E-ISSN: 3062-2107
Volume 1
Issue 1
2025

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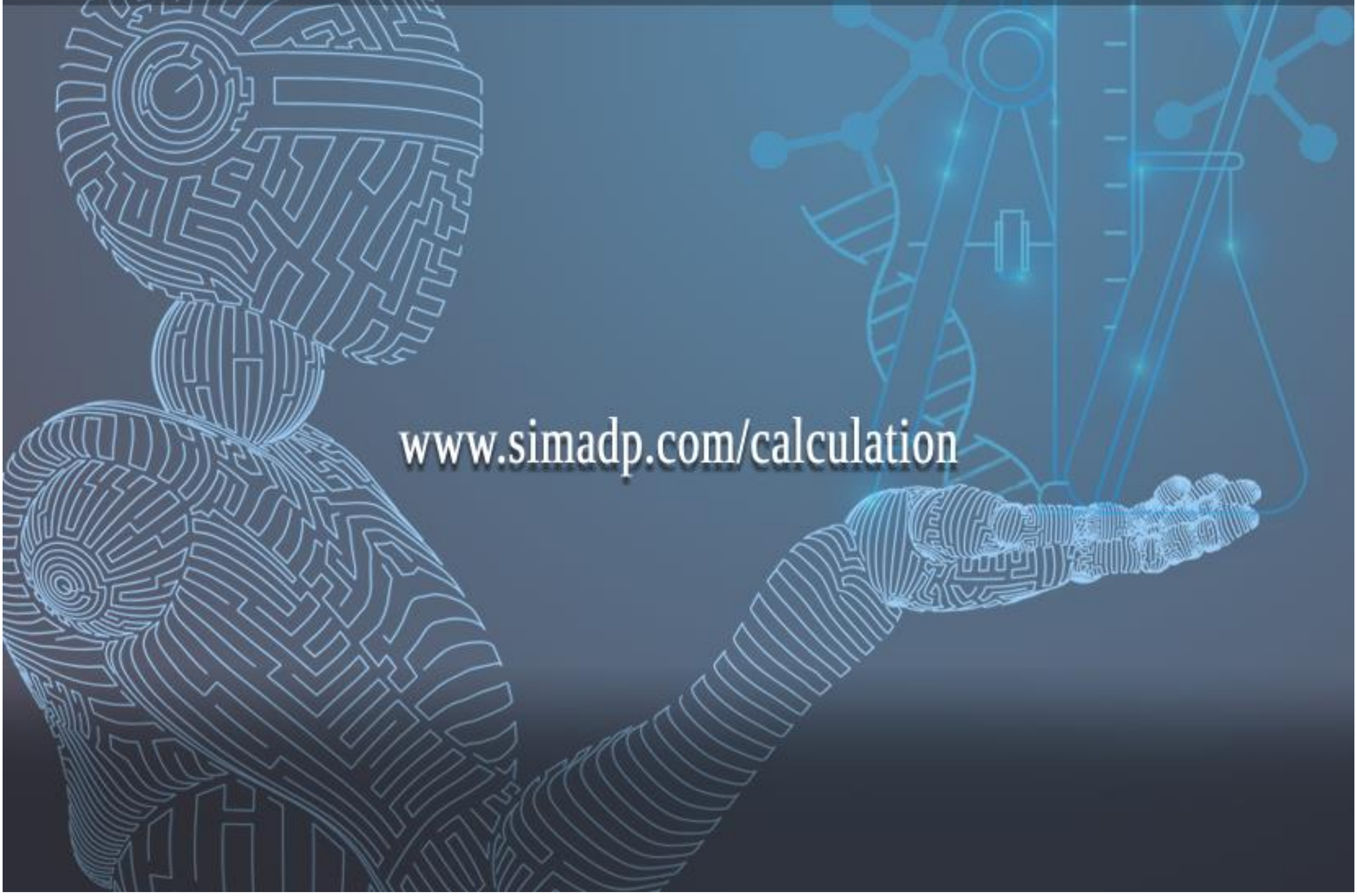

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VOLUME 1 ISSUE 1
E-ISSN: 3062-2107

January 2025
www.simadp.com/calculation

CALCULATION

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VOLUME 1 ISSUE 1
E-ISSN: 3062-2107

January 2025
www.simadp.com/calculation

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CALCULATION
*Volume **1**, Issue **1**, 2025, Pages:**1-1***
E-ISSN: 3062-2107
www.simadp.com/calculation

EDITORIAL

BAYRAM ŞAHIN

Dear readers and authors,

As SIMA Digital Publishing, we are here with a new journal. In addition to the **International Journal of Maps in Mathematics**, which is currently in publication, our new journal named **CALCULATION** will start its publication life with this issue. **CALCULATION** journal will appeal to a wide range of readers in terms of the scope. Essentially, papers will be published not only in the field of Mathematics, but also in all branches of Basic Sciences, Engineering Sciences and Social Sciences where Mathematics is used intensively.

