\in\mathbb{N}} \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \frac{\hat{\mathfrak{t}}}{2}\right), \inf_{n\in\mathbb{N}} \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{t}}}{2}\right)\right\} > 1 - \breve{\varrho},$$
$$\sup_{n\in\mathbb{N}} \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{t}})$$
$$\leq \max\left\{\sup_{n\in\mathbb{N}} \breve{\nu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \frac{\hat{\mathfrak{t}}}{2}\right), \sup_{n\in\mathbb{N}} \breve{\nu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{t}}}{2}\right)\right\} < \breve{\varrho}$$

and

$$\sup_{n\in\mathbb{N}} \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{t}})$$
$$\leq \max\left\{\sup_{n\in\mathbb{N}} \tilde{\omega}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \frac{\hat{\mathfrak{t}}}{2}\right), \sup_{n\in\mathbb{N}} \tilde{\omega}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{t}}}{2}\right)\right\} < \breve{\varrho}$$

$\forall\ \mathrm{T} > \min\{2\mathrm{T}, 2\mathrm{T}'\}$. Since a_n which is $\mathfrak{st}$- convergent towards $\mathfrak{U}$, we get that
$\breve{\delta}(N_{\hat{\mu}}(\breve{\varrho}, \hat{\mathfrak{t}})) = \breve{\delta}(N_{\breve{\nu}}(\breve{\varrho}, \hat{\mathfrak{t}})) = \breve{\delta}(N_{\tilde{\omega}}(\breve{\varrho}, \hat{\mathfrak{t}})) = 0$ for any $\hat{\mathfrak{t}} > 0$,
where
$N_{\hat{\mu}}(\breve{\varrho}, \hat{\mathfrak{t}}) = \{n \in \mathbb{N} : \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{t}}) \leq 1 - \breve{\varrho}\}$,
$N_{\breve{\nu}}(\breve{\varrho}, \hat{\mathfrak{t}}) = \{n \in \mathbb{N} : \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{t}}) \geq \breve{\varrho}\}$ and
$N_{\tilde{\omega}}(\breve{\varrho}, \hat{\mathfrak{t}}) = \{n \in \mathbb{N} : \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{t}}) \geq \breve{\varrho}\}$.
Describe the sets
$\mathfrak{D} = \{k \in \mathbb{N} : k \in N_{\hat{\mu}}(\breve{\varrho}, \hat{\mathfrak{t}})\}, \mathfrak{D}' = \{k \in \mathbb{N} : k \in N^c_{\hat{\mu}}(\breve{\varrho}, \hat{\mathfrak{t}})\}$, and
$\mathfrak{E} = \{k \in \mathbb{N} : k \in N_{\breve{\nu}}(\tilde{\varrho}, \hat{\mathfrak{t}})\}, \mathfrak{E}' = \{k \in \mathbb{N} : k \in N^c_{\breve{\nu}}(\breve{\varrho}, \hat{\mathfrak{t}})\}$, and
$\mathfrak{F} = \{k \in \mathbb{N} : k \in N_{\tilde{\omega}}(\tilde{\varrho}, \hat{\mathfrak{t}})\}, \mathfrak{F}' = \{k \in \mathbb{N} : k \in N^c_{\tilde{\omega}}(\breve{\varrho}, \hat{\mathfrak{t}})\}$
which means $|\mathfrak{D}| + |\mathfrak{E}| + |\mathfrak{F}| = n + 1 = |\mathfrak{D}'| + |\mathfrak{E}'| + |\mathfrak{F}'|$, in which $|\cdot|$ indicates the cardinality among a set.
Therefore, $\mathfrak{D} \cap \mathfrak{E} \cap \mathfrak{F} = \phi = \mathfrak{D}' \cap \mathfrak{E}' \cap \mathfrak{F}'$, we can determine. We determine that there is a number $n_0 \in \mathbb{N}$ which corresponds with the information above,

$$\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{t}})$$
$$= \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \frac{1}{n+1}\sum_{k=0}^{n}(\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{t}}\right)$$

$$= \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \sum_{k\in\mathbb{N}_{\hat{\mu}}}(\mathrm{a}_k - \mathrm{a}) + \sum_{k\in\mathbb{N}^c_{\hat{\mu}}}(\mathrm{a}_k - \mathrm{a}), (n+1)\hat{\mathfrak{r}}\right)$$
$$\geq \min\left\{\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \sum_{k\in\mathbb{N}_{\hat{\mu}}}(\mathrm{a}_k - \mathrm{a}), |\mathfrak{D}|\hat{\mathfrak{r}}\right), \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \sum_{k\in\mathbb{N}^c_{\hat{\mu}}}(\mathrm{a}_k - \mathrm{a}), |\mathfrak{D}'|\hat{\mathfrak{r}}\right)\right\}$$
$$\geq \min\left\{\min_{k\in\mathbb{N}_{\hat{\mu}}}\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}\right), \min_{k\in\mathbb{N}^c_{\hat{\mu}}}\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}\right)\right\}$$
$$\geq \min\left\{\inf_{k\in\mathbb{N}_{\hat{\mu}}}\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}\right), \min_{k\in\mathbb{N}^c_{\hat{\mu}}}\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}\right)\right\}$$
$$\geq \min\{1-\breve{\varrho}, 1-\breve{\varrho}\}$$
$$= 1-\breve{\varrho}$$

and also

$$\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}})$$
$$\leq \max\left\{\max_{k\in\mathbb{N}_{\breve{\nu}}}\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}), \max_{k\in\mathbb{N}^c_{\breve{\nu}}}\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}})\right\}$$
$$\leq \max\left\{\sup_{k\in\mathbb{N}_{\breve{\nu}}}\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}), \max_{k\in\mathbb{N}^c_{\breve{\nu}}}\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}})\right\}$$
$$\leq \breve{\varrho}$$

and

$$\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}})$$
$$\leq \max\left\{\max_{k\in\mathbb{N}_{\tilde{\omega}}}\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}), \max_{k\in\mathbb{N}^c_{\tilde{\omega}}}\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}})\right\}$$
$$\leq \max\left\{\sup_{k\in\mathbb{N}_{\tilde{\omega}}}\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}), \max_{k\in\mathbb{N}^c_{\tilde{\omega}}}\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}})\right\}$$
$$\leq \breve{\varrho}$$

for each $\hat{\mathfrak{r}} > \min\{2\mathrm{T}, 2\mathrm{T}'\} > 0$ along with $n \geq n_0$. It implies that the set is as follows:

$$\mathfrak{G} = \left\{\begin{array}{c} n \in \mathbb{N} : \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}}) \leq 1-\breve{\varrho} \text{ or} \\ \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}}) \geq \breve{\varrho}, \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}}) \geq \breve{\varrho} \end{array}\right\}$$

containing, at most, a finite number of terms. The sequence a_n is $\mathfrak{st}$-Cesaro summable towards $\mathfrak{U}$ in relation to $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$ since a finite subset among natural numbers contains zero density, as observed by the observation that $\breve{\delta}(\mathfrak{G}) = 0$. □

We demonstrate in the following example that the converse among Theorem (3.1) does not necessarily have to be true.