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We hope that **CALCULATION** journal will attract the intense interest of readers and authors as soon as possible and find a good place for itself in scientific publishing with the support of authors and readers.

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CALCULATION
Volume ***1****, Issue* ***1****, 2025, Pages:****2-18***
E-ISSN: 3062-2107
www.simadp.com/calculation

WARPED PRODUCT POINTWISE SEMI-SLANT SUBMANIFOLDS OF NEARLY PARA-KAEHLER MANIFOLDS

SEDAT AYAZ * AND YILMAZ GÜNDÜZALP

Abstract. In this article, firstly we introduce pointwise slant and pointwise semi-slant submanifolds in nearly para-Kaehler manifolds. We demonstrate that there exist pointwise semi-slant non-trivial warped product submanifold $\mathcal{M}^T \times_k \mathcal{M}^\theta$ in nearly para-Kaehler manifolds by giving an example. We get a characterization, give certain theorems depending on the casual characters and we reach an optimal inequality.

Keywords: Nearly para-Kaehler manifold, pointwise semi-slant submanifold, warped product submanifold

2020 Mathematics Subject Classification: 53C43,53C15,53C40.

1. Introduction

Pointwise slant submanifolds were first introduced by F. Etayo in [11] as quasi-slant submanifolds. Such submanifolds have been studied extensively by B.-Y. Chen and O.J. Garay [10]. Then P.Alegre and A.Carriazo studied slant submanifolds in para-Hermitian manifolds and detailed definitions of types of submanifolds in semi-Riemannian setting were provided by them [3, 4].

Warped products emerged in the mathematical and physical subjects before 1969, for example, semi-reducible space, which is utilized for the warped product by Kruchkovich in 1957 [19]. It has been succesfully used in the general theory of relativity, string theory and black holes. On the other hand, warped product manifolds was introduced and studied by R.L. Bishop and B. O'Neill [9]. Later, many authors researched on warped product and submanifolds [1, 2, 5, 7, 8, 12, 14, 15, 20]

B. Sahin studied warped product pointwise semi-slant submanifolds in Kaehler manifolds [23]. He researched that there exist of the second form $\mathcal{M}^T \times_k \mathcal{M}^\theta$ in Kaehler manifold $\bar{\mathcal{M}}$. Also he found a characterization, theorem, interesting results, inequality and he obtained examples of such submanifolds. Later, S. Ayaz and Y. Gündüzalp studied warped product pointwise hemi-slant submanifolds whose ambient spaces are nearly para-Kaehler manifolds [6].

Received: 2024.11.11 *Revised: 2024.12.04* *Accepted: 2024.12.10*

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Nearly Kaehler manifolds were studied by Tachibana in [25]. For example, S^6 (six dimensional sphere) is a example of nearly Kaehler non-Kaehler manifold.

Every nearly para-Kaehler manifold isn't a para-Kaehler. But Every para-Kaehler manifold is a nearly para-Kaehler [24]. So, we give some examples for both nearly para-Kaehler and para-Kaehler manifold and we research pointwise semi-slant warped product submanifolds in nearly para-Kaehler manifolds in this paper.

This article is organized as follows. In section 2, we recall some fundamental notins for the paper. In section 3, we introduce pointwise semi-slant submanifolds of nearly para-Kaehler manifold and give some examples. In section 4, we introduce pointwise semi-slant non-trivial warped product submanifolds in nearly para-Kaehler manifold. We also provide an example. In section 5, we obtain an inequality in terms of the second fundamental form.

2. Preliminaries

Let $(\bar{\mathcal{M}}, \breve{g})$ be a $\mathtt{2n}$-dimensional semi-Riemannian manifold. If there is a tensor field $\mathcal{P}$ of type $(\mathtt{1},\mathtt{1})$ on $\bar{\mathcal{M}}$, such that

$$\breve{g}(\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{Y}_b) = -\breve{g}(\mathcal{X}_a, \mathcal{Y}_b), \quad \mathcal{P}^2\mathcal{X}_a = \mathcal{X}_a \tag{2.1}$$

for any vector fields $\mathcal{X}_a, \mathcal{Y}_b$ on $\bar{\mathcal{M}}$, it is said a para-Hermitian manifold. In addition, it is called to be para-Kaehler manifold, if it satisfies $\bar{\nabla}\mathcal{P} = \mathtt{0}$ identically [17].

Let $\mathcal{T}\bar{\mathcal{M}}$ be the tangent bundle of $\bar{\mathcal{M}}$ and $\bar{\nabla}$, the covariant differential operator on $\bar{\mathcal{M}}$ with respect to $\breve{g}$. If

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{X}_a = 0 \tag{2.2}$$

for any $\mathcal{T}\bar{\mathcal{M}}$, then an almost para Hermitian manifold is called nearly para-Kaehler structure. Equation (2) is equivalent to

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b + (\bar{\nabla}_{\mathcal{Y}_b}\mathcal{P})\mathcal{X}_a = 0 \tag{2.3}$$

for any vector fields $\mathcal{X}_a, \mathcal{Y}_b$ on $\bar{\mathcal{M}}$

Let $\mathcal{M}$ be a submanifold of $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$. The Gauss and Weingarten equations are

$$\bar{\nabla}_{\mathcal{X}_\mathtt{a}}\mathcal{Y}_\mathtt{b} = \nabla_{\mathcal{X}_\mathtt{a}}\mathcal{Y}_\mathtt{b} + \breve{h}(\mathcal{X}_\mathtt{a}, \mathcal{Y}_\mathtt{b}), \tag{2.4}$$

$$\bar{\nabla}_{\mathcal{X}_\mathtt{a}}\mathcal{V}_c = -A_{\mathcal{V}_c}\mathcal{X}_\mathtt{a} + \nabla^{\perp}_{\mathcal{X}_\mathtt{a}}\mathcal{V}_c, \tag{2.5}$$

for $\mathcal{X}_\mathtt{a}, \mathcal{Y}_\mathtt{b} \in \Gamma(\mathcal{T}\mathcal{M})$ and $\mathcal{V}_c \in \Gamma(\mathcal{T}\mathcal{M}^{\perp})$, that $\breve{h}$ is the second fundamental form of $\mathcal{M}$, $A_{\mathcal{V}_c}$ is the Weingarten endomorphism with $\mathcal{V}_c$ and $\nabla^{\perp}$ is the normal connection. $A_{\mathcal{V}_c}$ and $\breve{h}$ are related by

$$\breve{g}(A_{\mathcal{V}_c}\mathcal{X}_a, \mathcal{Y}_b) = \breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{V}_c), \tag{2.6}$$

here $\breve{g}$ states the semi-Riemannian metric on $\mathcal{M}$. For any tangent vector field $\mathcal{X}_a$, we denote

$$\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a + S\mathcal{X}_a, \tag{2.7}$$

that $R\mathcal{X}_a$ is the tangential part of $\mathcal{P}\mathcal{X}_a$ and $S\mathcal{X}_a$ is the normal part.
For any normal vector field $\mathcal{V}_c$,

$$\mathcal{P}\mathcal{V}_c = r\mathcal{V}_c + s\mathcal{V}_c, \tag{2.8}$$

that $r\mathcal{V}_c$ and $s\mathcal{V}_c$ are the tangential and normal vectors of $\mathcal{P}\mathcal{V}_c$.

Now, denote by $\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b$ and $\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b$ the tangential and normal parts of $(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b$, i.e.,

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b = \mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, \tag{2.9}$$

for any $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{TM}_x)$. Using of (2.7), (2.8), (2.9) and the Weingarten and Gauss formulas, we obtain

$$\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b = (\bar{\nabla}_{\mathcal{X}_a}\mathcal{R})\mathcal{Y}_b - \mathcal{A}_{\mathcal{S}\mathcal{Y}_b}\mathcal{X}_a - r\breve{h}(\mathcal{X}_a, \mathcal{Y}_b) \tag{2.10}$$

and

$$\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b = (\bar{\nabla}_{\mathcal{X}_a}\mathcal{S})\mathcal{Y}_b - \breve{h}(\mathcal{X}_a, \mathcal{R}\mathcal{Y}_b) - s\breve{h}(\mathcal{X}_a, \mathcal{Y}_b). \tag{2.11}$$