Example 3.1. *Let* $\mathrm{b}_{\mathfrak{k}} = \begin{cases} 1+(-1)^{\mathfrak{k}}+\mathfrak{k}^2, & if \mathfrak{k}=m^2 \\ 1+(-1)^{\mathfrak{k}}-(\mathfrak{k}-1)^2, & if \mathfrak{k}=m^2+1 \\ 1+(-1)^{\mathfrak{k}}, & otherwise, \end{cases}$ *, for* $m \in \mathbb{N}$.
be in $\mathfrak{NnNLS}(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$. *At* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$, *the sequence* $(\mathrm{b}_{\mathfrak{k}})$ *is neither convergent nor* $\mathfrak{st}$*-convergent. Furthermore, it is also not Cesaro summable.*
To reach a limit, let's use $\mathfrak{st}$*-Cesaro summability. Cesaro means* $(\mathrm{a}_{\mathfrak{k}})$ *of sequence* $(\mathrm{b}_{\mathfrak{k}})$ *is*

$$\mathrm{a}_{\mathfrak{k}} = \begin{cases} 1+\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j+\mathfrak{k}, & if \mathfrak{k}=m^2 \\ 1+\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j, & otherwise. \end{cases}$$

Sequence $(\mathrm{a}_{\mathfrak{k}})$ *is* $\mathfrak{st}$*-convergent to 1 since for each* $\hat{\mathfrak{r}}>0$, *we have*
$\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim\hat{\mu}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}})=1$, $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim\breve{\nu}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}})=0$ *and* $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim\tilde{\omega}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}})=0$
where

$$\hat{\mu}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}}) = \begin{cases} \frac{\hat{\mathfrak{r}}}{\hat{\mathfrak{r}}+|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j+\mathfrak{k}|}, \mathfrak{k}=m^2 \\ \frac{\hat{\mathfrak{r}}}{\hat{\mathfrak{r}}+|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j|}, otherwise \end{cases}$$

$$\breve{\nu}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}}) = \begin{cases} \frac{|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j|}{\hat{\mathfrak{r}}+|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j+\mathfrak{k}|}, \mathfrak{k}=m^2 \\ \frac{|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j|}{\hat{\mathfrak{r}}+|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j|}, otherwise \end{cases}$$

$$\tilde{\omega}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}}) = \begin{cases} \frac{|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j+\mathfrak{k}|}{\hat{\mathfrak{r}}}, \mathfrak{k}=m^2 \\ \frac{|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j|}{\hat{\mathfrak{r}}}, otherwise \end{cases}$$

Hence, sequence $(\mathrm{b}_{\mathfrak{k}})$ *is* $\mathfrak{st}$*-Cesaro summable to 1 in* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$

4. Related Studies Lead to the Tauberian Theorems

The following lemma establishes homogeneity and additivity among the limit of statistical within a $\mathfrak{NnNLS}$.

Lemma 4.1. *Let* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$ *be a* $\mathfrak{NnNLS}$ *and* $\mathrm{u}=\{\mathrm{u}_k\}, \mathrm{v}=\{\mathrm{v}_k\}$ *be sequences in* $\mathfrak{U}$. *After that, the given are true:*
(i) When the limit of $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$*-statistical among* u *indicate* $\breve{\xi}$, *together with the* $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$*-*$\mathfrak{st}$*-limit among* v *is* ρ, *after that the limit of* $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$*-statistical among the sum* $(\mathrm{u}+\mathrm{v})$ *represent* $\breve{\xi}+\dot{\rho}$.
(ii) When the limit of $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$*-statistical among* u *is* $\breve{\xi}$, *along with* α *represent any real number, after that the limit of* $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$*-statistical among* $\alpha\mathrm{u}$ *is* $\alpha\breve{\xi}$.

Theorem 4.1. *Let* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$ *be a* $\mathfrak{NnNLS}$ *along with* $\{\mathrm{a}_n\}$ *denote a sequence within* $\mathfrak{U}$. *When* $\{\mathrm{a}_n\}$ *is a* $\mathfrak{st}$*-Cesaro summable towards* a *in relation to* $\mathfrak{NnN}$ $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$, *after that* $\breve{\mathcal{X}}_{\mathfrak{y}_n}$ *which is* $\mathfrak{st}$*-convergent towards* a *for every* $\mathfrak{y}>0$, *That is* $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim_{n\to\infty}\breve{\mathcal{X}}_{\mathfrak{y}_n}=\mathrm{a}$, *in which* $\mathfrak{y}_n=[\mathfrak{y}n]$.

Proof. Consider $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim_{n\to\infty}\breve{\mathcal{X}}_n=\mathrm{a}$. After that, for a sufficiently large N, follwed $\breve{\varrho}>0$ together with put $\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1}\in\mathfrak{U}$, the given sets are described:

$$\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})=\{k\le\mathfrak{y}_N:\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_k-\mathrm{a},\hat{\mathfrak{r}})\le 1-\breve{\varrho}\},$$
$$\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})=\{k\le\mathfrak{y}_N:\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_k-\mathrm{a},\hat{\mathfrak{r}})\ge\breve{\varrho}\},$$
$$\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})=\{k\le\mathfrak{y}_N:\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_k-\mathrm{a},\hat{\mathfrak{r}})\ge\breve{\varrho}\},$$
$$\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})=\{k\le\mathfrak{y}_N:\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_k}-\mathrm{a},\hat{\mathfrak{r}})\le 1-\breve{\varrho}\},$$

$$\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}}) = \{k \le \mathfrak{y}_N : \breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_k} - \mathrm{a},\hat{\mathfrak{r}}) \ge \breve{\varrho}\},$$
$$\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}}) = \{k \le \mathfrak{y}_N : \tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_k} - \mathrm{a},\hat{\mathfrak{r}}) \ge \breve{\varrho}\}.$$

We then examine the cases given here.

Case 1: $\mathfrak{y} > 1$.

The case is easy to observe that $\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}}) \subseteq \kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})$, $\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}}) \subseteq \kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})$ and also $\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}}) \subseteq \kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})$ for every $\hat{\mathfrak{r}} > 0$. It suggests that which follows:

$$\frac{|\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} = \frac{\mathfrak{y}|\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\hat{\lambda}_N+\mathfrak{y}} \le \frac{\mathfrak{y}|\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1} \le \frac{\mathfrak{y}|\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1}$$
$$\frac{|\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} = \frac{\mathfrak{y}|\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+\mathfrak{y}} \le \frac{\mathfrak{y}|\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1} \le \frac{\mathfrak{y}|\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1} \text{ and}$$
$$\frac{|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} = \frac{\mathfrak{y}|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+\mathfrak{y}} \le \frac{\mathfrak{y}|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1} \le \frac{\mathfrak{y}|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1}$$

By applying the mentioned inequalities, accordingly, we can establish that
$\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \le \mathfrak{y}\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$, $\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \le \mathfrak{y}\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$, and $\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \le \mathfrak{y}\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$.
Therefore, for each $\hat{\mathfrak{r}} > 0$, we obtain $\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) = 0$.
Consequently, we may show that $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty} \breve{\mathcal{X}}_{\mathfrak{y}_n} = \mathrm{a}$.

Case 2: $\mathfrak{y} \in (0,1)$.