Similarly, for any $\mathcal{V}_c \in \mathcal{T}^{\perp}\mathcal{M}$, denote the tangential and normal parts of $(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b$ by $\mathcal{G}_{\mathcal{V}_c}\mathcal{Y}_b$ and $\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b$ respectively, we get

$$\mathcal{G}_{\mathcal{X}_a}\mathcal{V}_c = (\bar{\nabla}_{\mathcal{X}_a}r)\mathcal{V}_c + \mathcal{R}\mathcal{A}_{\mathcal{V}_c}\mathcal{X}_a - \mathcal{A}_{s\mathcal{V}_c}\mathcal{X}_a \tag{2.12}$$

and

$$\mathcal{U}_{\mathcal{X}_a}\mathcal{V}_c = (\bar{\nabla}_{\mathcal{X}_a}s)\mathcal{V}_c + \breve{h}(r\mathcal{V}_c, \mathcal{X}_a) + \mathcal{S}\mathcal{A}_{\mathcal{V}_c}\mathcal{X}_a \tag{2.13}$$

where the covariant derivative of $\mathcal{R}, \mathcal{S}, r, s$ are defined by

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R})\mathcal{Y}_b = \nabla_{\mathcal{X}_a}\mathcal{R}\mathcal{Y}_b - \mathcal{R}\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \quad (\bar{\nabla}_{\mathcal{X}_a}\mathcal{S})\mathcal{Y}_b = \nabla^{\perp}_{\mathcal{X}_a}\mathcal{S}\mathcal{Y}_b - \mathcal{S}\nabla_{\mathcal{X}_a}\mathcal{Y}_b,$$

$$(\bar{\nabla}_{\mathcal{X}_a}r)\mathcal{V}_c = \nabla_{\mathcal{X}_a}r\mathcal{V}_c - r\nabla^{\perp}_{\mathcal{X}_a}\mathcal{V}_c, \quad (\bar{\nabla}_{\mathcal{X}_a}s)\mathcal{V}_c = \nabla^{\perp}_{\mathcal{X}_a}s\mathcal{V}_c - s\nabla^{\perp}_{\mathcal{X}_a}\mathcal{V}_c$$

for any $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{TM}$ and $\mathcal{V}_c \in \Gamma(\mathcal{TM}^{\perp})$

For the proporties of $\mathcal{G}$ and $\mathcal{U}$ we refer [18], which we express here for later use.

$$(m_1) \quad (a) \quad \mathcal{G}_{\mathcal{X}_a+\mathcal{Y}_b}\mathcal{W}_c = \mathcal{G}_{\mathcal{X}_a}\mathcal{W}_c + \mathcal{G}_{\mathcal{Y}_b}\mathcal{W}_c$$
$$(b) \quad \mathcal{U}_{\mathcal{X}_a+\mathcal{Y}_b}\mathcal{W}_c = \mathcal{U}_{\mathcal{X}_a}\mathcal{W}_c + \mathcal{U}_{\mathcal{Y}_b}\mathcal{W}_c$$

$$(m_2) \quad (a) \quad \mathcal{G}_{\mathcal{X}_a}(\mathcal{Y}_b + \mathcal{W}_c) = \mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{G}_{\mathcal{X}_a}\mathcal{W}_c$$
$$(b) \quad \mathcal{U}_{\mathcal{X}_a}(\mathcal{Y}_b + \mathcal{W}_c) = \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{W}_c$$

$$(m_3) \quad (a) \quad \breve{g}(\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{W}_c) = -\breve{g}(\mathcal{Y}_b, \mathcal{A}_{\mathcal{X}_a}\mathcal{W}_c)$$
$$(b) \quad \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{V}_f) = -\breve{g}(\mathcal{Y}_b, \mathcal{G}_{\mathcal{X}_a}\mathcal{V}_f)$$

$$(m_4) \quad \mathcal{G}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b = -\mathcal{P}(\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b)$$

for any $\mathcal{X}_a, \mathcal{Y}_b, \mathcal{W}_c \in \Gamma(\mathcal{TM}_x)$ and $V_f \in \Gamma(\mathcal{TM}_b^{\perp})$

On a nearly para-Kaehler manifold $\bar{\mathcal{M}}_x$. by equations (2.2) and (2.9), we get

$$(a)\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{G}_{\mathcal{Y}_b}\mathcal{X}_a = 0 \quad (b)\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{Y}_b}\mathcal{X}_a = 0 \tag{2.14}$$

for any $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{TM}_x)$

The mean curvature vector field is defined by

$$H = \frac{1}{n} trace\breve{h}. \tag{2.15}$$

We now introduce the following notions in a nearly para-Kaehler manifolds.

Definition 2.1. *We call that a submanifold $\mathcal{M}$ in a nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$ is pointwise slant, if for all timelike or spacelike tangent vector field $\mathcal{X}_a$, the ratio $\breve{g}(R\mathcal{X}_a, R\mathcal{X}_a)/\breve{g}(\mathcal{P}\mathcal{X}_a, P\mathcal{X}_a)$ is a function. Moreover, a submanifold $\mathcal{M}$ of nearly para-Kaehler manifold $\bar{\mathcal{M}}$ is said pointwise slant [13], if at each point $\mathtt{p} \in \mathcal{M}$, the Wirtinger angle $\mathcal{P}X$ between $\theta(X)$ and $\mathcal{T}_p\mathcal{M}$ is dependent of the choice of the non-zero $X \in \mathcal{T}_p\mathcal{M}$. In this instance, the Wirtinger angle causes a real-valued function $\theta : \mathcal{TM} - \{0\} \to \mathtt{R}$ which is said slant function or the Wirtinger function.*

It is easy to see that a pointwise slant submanifold in nearly para-Kaehler manifold is said slant, if its Wirtinger function α is globally constant. Also we notice that all slant submanifolds are pointwise slant submanifolds.

If $\mathcal{M}$ is a para-complex (para-holomorphic) submanifold, in that case, $\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a$ and the above ratio is equal to 1. Moreover if $\mathcal{M}$ is totaly real (anti-invariant), then $R = 0$, so $\mathcal{P}\mathcal{X}_a = S\mathcal{X}_a$ and the above ratio equals 0. Hence, both totally real and para-complex submanifolds are the private situations of pointwise slant submanifolds. Neither totally real nor para-complex pointwise slant submanifold can be called a proper pointwise slant. These manifolds are proper manifolds.

Definition 2.2. *Let $\mathcal{M}$ be a proper pointwise slant submanifold in nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$. We call that it is of*
type-1 if for any space-like or time-like vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is time-like or space-like, and $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} > 1$.
type-2 if for any space-like or time-like vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is time-like or space-like, and $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} < 1$.

Similar to the way of P. Alegre and A. Carriazo used [4], the following theorem and results are obtained.

Theorem 2.1. *Let $\mathcal{M}$ be a pointwise slant submanifold in nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$. So,*
(a) *$\mathcal{M}$ is pointwise slant submanifold of type-1 if and only if for any spacelike or timelike vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike or spacelike, also arise a function $\mu \in (1, +\infty)$. Therefore,*

$$R^2 = \mu Id. \tag{2.16}$$

If θ indicates the slant function of $\mathcal{M}$, $\mu = \cosh^2\theta$.
(b) *$\mathcal{M}$ is pointwise slant submanifold of type-2 if and only if for any spacelike or timelike vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike or spacelike, also arise a function $\mu \in (0, 1)$. Therefore,*

$$R^2 = \mu Id. \tag{2.17}$$

If θ indicates the slant function of $\mathcal{M}$, $\mu = \cos^2\theta$.

Proof. Firstly, if $\mathcal{M}$ is the pointwise slant submanifold of type-1 for any spacelike tangent vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike and by the equation of (2.1), $\mathcal{P}\mathcal{X}_a$ is too. Furthermore, they supply $|R\mathcal{X}_a|/|\mathcal{P}\mathcal{X}_a| > 1$. Therefore, it follows that the slant function θ. Because of,

$$\cosh\theta = \frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} = \frac{\sqrt{-\breve{g}(R\mathcal{X}_a, R\mathcal{X}_a))}}{\sqrt{-\breve{g}(\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a)}}. \tag{2.18}$$

Using (2.1) and (2.18), we have

$$\breve{g}(R^2\mathcal{X}_a, \mathcal{X}_a) = \cosh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{X}_a).$$

Thus, we get $R^2\mathcal{X}_a = \mathcal{X}_a I$. So, from (2.18), we get $\mu = \cosh^2\theta$.

In a similar method for any timelike tangent vector field $\mathcal{Z}$, now, $R\mathcal{Z}$ and $\mathcal{P}\mathcal{Z}$ are spacelike and therefore, instead of (2.18) we get

$$\cosh\theta = \frac{|R\mathcal{Z}|}{|\mathcal{P}\mathcal{Z}|} = \frac{\sqrt{\breve{g}(R\mathcal{Z}, R\mathcal{Z}))}}{\sqrt{\breve{g}(\mathcal{P}\mathcal{Z}, \mathcal{P}\mathcal{Z})}}.$$

Because of $R^2\mathcal{Z} = \mu\mathcal{Z}$, for any spacelike and timelike $\mathcal{Z}$ it further provides for lightlike vector fields and therefore we get $R^2 = \mu Id$. The converse is (a) direct calculation.
Similarly, we have (b).