To conclude our case, we now demonstrate that the expression $\breve{\mathcal{X}}_n$, in the sequence $\breve{\mathcal{X}}_{\mathfrak{y}_n}$, never occurs beyond $1 + \frac{1}{\mathfrak{y}}$ times.
Assume that for few $p, q \in \mathbb{N}$, we get $n = \mathfrak{y}_p = \hat{\mathfrak{y}}_{p+1} = \dots = \mathfrak{y}_{p+q-1} < \mathfrak{y}_{p+q}$,
or similarly,
$n \le \mathfrak{y}p < \mathfrak{y}(p+1) < \dots < \mathfrak{y}(p+q-1) < n+1 \le \mathfrak{y}(p+q)$.
Thus, we've been given $n + \mathfrak{y}(q-1) < \mathfrak{y}p + \mathfrak{y}(q-1) = \mathfrak{y}(p+q-1) < n+1$,
which gives $\mathfrak{y}(q-1) < 1$, i.e., $q < 1 + \frac{1}{\mathfrak{y}}$. According to this field, we get for any $\breve{\varrho} > 0$ and also $\hat{\mathfrak{r}} > 0$ that
$\frac{|\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} \le \left(1+\frac{1}{\mathfrak{y}}\right)\frac{\mathfrak{y}_N+1}{N+1}\frac{|\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1} \le 2(\mathfrak{y}+1)\frac{|\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1}$ and
$\frac{|\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} \le \left(1+\frac{1}{\mathfrak{y}}\right)\frac{\mathfrak{y}_N+1}{N+1}\frac{|\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1} \le 2(\mathfrak{y}+1)\frac{|\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1}$ also
$\frac{|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} \le \left(1+\frac{1}{\mathfrak{y}}\right)\frac{\mathfrak{y}_N+1}{N+1}\frac{|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1} \le 2(\mathfrak{y}+1)\frac{|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1}$
for which N is large enough, such that $\frac{(\mathfrak{y}_{n+1})}{N+1} \le 2\mathfrak{y}$.
Consequently, it follows that
$\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \le 2(\mathfrak{y}+1)\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$
$\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \le 2(\mathfrak{y}+1)\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$ and
$\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \le 2(\mathfrak{y}+1)\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$ correspondingly.
Considering that $\{\breve{\mathcal{X}}_n\}$ is $\mathfrak{st}$-convergent toward $\mathfrak{U}$,
$\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})) = 0$ for any $\hat{\mathfrak{r}} > 0$.
Therefore, $\forall \hat{\mathfrak{r}} > 0$, $\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})) = 0$.
We have therefore also demonstrated that $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty} \breve{\mathcal{X}}_{\mathfrak{y}_n} = \mathrm{a}$ in this instance. □

Theorem 4.2. *Let* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$ *indicate a* $\mathfrak{NnNLS}$ *and let* $\{\mathrm{a}_n\}$ *be a sequence within* $\mathfrak{U}$. *If* $\{\mathrm{a}_n\}$ *represents a* $\mathfrak{st}$*-Cesaro summable toward* a *in relation to* $\mathfrak{NnN}$ $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$. *After*

that, $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty} \dfrac{1}{\mathfrak{y}_n - n} \sum_{k=n+1}^{\mathfrak{y}_n} \mathrm{a}_k = \mathrm{a}$, *for any* $\mathfrak{y} > 1$ *along with*
$\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty} \dfrac{1}{n - \mathfrak{y}_n} \sum_{k=\mathfrak{y}_{n+1}}^{n} \mathrm{a}_k = \mathrm{a}$, *for all* $0 < \mathfrak{y} < 1$.

Proof. Consider $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty} \breve{\mathcal{X}}_n = \mathrm{a}$. Select $\iota_1, \iota_2 > 0$ with a given $\breve{\varrho} > 0$ so that $\max\{\hat{\mathfrak{r}}_1, \hat{\mathfrak{r}}_2\} < \breve{\varrho}$ and $\min\{1-\iota_1, 1-\iota_2\} > 1-\breve{\varrho}$. Next, define the following sets for each $\hat{\mathfrak{r}} > 0$ and a sufficiently large N:

$$\begin{aligned}
\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1, \hat{\mathfrak{r}}) &= \{k \le N : \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_k - \mathrm{a}, \hat{\mathfrak{r}}) \le 1 - \iota_1\},\\
\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\iota_1, \hat{\mathfrak{r}}) &= \{k \le N : \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_k - \mathrm{a}, \hat{\mathfrak{r}}) \ge \iota_1\},\\
\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\iota_1, \hat{\mathfrak{r}}) &= \{k \le N : \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_k - \mathrm{a}, \hat{\mathfrak{r}}) \ge \iota_1\},\\
\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2, \hat{\mathfrak{r}}) &= \{k \le N : \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_k} - \breve{\mathcal{X}}_k, \hat{\mathfrak{r}}) \le 1 - \hat{\mathfrak{r}}_2\},\\
\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2, \hat{\mathfrak{r}}) &= \{k \le N : \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_k} - \breve{\mathcal{X}}_k, \hat{\mathfrak{r}}) \ge \hat{\mathfrak{r}}_2\},\\
\kappa_{\tilde{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2, \hat{\mathfrak{r}}) &= \{k \le N : \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_k} - \breve{\mathcal{X}}_k, r) \ge \hat{\mathfrak{r}}_2\}.
\end{aligned}$$

We now explain the cases listed here.

Case I: $\mathfrak{y} > 1$. Define the following sets for any $\hat{\mathfrak{r}} > 0$ and given $\breve{\varrho} > 0$:

$$\begin{aligned}
\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho}, \hat{\mathfrak{r}}) &= \{k \le N : \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathfrak{J}_n(w) - \mathrm{a}, \hat{\mathfrak{r}}) \le 1 - \breve{\varrho}\},\\
\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho}, \hat{\mathfrak{r}}) &= \{k \le N : \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathfrak{J}_n(w) - \mathrm{a}, \hat{\mathfrak{r}}) \ge \breve{\varrho}\},\\
\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho}, \hat{\mathfrak{r}}) &= \{k \le N : \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathfrak{J}_n(w) - \mathrm{a}, \hat{\mathfrak{r}}) \ge \breve{\varrho}\},
\end{aligned}$$

in which $\mathfrak{J}_n(w) = \frac{1}{\mathfrak{y}_n - n} \sum_{k=n+1}^{\mathfrak{y}_n} a_k$ for each $n \in \mathbb{N}$.
For each $\mathfrak{y} > 1$ and also sufficiently large $n \in \mathbb{N}\backslash\{0\}$ so that $n < \mathfrak{y}_n$ along with $n \ge \frac{3\mathfrak{y}-1}{\mathfrak{y}(\mathfrak{y}-1)}$, we get that for every $\hat{\mathfrak{r}} > 0$ along with $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1} \in \mathfrak{U}$,

$$\begin{aligned}
&\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \frac{1}{\mathfrak{y}_n - n} \sum_{k=n+1}^{\mathfrak{y}_n} \mathrm{a}_k - \mathrm{a}, \hat{\mathfrak{r}}\right)\\
&= \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \frac{\mathfrak{y}_n + 1}{\mathfrak{y}_n - n} \frac{1}{\mathfrak{y}_{n+1}} \sum_{k=0}^{\mathfrak{y}_n} \mathrm{a}_k - \frac{1}{\mathfrak{y}_n - n} \sum_{k=0}^{n} \mathrm{a}_k - \mathrm{a}, \hat{\mathfrak{r}}\right)\\
&= \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \frac{\mathfrak{y}_n + 1}{\mathfrak{y}_n - n} \breve{\mathcal{X}}_{\mathfrak{y}_n} - \frac{\mathfrak{y}_n + 1 - \mathfrak{y}_n + n}{\mathfrak{y}_n - n} \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}}\right)\\
&\ge \min\left\{\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_n} - \breve{\mathcal{X}}_n, \frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{\mathfrak{y}_n - n}}\right), \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&\ge \min\left\{\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_n} - \breve{\mathcal{X}}_n, \frac{(\mathfrak{y} - 1)\hat{\mathfrak{r}}}{4\mathfrak{y}}\right), \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&= \min\left\{\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_n} - \breve{\mathcal{X}}_n, \hat{\mathfrak{r}}_0), \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&> \min\{1 - \iota_2, 1 - \iota_1\}\\
&> 1 - \breve{\varrho}
\end{aligned}$$

and

$$
\begin{aligned}
&\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\frac{1}{\mathfrak{y}_n-n}\sum_{k=n+1}^{\mathfrak{y}_n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&=\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\frac{\mathfrak{y}_n+1}{\mathfrak{y}_n-n}\frac{1}{\mathfrak{y}_n+1}\sum_{k=0}^{\mathfrak{y}_n}\mathrm{a}_k-\frac{1}{\mathfrak{y}_n-n}\sum_{k=0}^{n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&=\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\frac{\mathfrak{y}_n+1}{\mathfrak{y}_n-n}\breve{\mathcal{X}}_{\mathfrak{y}_n}-\frac{\mathfrak{y}_n+1-\mathfrak{y}_n+n}{\mathfrak{y}_n-n}\breve{\mathcal{X}}_n-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&\le\max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{\mathfrak{y}_n-n}}\right),\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&\le\max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{(\mathfrak{y}-1)\hat{\mathfrak{r}}}{4\mathfrak{y}}\right),\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&=\max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\hat{\mathfrak{r}}_0\right),\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&<\max\{\iota_2,\iota_1\}\\
&<\breve{\varrho},
\end{aligned}
$$

and

$$
\begin{aligned}
&\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\frac{1}{\mathfrak{y}_n-n}\sum_{k=n+1}^{\mathfrak{y}_n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&=\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\frac{\mathfrak{y}_n+1}{\mathfrak{y}_n-n}\frac{1}{\mathfrak{y}_n+1}\sum_{k=0}^{\mathfrak{y}_n}\mathrm{a}_k-\frac{1}{\mathfrak{y}_n-n}\sum_{k=0}^{n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&=\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\frac{\mathfrak{y}_n+1}{\mathfrak{y}_n-n}\breve{\mathcal{X}}_{\mathfrak{y}_n}-\frac{\mathfrak{y}_n+1-\mathfrak{y}_n+n}{\mathfrak{y}_n-n}\breve{\mathcal{X}}_n-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&\le\max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{\mathfrak{y}_n-n}}\right),\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&\le\max\left\{\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{(\mathfrak{y}-1)\hat{\mathfrak{r}}}{4\mathfrak{y}}\right),\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&=\max\left\{\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\hat{\mathfrak{r}}_0\right),\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\dots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&<\max\{\iota_2,\iota_1\}\\
&<\breve{\varrho},
\end{aligned}
$$

where $\hat{\mathfrak{r}}_0=\frac{r(\mathfrak{y}-1)}{4\mathfrak{y}}>0$. Therefore, we get for all $\hat{\mathfrak{r}}>0$,
$\kappa^c_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})\cup\kappa^c_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\subseteq\kappa^c_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})$,
$\kappa^c_{\breve{\nu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})\cup\kappa^c_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\subseteq\kappa^c_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})$,
$\kappa^c_{\tilde{\omega},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})\cup\kappa^c_{\tilde{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\subseteq\kappa^c_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})$
or equivalently,

$$
\begin{aligned}
\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})&\subseteq\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\cap\kappa_{\hat{\mu},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}),\\
\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})&\subseteq\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\cap\kappa_{\breve{\nu},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}),
\end{aligned}
$$

$$\kappa_{\breve{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) \subseteq \kappa_{\tilde{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}) \cap \kappa_{\tilde{\omega},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}). \tag{4.1}$$

For any $r > 0$ for which we take the asymptotic densities of both sides of (4.1), we get

$$\begin{aligned}
0 &\le \breve{\delta}(\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))\\
&\le \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}}) \cup \kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}))\\
&= \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) + \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) - \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}}) \cap \kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}))\\
&\le \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) + \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}))\\
0 &\le \breve{\delta}(\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))\\
&\le \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) + \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) \text{ and}\\
0 &\le \breve{\delta}(\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))\\
&\le \breve{\delta}(\kappa_z\tilde{\omega},\breve{\mathcal{X}}(\iota_1,\hat{\mathfrak{r}})) + \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})).
\end{aligned}$$

Since $\{\breve{\mathcal{X}}_n\}$ is $\mathfrak{st}$-convergent towards $\mathrm{a} \in \mathfrak{U}$,
$\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) = 0$ for every $\hat{\mathfrak{r}} > 0$. Therefore, $\{\breve{\mathcal{X}}_{\mathfrak{y}_n}\}$ also $\mathfrak{st}$-converges towards $\mathfrak{U}$.
Using the argument above, $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim\limits_{n\to\infty}(\breve{\mathcal{X}}_{\mathfrak{y}_n} - \breve{\mathcal{X}}_n) = 0$ is implied. Therefore, we get $\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) = 0 \ \forall\ \hat{\mathfrak{r}} > 0$.
Now, we can determine that $\breve{\delta}(\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})) = 0$.
Therefore, $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim\limits_{n\to\infty} \dfrac{1}{\mathfrak{y}_n - n} \displaystyle\sum_{k=n+1}^{\mathfrak{y}_n} \mathrm{a}_k = \mathrm{a}$, for each $\mathfrak{y} > 1$.

Case II: $\mathfrak{y} \in (0,1)$.
The following sets should be described for any $\hat{\mathfrak{r}} > 0$ and given $\breve{\varrho} > 0$:

$$\begin{aligned}
\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) &= \{k \le N : \hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\mathfrak{J}_k(w) - \mathrm{a},\hat{\mathfrak{r}}) \le 1 - \breve{\varrho}\},\\
\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) &= \{k \le N : \breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\mathfrak{J}_k(w) - \mathrm{a},\hat{\mathfrak{r}}) \ge \breve{\varrho}\},\\
\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) &= \{k \le N : \tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\mathfrak{J}_k(w) - \mathrm{a},\hat{\mathfrak{r}}) \ge \breve{\varrho}\},
\end{aligned}$$

in which $\mathfrak{J}_k(w) = \frac{1}{n-\mathfrak{y}_n}\sum_{k=\mathfrak{y}_n+1}^{\mathfrak{y}_n} \mathrm{a}_k$ for any $n \in \mathbb{N}$.
For all sufficiently large $n \in \mathbb{N}\backslash\{0\}$ together with $0 < \mathfrak{y} < 1$ in a way that $n > \mathfrak{y}_n$ along with $n > \frac{1}{\mathfrak{y}}$, we get that $\forall \hat{\mathfrak{r}} > 0$ along with $\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1} \in \mathfrak{U}$, that

$$\begin{aligned}
&\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{1}{n-\mathfrak{y}_n}\sum_{k=\mathfrak{y}_{n+1}}^{\mathfrak{y}_n} \mathrm{a}_k - \mathrm{a},\hat{\mathfrak{r}}\right)\\
&\ge \min\left\{\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n} - \breve{\mathcal{X}}_n,\frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{n-\mathfrak{y}_n}}\right),\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n - \mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&> \min\{1-\iota_2, 1-\iota_1\}\\
&> 1 - \breve{\varrho}
\end{aligned}$$

and

$$\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{1}{n-\mathfrak{y}_n}\sum_{k=\mathfrak{y}_{n+1}}^{\mathfrak{y}_n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)$$
$$\leq \max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{n-\mathfrak{y}_n}}\right),\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}$$
$$\leq \max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\hat{\mathfrak{r}}_1\right),\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}$$
$$< \max\{\iota_2,\iota_1\}$$
$$< \breve{\varrho},$$

and

$$\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{1}{n-\mathfrak{y}_n}\sum_{k=\mathfrak{y}_{n+1}}^{\mathfrak{y}_n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)$$
$$\leq \max\left\{\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{n-\mathfrak{y}_n}}\right),\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}$$
$$\leq \max\left\{\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\hat{\mathfrak{r}}_1\right),\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}$$
$$< \max\{\iota_2,\iota_1\}$$
$$< \breve{\varrho},$$

where $\hat{\mathfrak{r}}_1=\frac{(1-\mathfrak{y})\hat{\mathfrak{r}}}{4\mathfrak{y}}>0$. Therefore, for all $\iota>0$, we obtain

$$\begin{aligned}\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})&\subseteq\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\cup K_{\hat{\mu},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}),\\ \kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})&\subseteq\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\cup K_{\breve{\nu},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}),\\ \kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})&\subseteq\kappa_{\tilde{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\cup \kappa_{\tilde{\omega},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}).\end{aligned}\tag{4.2}$$