Finally, for both pointwise slant submanifolds of type-1 and type-2, if $\mathcal{X}_a$ is space-like, in that case, $\mathcal{P}\mathcal{X}_a$ is timelike. Thus, all pointwise slant submanifold of type-1 and type-2 should be a neutral semi-Riemann manifold.

□

Using (2.1),(2.7) and Theorem 2.1, we obtain the following corollary.

Corollary 2.4. Let $\mathcal{M}$ be a pointwise slant submanifold of a nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$ with the slant function θ. For any non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{M})$, we obtain:
If $\mathcal{M}$ is of type-1, then

$$\breve{g}(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cosh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b), \;\; \breve{g}(S\mathcal{X}_a, S\mathcal{Y}_b) = \sinh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b). \tag{2.19}$$

If $\mathcal{M}$ is of type-2, then

$$\breve{g}(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cos^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b), \;\; \breve{g}(S\mathcal{X}_a, S\mathcal{Y}_b) = -\sin^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b). \tag{2.20}$$

Using (2.1),(2.7),(2.8) and Theorem 2.1, we get the following corollary.

Corollary 2.1. *Let $\mathcal{M}$ be a pointwise slant submanifold in nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$. $\mathcal{M}$ is a pointwise slant submanifold of*
**type-1 if and only if $rS\mathcal{X}_a = -\sinh^2\theta\mathcal{X}_a$ and $SR\mathcal{X}_a = -sS\mathcal{X}_a$ for all timelike (spacelike) vector field $\mathcal{X}_a$.*
**type-2 if and only if $rS\mathcal{X}_a = \sin^2\theta\mathcal{X}_a$ and $SR\mathcal{X}_a = -sS\mathcal{X}_a$ for all timelike (spacelike) vector field $\mathcal{X}_a$.*

3. Pointwise Semi-Slant Submanifolds in Nearly Para-Kaehler Manifolds

In this section, we introduce and study pointwise semi-slant submanifolds in nearly para-Kaehler manifold. Also we give some examples.

Definition 3.1. *A semi-Riemannian submanifold $\mathcal{M}$ of a nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$ is called pointwise semi-slant submanifold, if there are two orthogonal distributions D^T , D^θ on $\mathcal{M}$ at the point $q \in \mathcal{M}$ such that the following conditions are satisfied.*
1) $\mathcal{TM} = \mathcal{D}^T \oplus \mathcal{D}^\theta$;
2) $\mathcal{D}^T$ is an invariant (para-holomorphic) distribution, $\mathcal{P}\mathcal{D}^T = \mathcal{D}^T$;
3) $\mathcal{D}^\theta$ is a pointwise slant distribution.

Then, we say the θ is the semi-slant function with the pointwise slant distribution $\mathcal{D}^\theta$. The invariant distribution $\mathcal{D}^T$ of a pointwise semi-slant submanifold is a pointwise slant distribution with slant function $\theta = 0$.
In the above definition, if we suppose that the dimensions $a = dim\mathcal{D}^T$ and $b = dim\mathcal{D}^\theta$, then we get
*) If $a = 0$ and θ is globally constant, $\mathcal{M}$ is a slant submanifold.
*) If $a = 0$, $\mathcal{M}$ is a pointwise slant submanifold.
*) If $b = 0$, $\mathcal{M}$ is an invariant submanifold.
*) If $a = 0$ and $\theta = \frac{\pi}{2}$, $\mathcal{M}$ is an anti-invariant submanifold.
*) If $a \neq 0$ and θ is constant on $\mathcal{M}$, $\mathcal{M}$ is a semi-slant submanifold.
*) If $a \neq 0$, $b \neq 0$ and $\theta = \frac{\pi}{2}$, $\mathcal{M}$ is a semi invariant submanifold.
A pointwise semi-slant submanifold $\mathcal{M}$ is called proper if its semi-slant function satisfies $\theta \neq 0, \frac{\pi}{2}$, also θ is nonconstant on $\mathcal{M}$.

Remark 3.1. *Pointwise slant submanifold is a generalization of slant submanifold.*

Using (1),(5),(6), Theorem 2.1 and Remark 3.1, we have the following result.

Corollary 3.1. *Let $\mathcal{M}$ be a pointwise semi-slant submanifold in nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$ with semi-slant function θ. Then, for any non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{D}^\theta)$, we obtain*
If $\mathcal{M}$ is of type-1, then

$$\breve{g}(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cosh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b),\ \ \breve{g}(S\mathcal{X}_a, S\mathcal{Y}_b) = \sinh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b). \tag{3.21}$$

If $\mathcal{M}$ is of type-2, then

$$\breve{g}(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cos^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b),\ \breve{g}(S\mathcal{X}_a, S\mathcal{Y}_b) = -\sin^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b). \tag{3.22}$$

Now, we give two lemmas for using next section.