For any $r>0$ for which we assume the asymptotic densities among both sides of (4.2), we get
$0\leq\breve{\delta}(\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))\leq\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}}))+\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}))$,
$0\leq\breve{\delta}(\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))\leq\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}}))+\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}))$ together with
$0\leq\breve{\delta}(\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))\leq\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}}))+\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}))$.
Since $\{\breve{\mathcal{X}}_n\}$ is $\mathfrak{st}$-convergent toward $\mathrm{a}\in\mathfrak{U}$, we obtain $\{\breve{\mathcal{X}}_{\mathfrak{y}_n}\}$ is also $\mathfrak{st}$-convergent towards a.
According to the proof provided, $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim_{n\to\infty}(\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n)=0$.
Therefore, we get $\breve{\delta}(\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))=\breve{\delta}(\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))=\breve{\delta}(\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))=0$.
We can so demonstrate that $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim_{n\to\infty}\frac{1}{n-\mathfrak{y}_n}\sum_{\mathfrak{y}_{n+1}}^{n}\mathrm{a}_k=\mathrm{a}$, for each $\mathfrak{y}\in(0,1)$. □

5. Conclusion

In this stydy, we extended classical Tauberian theorems to the framework of neutrosophic n-normed linear spaces by employing the concept of statistical Cesaro summability. This integration offers a novel perspective for analyzing convergence behaviors within uncertain and imprecise environments, which are effectively modeled using neutrosophic structures. The established results not only generalize known theorems in normed linear spaces but also provide a robust mathematical foundation for further applications in areas such as functional analysis, information theory and decision-making under uncertainty. Future research can explore analogous results using other summability methods and extend the framework to more generalized neutrosophic spaces, enriching the theoretical development of both summability theory and neutrosophic analysis.

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INTEGRABILITY FOR THE DERIVATIVE FORMULAS OF THE TYPE-2 BISHOP FRAME AND ITS APPLICATIONS

EVREN ERGÜN *

Abstract. The main objective of the work is to examine the integrability of the derivative formulae for the type-2 Bishop frame in three-dimensional Euclidean space. We use the coordinate system introduced in [12], which allows for the examination of integration. As an application, we analyze the position vectors of certain curves that are important in mathematics and physical study.

Keywords: Darboux helix, Position vector field, Type-2 Bishop frame, Euclidean 3-space

2020 Mathematics Subject Classification: 53B25, 53B30, 53C50.

1. Introduction

The theory of curves has gone through a long period of development until it reaches a truly modern manner: from the theory of plane curves, with the beginning of calculus, in 1684, the year in which Gottfried Wilhelm Leibniz created it in his Meditaito nova de natura anguli contactus et osculi, to the theory of space curves, reached to the peak point with the infinitesimal calculus. In this development, we have to mention two important things. The first one is the notion of moving frame, as we know it today, created by Gaston Darboux. The second one is the term binormal mentioned in a treatise on space curves by Barre de Saint-Venant. The Frenet frame is a well-known example of a moving frame utilized to describe a space curve in three-dimensional ambient spaces, including Euclidean and Lorentz-Minkowski spaces. The Frenet equations, or Frenet formulae, were first proposed in 1831 by Karl Eduard Senff and Johann Martin Bartels, enhancing the simplicity and utility of the theory of space curves. The scientists were once again discussed in Jean Frederic Frenet's thesis in 1847, published in 1852. Shortly thereafter, those equations were independently discovered by Joseph Alfred Serret in 1851 and are sometimes referred to as the Frenet-Serret equations (for more information at this early stage in history, , see [7]). On the other hand, mathematicians have done a great number of surveys involving the concept of binormal. But it was not until 2010 that the survey of the first moving frame established by the binormal was published by Yılmaz and Turgut. The authors were the first to create the idea of the

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moving frame in a more novel manner than usual in their "A new version of Bishop frame and an application to spherical images." The main principle is that they do this using a common vector field as the binormal vector field of Frenet-Serret frame (for details, see [9]). Later, an analogue of this survey is done in Lorentz-Minkowski 3-space. [8, 10]

The determination of the position vector field of a smooth curve with a certain property—that is, a slant helix, where the principal normal vector field forms a constant angle with a fixed straight line—was investigated in 2010 by Ali and Turgut. They discovered a third-order vector differential equation. By solving the vector differential equation, they obtained the position vector field of a timelike slant helix in Minkowski space, where the straight line is parallel to e_3 [1]. Refer to [2] for slant helices in Euclidean 3-space. In 2011, researchers conducted analogous investigations to ascertain the position vector field of a generic helix using both the Frenet and standard frames in Euclidean three-dimensional space [3]. Refer to [4, 5] for timelike and spacelike generic helices in Minkowski 3-space.

In the past two decades, the problem of determining the position vectors has emerged as an attractive field of study. In recent years, Yerlikaya and his coauthor [12, 13] have approached this problem from a different perspective than those mentioned in the literature. This approach is based on a new coordinate system that will facilitate the integrability of derivative formulas of the Bishop frame. Inspired by these studies, we focus on that of the type-2 Bishop frame and examine the position vector field of several special curves.

2. Preliminaries

When the real vector space $\mathbb{R}^3$ is endowed with the standard flat metric, known as the Euclidean metric, represented by $g = dx_1^2 + dx_2^2 + dx_3^2$, the corresponding space is known as Euclidean space and denoted as $\mathbb{E}^3$, where (x_1, x_2, x_3) constitutes the usual coordinate system of $\mathbb{E}^3$. The norm of an arbitrary vector $w \in \mathbb{E}^3$ is defined as $\|u\| = \sqrt{g(u,u)}$. Furthermore, given two non-zero vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{E}^3$, it is important to note that the cross product of u and v is denoted as

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Let $\gamma : J \to \mathbb{E}^3$ be a smooth curve parametrized by the arbitrary parameter t, where J is an open subset of $\mathbb{R}$. The curve γ is referred as a unit speed curve parametrized by the arc lenght s if its velocity vector γ', the first derivative of the curve, fulfills the condition $\|\gamma'\| = 1$. The parameter of γ shall hereafter be denoted as s. In Euclidean 3-space, the Frenet-Serret frame along the curve γ, denoted by $\{t, n, b\}$, has the derivative formula expressed as

$$\begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix},$$

where the curvature and the torsion functions of γ are denoted by κ and τ, respectively.

The derivative formulae for the type-2 Bishop frame represented by $\{\zeta_1, \zeta_2, b\}$ along γ are as follows:

$$\begin{pmatrix} \zeta_1' \\ \zeta_2' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\epsilon_1 \\ 0 & 0 & -\epsilon_2 \\ \epsilon_1 & \epsilon_2 & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ b \end{pmatrix}, \tag{2.1}$$

where ϵ_1 and ϵ_2 are the type-2 Bishop curvature functions of γ and ζ_1, ζ_2 are arbitrary unit vector fields in $\mathbb{E}^3$. The geometric apparatus between the type-2 Bishop frame and the Frenet-Serret frame, which we referred to before, is given by

$$\begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) & 0 \\ \sin\theta(s) & \cos\theta(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ b \end{pmatrix}, \tag{2.2}$$

$$\kappa = -\theta^{'}(s), \ \tau = \sqrt{\epsilon_1^2 + \epsilon_2^2}, \tag{2.3}$$

where $\theta(s) = \arctan\left(\frac{\epsilon_1}{\epsilon_2}\right)$. Note that the above apparatus differs from that of the study of Yılmaz and Turgut [9]. By Eq. (2.3) and the angle θ, there exists the following theorem:

Theorem 2.1. [6] *Let $\gamma = \gamma(s)$ be a smooth curve with curvatures $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$. γ is a general helix if and only if type-2 Bishop curvatures of the curve satisfy*

$$\frac{\epsilon_1^2}{\left(\epsilon_1^2 + \epsilon_2^2\right)^{\frac{3}{2}}} \left(\frac{\epsilon_2}{\epsilon_1}\right)^{'} = constant.$$

Remark 2.1. *A necessary condition for the type-2 Bishop frame to exist at all points along a curve is that the curvature of the curve should not be zero. If $\kappa = 0$, then the principal normal vector field of the curve denoted by n becomes $(0, 0, 0)$. This means that the binormal vector field b becomes $(0, 0, 0)$. This causes a contradiction in the fact that the system $\{\zeta_1, \zeta_2, b\}$ is orthogonal.*

3. Conclusion

Let $\mathbb{E}^3$ endow the Euclidean 3-space and its basis be $B = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$. Let the coordinates of a vector relative to the basis B be $\{x_1, x_2, x_3\}$. In [12], the authors established an ordered orthonormal basis $B'' = \{e_1'', e_2'', e_3''\}$ and the corresponding new coordinate system $\{x_1'', x_2'', x_3''\}$ such that

$$e_j'' = \frac{e_3'' \times e_i}{\left\| e_3'' \times e_i \right\|}, \quad j = 1, 2 \ \ i = 1, 2, 3$$

$$\left(e_j'' \times e_3''\right) = \begin{cases} -e_2'' & , \ j = 1 \\ e_1'' & , \ j = 2. \end{cases}$$

Let $\gamma : I \to \mathbb{E}^3$ be a smooth curve parameterized by arc length s, where $s \in I$, and its type-2 Bishop apparatus $\{\zeta_1, \zeta_2, b, \epsilon_1, \epsilon_2\}$ at the point $\gamma(s)$. Let us consider an any curve $\overline{\gamma}$ obtained from γ through a rigid motion, in such a way that the binormal vector field $\overline{b}$ at the point $\overline{\gamma}(s_0)$ of $\overline{\gamma}$ aligns with e_3''. Consequently, due to this motion, $\overline{\zeta}_1$ and $\overline{\zeta}_2$ sit in the plane defined by e_1'' and e_2''. The other vector fields of $\overline{\gamma}$ are designated as $\overline{\zeta}_1$ and $\overline{\zeta}_2$, respectively. Consequently, it is appropriate to discuss the transition matrix between the systems $\left\{\bar{\zeta}_1, \bar{\zeta}_2, \bar{b}\right\}$ and $\{e_1'', e_2'', e_3''\}$, which is structured as follows:

$$\begin{pmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \\ \bar{b} \end{pmatrix} = \begin{pmatrix} \cos\mu(s) & -\sin\mu(s) & 0 \\ \sin\mu(s) & \cos\mu(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1'' \\ e_2'' \\ e_3'' \end{pmatrix}, \tag{3.4}$$

where the angle between the vector fields $\bar{b}$ and e_3'' denotes μ.

Furthermore, it is noteworthy that the rigid motion transforming $\bar{\gamma}(s_0)$ into $\gamma(s_0)$ and $\bar{\zeta}_1, \bar{\zeta}_2, \bar{b}$ into ζ_1, ζ_2, b is, in fact, identical to the aforementioned rigid motion. Hence, we write

$$\zeta_1 = \bar{\zeta}_1, \ \ \zeta_2 = \bar{\zeta}_2, \ \ b = \bar{b}$$

for any $s = s_0$.

By establishing $i = 2$ and $j = 2$ by the argument that $e_3'' = b = (b_1, b_2, b_3)$, we derive

$$e_1'' = \frac{1}{\sqrt{1-b_2^2}}\left(-b_1b_2, 1-b_2^2, -b_2b_3\right) \tag{3.5}$$

and

$$e_2'' = \frac{1}{\sqrt{1-b_2^2}}\left(-b_3, 0, b_1\right). \tag{3.6}$$

Theorem 3.1. *Let $\{e_1'', e_2'', e_3''\}$ be the new ordered orthonormal basis obtained from the natural ordered orthonormal basis of $\mathbb{E}^3$ and $\epsilon_1(s)$, $\epsilon_2(s)$ be differentiable functions, where s belongs to an open interval in $\mathbb{R}$. According to the new coordinate system, the binormal vector field $b = (b_1, b_2, b_3)$ in an indirect solution triplet of Eq. (2.1), which is determined by Eqs. (3.7) and (3.8) is given by*

$$\begin{cases} b_1(s) &= \cos f_1(s)\cos f_2(s) \\ b_2(s) &= \sin f_1(s) \\ b_3(s) &= \cos f_1(s)\sin f_2(s) \end{cases}$$

where

$$f_1(s) = c_1 + \int \left(\cos\mu(s)\ \epsilon_1(s) + \sin\mu(s)\ \epsilon_2(s)\right) ds, \tag{3.7}$$

$$f_2(s) = c_2 + \int \frac{-\sin\mu(s)\ \epsilon_1(s) + \cos\mu(s)\ \epsilon_2(s))}{\cos f_1(s)}\, ds \tag{3.8}$$

and c_1, c_2 are constants.

Proof. Let $\{e_1'', e_2'', e_3''\}$ be the new ordered orthonormal basis derived from the natural ordered orthonormal basis in the Euclidean 3-space. Thus, Eqs. (3.5) and (3.6) are valid. Using Eq. (3.4), a relationship between type-2 Bishop vector fields and the vector fields of the new system is appeared as

$$\begin{aligned} \zeta_1(s) &= \cos\mu(s)\ e_1''(s) - \sin\mu(s)\ e_2''(s) \\ \zeta_2(s) &= \sin\mu(s)\ e_1''(s) + \cos\mu(s)\ e_2''(s). \end{aligned} \tag{3.9}$$

We will now compute the elements of the binormal vector field $b(s)$. Substituting Eqs. (3.5) and (3.6) into Eq. (3.9) and putting it into Eq. (2.1), we get

$$\frac{db_1}{ds} = \frac{-1}{\sqrt{1-b_2^2}}\Big[\{\cos\mu(s)\ \epsilon_2 - \sin\mu(s)\ \epsilon_1\}\, b_3 + \{\cos\mu(s)\ \epsilon_1 + \sin\mu(s)\ \epsilon_2\}\, b_1 b_2\Big] \tag{3.10}$$

$$\frac{db_2}{ds} = \{\cos\mu(s)\ \epsilon_1 + \sin\mu(s)\ \epsilon_2\}\sqrt{1-b_2^2}, \tag{3.11}$$

$$\frac{db_3}{ds} = \frac{-1}{\sqrt{1-b_2^2}}\Big[\{\sin\mu(s)\ \epsilon_1 - \cos\mu(s)\ \epsilon_2\}\, b_1 + \{\cos\mu(s)\ \epsilon_1 + \sin\mu(s)\ \epsilon_2\}\, b_2 b_3\Big] \tag{3.12}$$

Due to the fact that Eq. (3.11) is a type of separable equations, it is simpler to solve compared to other equations, and as a result, the answer ends up being

$$b_2 = \sin \underbrace{\left[c_1 + \int (\cos\mu(s)\ \epsilon_1 + \sin\mu(s)\ \epsilon_2)\ ds\right]}_{=f_1(s)}. \tag{3.13}$$

On the other hand, especially since Eqs. (3.10) and (3.12) are non-linear differential equations, it is beneficial to introduce a new variable $g(s)$ rather than solving them directly, which adheres to the following situation:

$$b_1^2 + b_2^2 + b_3^2 = 1,$$

from which

$$b_1 = \cos f_1(s)\ \cos f_2(s), \quad b_3 = \cos f_1(s)\ \sin f_2(s). \tag{3.14}$$

Substituting Eqs. (3.13) and (3.14) into Eq. (3.10), we obtain

$$f_2(s) = c_2 + \int \frac{(-\sin\mu(s)\ \epsilon_1 + \cos\mu(s)\ \epsilon_2)}{\cos f_1(s)}\ ds, \tag{3.15}$$

which completes the proof.