Lemma 3.1. *Let $\mathcal{M}$ be a proper pointwise semi-slant type-1-2 submanifold whose ambient spaces are nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$. $\mathcal{D}^\theta$ is slant distribution and $(\mathcal{D}^{\mathrm{T}})$ is holomorphic distribution. Then we get*
1) (for type-1)

$$\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}) = -csch^2\theta\{\breve{g}(\check{h}(\mathcal{X}_a, \mathcal{P}\mathcal{Y}_b), S\mathcal{Z}) - \breve{g}(\check{h}(\mathcal{X}_a, \mathcal{Y}_b), SR\mathcal{Z}) - \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, S\mathcal{Z})\} \tag{3.23}$$

2) (for type-2)

$$\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z})=csc^2\theta\{\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),S\mathcal{Z})-\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Y}_b),SR\mathcal{Z})-\breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z})\} \quad (3.24)$$

for any non-null vector fields $\mathcal{X}_a,\mathcal{Y}_b\in\Gamma(\mathcal{D}^T)$, $\mathcal{Z}\in\Gamma(\mathcal{D}^\theta)$.

Proof. **1)** (for type-1)

$$\begin{aligned}\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}) &= -\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{P}\mathcal{Z})\\ &= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b,\mathcal{P}\mathcal{Z})+\breve{g}((\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b,\mathcal{P}\mathcal{Z})\end{aligned}$$

By using (7),(8) and (9), we get

$$\begin{aligned}\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}) &= \breve{g}(\mathcal{P}\mathcal{Y}_b,\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Z})+\breve{g}(\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b,R\mathcal{Z})+\breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z})\\ &= \breve{g}(\mathcal{Y}_b,\mathcal{P}\bar{\nabla}_{\mathcal{X}_a}R\mathcal{Z})+\breve{g}(\mathcal{P}\mathcal{Y}_b,\bar{\nabla}_{\mathcal{X}_a}S\mathcal{Z})-\breve{g}(\mathcal{Y}_b,\mathcal{G}_{\mathcal{X}_a}R\mathcal{Z})\\ &+ \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z})\\ &= \breve{g}(\mathcal{Y}_b,\bar{\nabla}_{\mathcal{X}_a}R^2\mathcal{Z})-\breve{g}(\mathcal{Y}_b,\bar{\nabla}_{\mathcal{X}_a}SR\mathcal{Z})+\breve{g}(\mathcal{Y}_b,(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})S\mathcal{Z})\\ &- \breve{g}(\mathcal{P}\mathcal{Y}_b,A_{S\mathcal{Z}}\mathcal{X}_a)-\breve{g}(\mathcal{Y}_b,\mathcal{G}_{\mathcal{X}_a}R\mathcal{Z})+\breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z}).\end{aligned}$$

By using (9),(4),(5),(6),(16) and (17) we get

$$\begin{aligned}\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}) &= -cosh^2\theta\breve{g}(\mathcal{Y}_b,\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z})+\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Y}_b),SR\mathcal{Z})\\ &- \breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),S\mathcal{Z})+\breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z})\\ &= cosh^2\theta\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z})+\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Y}_b),SR\mathcal{Z})\\ &- \breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),S\mathcal{Z})+\breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z}).\end{aligned}$$

From the above relation, we get (1) and using similar method, we obtain (2). □

Also, we find the following result.

Corollary 3.2. *Let* $\mathcal{M}$ *be a proper pointwise semi-slant type-1,2 submanifold in nearly para-Kaehler manifold* $\bar{\mathcal{M}}$. *Holomorphic distribution* $\mathcal{D}^{\mathrm{T}}$ *defines a totally geodesic foliation if and only if*

$$-A_{S\mathcal{Z}}\mathcal{P}\mathcal{X}_a+A_{SR\mathcal{Z}}\mathcal{X}_a+\mathcal{U}_{\mathcal{X}_a}S\mathcal{Z}\in\mathcal{D}^\theta$$

for any non-null vector fields $\mathcal{X}_a\in\Gamma(\mathcal{D}^T)$ *and* $\mathcal{Z}\in\Gamma(\mathcal{D}^\theta)$.