□

The other important that this work will attain can be understood by finding its tangent vector field for the position vector field of a curve. By the proposition that we have just achieved, it may easily be calculated:

For this, we begin by getting the type-2 Bishop vector fileds ζ_1 and ζ_2. Substituting Eqs. (3.5) and (3.6) into Eq. (3.9), for $\zeta_1 = (\zeta_{1_1}, \zeta_{1_2}, \zeta_{1_3})$ and $\zeta_2 = (\zeta_{2_1}, \zeta_{2_2}, \zeta_{2_3})$, we get

$$\zeta_{1_1} = \sin\mu(s)\sin f_2(s) - \cos\mu(s)\sin f_1(s)\cos f_2(s) \tag{3.16}$$

$$\zeta_{1_2} = \cos\mu(s)\cos f_1(s) \tag{3.17}$$

$$\zeta_{1_3} = -\cos\mu(s)\sin f_1(s)\sin f_2(s) - \sin\mu(s)\cos f_2(s) \tag{3.18}$$

and

$$\zeta_{2_1} = -\sin\mu(s)\sin f_1(s)\cos f_2(s) - \cos\mu(s)\sin f_2(s) \tag{3.19}$$

$$\zeta_{2_2} = \sin\mu(s)\cos f_1(s) \tag{3.20}$$

$$\zeta_{2_3} = \cos\mu(s)\cos f_2(s) - \sin\mu(s)\sin f_1(s)\sin f_2(s). \tag{3.21}$$

From Eq. (2.2) taking into account ζ_1 and ζ_2, we have the following remark.

Remark 3.1. *When referring to the position vector field, represented as γ, it is important to remember the following equation:*

$$\frac{d\gamma}{ds} = t. \tag{3.22}$$

With this relation, it is more convenient to perform the operation with the tangent vector field than the binormal vector field. Referring to the proposition 3.1, we have the following relations:

$$t_1 = \frac{-1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\sin f_1\cos f_2\left\{\epsilon_2\cos\mu - \epsilon_1\sin\mu\right\} - \sin f_2\left\{\epsilon_1\cos\mu + \epsilon_2\sin\mu\right\}\right)$$

$$t_2 = \frac{\cos f_2}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\left\{\epsilon_2\cos\mu - \epsilon_1\sin\mu\right\}\right) \tag{3.23}$$

$$t_3 = \frac{-1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\sin f_1\sin f_2\left\{\epsilon_2\cos\mu - \epsilon_1\sin\mu\right\} + \cos f_2\left\{\epsilon_1\cos\mu + \epsilon_2\sin\mu\right\}\right)$$

4. Applications

Some remarkable curves share the characteristic that a vector field makes a constant angle with a fixed line in space. In the type-2 Bishop frame, two curves exhibit the specified property:

Inclined Curve: A smooth curve is classified as an inclined curve if the vector field ζ_1 (or ζ_2) within its osculating plane forms a constant angle with a fixed line in space. It is analytically defined by the constancy of the ratio of Bishop curvatures ϵ_1 and ϵ_2, as presented by Özyılmaz in the Euclidean 3-space $\mathbb{E}^3$. [6].
Darboux Helix: A smooth curve is classified as a Darboux helix if the Darboux vector $w = -\epsilon_2\zeta_1 + \epsilon_1\zeta_2$ makes a constant angle with a fixed line in space. The curvatures ϵ_1 and ϵ_2 of a Darboux helix adhere to the subsequent equation:

$$\frac{\left(\epsilon_1^2+\epsilon_2^2\right)^{\frac{3}{2}}}{\epsilon_1^2}\frac{1}{\left(\frac{\epsilon_2}{\epsilon_1}\right)'} = \text{constant} \tag{4.24}$$

[11]. In Eq. (4.24), remark that the ratio $\frac{\epsilon_2}{\epsilon_1}$ must not be constant. According to the theorem (2.1), an inclined curve is a general helix, but not a Darboux helix.

A new coordinate system is presented in the preceding section, facilitating the integrability of the derivative formulas for the type-2 Bishop frame. This results in a theorem that demonstrates only one of the triplets of the indirect solutions of Eq. (2.1). In this section, we examine the necessary conditions for the indirect solution to achieve stability. Alternatively, we assess the nature of the integration measure.

It is widely recognized that the curvatures of a curve remain constant until a rigid motion is encountered. Consequently, the type-2 Bishop curvatures $\bar{\epsilon_1}$ and $\bar{\epsilon_2}$ of $\bar{r}$ must satisfy the subsequent conditions:

$$\epsilon_1 = \bar{\epsilon_1}, \quad \epsilon_2 = \bar{\epsilon_2},$$

where ϵ_1 and ϵ_2 are the type-2 Bishop curvature functions of r.

From the theorem (3.1), we have Eqs. (3.16) and (3.19) mentioned the previous section. Consequently, we seek to determine the curvatures $\bar{\epsilon_1}$ and $\bar{\epsilon_2}$, respectively. By differentiating Equation (3.16) with regard to s, we obtain

$$\bar{\epsilon_1} = \sqrt{\left(\frac{d\mu}{ds} - \sin f_1 \, \frac{df_2}{ds}\right)^2 + \epsilon_1^2}. \tag{4.25}$$

Similarly, it is evident that another curvature is represented by

$$\bar{\epsilon_2} = \sqrt{\left(\frac{d\mu}{ds} - \sin f_1 \, \frac{df_2}{ds}\right)^2 + \epsilon_2^2}. \tag{4.26}$$

Lemma 4.1. *Let $\gamma(s)$ be a curve in the Euclidean 3-space and let s be its arc length parameter. Assume that the differentiable functions $\epsilon_1(s)$ and $\epsilon_2(s)$ be the type-2 Bishop curvatures of γ. If the following relation holds*

$$\frac{d\mu}{ds} - \sin f_1(s) \, \frac{df_2}{ds} = 0, \tag{4.27}$$

then there exist "steady" solutions satisfying Eq. (2.1), where $f_1(s)$ and $f_2(s)$ are given by Eqs. (3.7) and (3.8), respectively.

Based on Lemma 4.1, we can examine two possible situations.

Case 1: Assuming $\mu =$ constant, Eq. (4.27) is simplified to

$$\sin f_1(s)\frac{df_2}{ds} = 0. \tag{4.28}$$

Suppose that $\sin f_1(s)$ equals zero. Thus, we have $f_1 = 0$ or $f_1 = 2\pi k$, $k \in \mathbb{Z}$. Consequently, the integrand in Eq. (3.7) may be considered as

$$\cos\mu \; \epsilon_1(s) + \sin\mu \; \epsilon_2(s) = 0. \tag{4.29}$$

The following statements are deduced from the last equality.

- when $\sin\mu = 0$ (or $\cos\mu = 0$), we have $\epsilon_1 = 0$ (or $\epsilon_2 = 0$). This never occurs.
- If $\sin\mu \neq 0$ and $\cos\mu \neq 0$, the resulting position vector field is an inclined curve with

$$\frac{\epsilon_2}{\epsilon_1} = -\cot\mu.$$

Note that the function $f_2(s)$ can be determined from the aforementioned relation using Eq.(4.29), specifically $f_2(s) = c_2 + \frac{-1}{\sin\mu}\int \epsilon_1(s)\ ds$. Furthermore, when the position vector field refers to an inclined curve, its straight line can determine $d = (a, b, c)$ with the help of Eq. (3.16):

$$\langle\zeta_1, d\rangle = -\cos\mu\ b + \sin\mu\left\{a\sin f_2(s) - c\cos f_2(s)\right\}$$

The concept of inclined curves indicates that the necessary and sufficient condition for the previous inner product to remain constant is the achievement of the following relations.

$$a\sin f_2(s) - c\cos f_2(s) = 0,$$

$$b = \pm 1.$$

Hence, we obtain $d = (0, \pm1, 0)$. This provides knowledge about the plane where the straight line is spanned.

In light of this information, the position vector field of an inclined curve having a straight line spanned by e_2 is computed using Eq.(3.23) as the following:

$$\gamma(s) = (d_1, d_2 s, d_3), \tag{4.30}$$

where d_i $(i = 1, 2, 3)$ is a constant of integration. The last equality expresses to us that the above position vector field is a geodesic, which gives rise to a contradiction with the creation of the type-2 Bishop frame according to Remark 2.1.

Let $\frac{df_2}{ds} = 0$. Thus, it is evident that $f_2 =$ constant. Hence, the integrand in Eq. (3.8) is

$$-\sin\mu\ \epsilon_1(s) + \cos\mu\ \epsilon_2(s) = 0. \tag{4.31}$$

The following statements are deduced from the last equality.

- when $\cos\mu = 0$ (or $\sin\mu = 0$), we have $\epsilon_1 = 0$ (or $\epsilon_2 = 0$). This never occurs.
- If $\sin\mu \neq 0$ and $\cos\mu \neq 0$, the resulting position vector field is an inclined curve with

$$\frac{\epsilon_2}{\epsilon_1} = \tan\mu.$$

Using Eq.(4.29), we can get the function $f_1(s)$ as follows: $f_1(s) = c_1 + \frac{1}{\cos\mu}\int \epsilon_1(s)\ ds$. Also, if the position vector field corresponds to an inclined curve, its straight line may compute $d = (a, b, c)$ with the help of Eq. (3.16).

$$\langle\zeta_1, d\rangle = \cos\mu\sin f_1\left\{-a\cos m - c\sin m\right\} - \cos\mu\cos f_1\ b + \sin\mu\left\{a\sin m - c\cos m\right\},$$

where m depends on the constantity of f_2. From the definition of inclined curves, the previously mentioned dot product remains constant if and only if the subsequent relations are satisfied.:

$$a\cos m + c\sin m = 0,$$
$$a\sin m - c\cos m = 1,$$
$$b = 0,$$

from which we get $d = (\sin m, 0, -\cos m)$. This provides details about the plane where the straight line is located.

Similarly, it is easy to see the position vector field of an inclined curve having a straight line spanned by e_1 and e_3 as the following:

$$\gamma(s) = (\sin m\ s, d_1, \cos m\ s)\,.$$

This causes a contradiction for the same reason. Thus, we have the following result:

Corollary 4.1. *There does not exist any inclined curve with the type-2 Bishop curvatures* $\epsilon_1(s)$ *and* $\epsilon_2(s)$ *in* $\mathbb{E}^3$.

For the last one, we have

Case 2: When μ is not constant, three subcases emerge as

- $f_1 =$ constant, $f_2 =$ constant
- $f_1 \neq$ constant, $f_2 =$ constant
- $f_1 =$ constant, $f_2 \neq$ constant

Upon analyzing the first two items, we identify a contradiction with the claim that $\mu \neq$ constant. As a result, these subcases do not happen. We will now analyze the last item.

By the constancy of the function f_1, we have the following.

$$\cos\mu(s)\epsilon_1(s) + \sin\mu(s)\epsilon_2(s) = 0. \tag{4.32}$$

Combining the previous equation and Eq. (3.8), we get the function f_1 as

$$c_2 + \frac{-1}{n}\int\sqrt{\epsilon_1^2(s) + \epsilon_2^2(s)}\;ds, \tag{4.33}$$

where $n = \cos c_1$. Hence, Eq. (4.27) becomes

$$\frac{d\mu}{ds} + m\sqrt{\epsilon_1^2(s) + \epsilon_2^2(s)} = 0, \tag{4.34}$$

from which we get

$$\mu(s) = -m\int\sqrt{\epsilon_1^2(s) + \epsilon_2^2(s)}\;ds, \tag{4.35}$$

where $m = \frac{\sqrt{1-n^2}}{n}$. From Eqs. (4.32) and (4.35), we obtain

$$m = \frac{\epsilon_1^2\left(\frac{\epsilon_2}{\epsilon_1}\right)^{'}}{\left(\epsilon_1^2 + \epsilon_2^2\right)^{\frac{3}{2}}}. \tag{4.36}$$

Since the ratio of $\epsilon_2(s)$ to $\epsilon_1(s)$ is non-constant due to Eq. (4.32), we have $m \neq 0$. Therefore, Eq. (4.36) is

$$\frac{1}{m} = \frac{\left(\epsilon_1^2 + \epsilon_2^2\right)^{\frac{3}{2}}}{\epsilon_1^2} \frac{1}{\left(\frac{\epsilon_2}{\epsilon_1}\right)'} = \text{constant}.$$

By substituting Eqs. (4.32) and (4.33) into Eq. (3.23) with the help of Eq. (4.24), we get the position vector of a Darboux helix.

Proposition 4.1. *Let γ be a Darboux helix in $\mathbb{E}^3$ and $\epsilon_1(s)$, $\epsilon_2(s)$ be its type-2 Bishop curvatures. Thus, its position vector field is calculated:*

$$\gamma(s) = \Big(-\sqrt{1-n^2}\int \cos\left(c_2 + \int p(s)\, ds\right)\, ds + \ d_1,\ ns + \ d_2,$$
$$-\sqrt{1-n^2}\int \sin\left(c_2 + \int p(s)\, ds\right)\, ds + \ d_3\Big),$$

where $p(s) = \frac{-1}{n}\int \sqrt{\epsilon_1^2(s) + \epsilon_2^2(s)}ds$ and $n \neq 1$, c_2 and d_i for $i = 1, 2, 3$ are constant.

Example 4.1. *Substituting $\epsilon_1(s) = \tan\left(arcsin\frac{s}{5}\right)$ and $\epsilon_2(s) = 1$ in Proposition 4.1, we get the position vector of Darboux helix in the sense of type-2 Bishop frame as follows:*

$$r(s) = \Bigg(\frac{\sqrt{26}}{26}\ s + d_1, \frac{5}{\sqrt{26}}\int \cos\left[c_2 - \sqrt{26}\arcsin\frac{s}{5}\right] ds + d_2,$$
$$\frac{5}{\sqrt{26}}\int \sin\left[c_2 - \sqrt{26}\arcsin\frac{s}{5}\right] ds + d_3\Bigg).$$

Plotting for $d_1 = d_2 = d_3 = c_2 = 0$, we have the following figure.

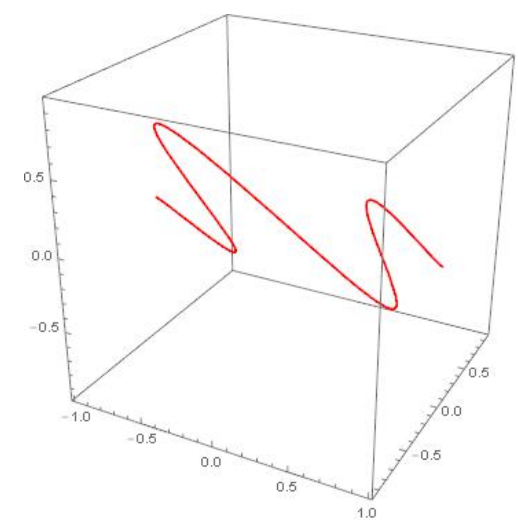

FIGURE 1. The Darboux helix with $k_1(s) = \tan(\arcsin ms)$, $k_2(s) = 1$ for $m = \frac{1}{5}$

Remark 4.1. *Taking $n = 1$ in the aforementioned statement shows that the position vector of the Darboux helix is expressed as Eq. (4.30). In light of theorem 2.1, we derive the following corollary based on result 4.1.*

Corollary 4.2. *A Darboux helix with the type-2 Bishop curvature functions* $\epsilon_1(s)$ *and* $\epsilon_2(s)$ *in* $\mathbb{E}^3$ *is a general helix, vice versa.*

Remark 4.2. *This study examines the outcomes when* $i = 2$. *The geometric interpretation of the results for* $i = 1$ *and* $i = 3$ *refers to the displacement of the components of the curve.*

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