Proof. By using (23), (24) and m_3 (b), we get corollary. □

Lemma 3.2. *Let* $\mathcal{M}$ *be a proper pointwise semi-slant type-1-2 submanifold in nearly para-Kaehler manifold* $(\bar{\mathcal{M}},\mathcal{P},\breve{g})$. *The distribution* $\mathcal{D}^{\mathrm{T}}$ *is holomorphic distribution and distribution* $\mathcal{D}^\theta$ *is slant distribution. Then we get*
1) (for type-1)

$$\begin{aligned}-\sinh^2\theta\breve{g}([\mathcal{Z},W],\mathcal{X}_a) &= \breve{g}(A_{S\mathcal{Z}}\mathcal{P}\mathcal{X}_a-A_{SR\mathcal{Z}}\mathcal{X}_a,W)-\breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Z},SW)\\ &+ \breve{g}(\mathcal{U}_{\mathcal{X}_a}W,S\mathcal{Z})-\breve{g}(A_{SW}\mathcal{P}\mathcal{X}_a-A_{SRW}\mathcal{X}_a,\mathcal{Z}),\end{aligned}$$

2)(for type-2)

$$\begin{aligned}\sin^2\theta\breve{g}([\mathcal{Z},W],\mathcal{X}_a) &= \breve{g}(A_{S\mathcal{Z}}\mathcal{P}\mathcal{X}_a - A_{SR\mathcal{Z}}\mathcal{X}_a, W) - \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Z}, SW)\\ &+ \breve{g}(\mathcal{U}_{\mathcal{X}_a}W, S\mathcal{Z}) - \breve{g}(A_{SW}\mathcal{P}\mathcal{X}_a - A_{SRW}\mathcal{X}_a, \mathcal{Z}),\end{aligned}$$

for any non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{D}^T)$ *and* $\mathcal{Z}, W \in \Gamma(\mathcal{D}^\theta)$.

Proof. 1)(for type-1)

$$\breve{g}([\mathcal{Z},\mathcal{W}],\mathcal{X}_a) = -\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a) + \breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{W}}\mathcal{Z},\mathcal{X}_a) \tag{3.25}$$

By using the two terms in the right hand side of (25), we obtain

$$\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a) = \breve{g}(\bar{\nabla}_{\mathcal{Z}}\mathcal{P}\mathcal{W},\mathcal{X}_a) - \breve{g}((\bar{\nabla}_{\mathcal{Z}}\mathcal{P})\mathcal{W},\mathcal{X}_a)$$

By using (7),(8) and (9), we have

$$\begin{aligned}\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a) &= \breve{g}(\bar{\nabla}_{\mathcal{Z}}R\mathcal{W},\mathcal{X}_a) + \breve{g}(\bar{\nabla}_{\mathcal{Z}}S\mathcal{W},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a)\\ &= -\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{Z}}R\mathcal{W},\mathcal{P}\mathcal{X}_a) - \breve{g}(A_{S\mathcal{W}}\mathcal{Z},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a)\\ &= cosh^2\theta\breve{g}(\bar{\nabla}_{\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a) - \breve{g}(\bar{\nabla}_{\mathcal{Z}}SR\mathcal{W},\mathcal{P}\mathcal{X}_a)\\ &+ \breve{g}((\bar{\nabla}_{\mathcal{Z}}\mathcal{P})R\mathcal{W},\mathcal{P}\mathcal{X}_a) - \breve{g}(A_{S\mathcal{W}}\mathcal{Z},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a)\\ &= cosh^2\theta\breve{g}(\bar{\nabla}_{\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a) + \breve{g}(A_{SR\mathcal{W}}\mathcal{Z},\mathcal{P}\mathcal{X}_a)\\ &+ \breve{g}(\mathcal{G}_{\mathcal{Z}}R\mathcal{W},\mathcal{P}\mathcal{X}_a) - \breve{g}(A_{S\mathcal{W}}\mathcal{Z},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a).\end{aligned} \tag{3.26}$$

Interchanging $\mathcal{W}$ and $\mathcal{Z}$ in (26). We have

$$\begin{aligned}\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{W}}\mathcal{Z},\mathcal{X}_a) &= cosh^2\theta\breve{g}(\bar{\nabla}_{\mathcal{W}}\mathcal{Z},\mathcal{P}\mathcal{X}_a) + \breve{g}(A_{SR\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a)\\ &+ \breve{g}(\mathcal{G}_{\mathcal{W}}R\mathcal{Z},\mathcal{P}\mathcal{X}_a) - \breve{g}(A_{S\mathcal{Z}}\mathcal{W},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{W}}\mathcal{Z},\mathcal{X}_a).\end{aligned} \tag{3.27}$$

By using (25),(26) and (27), we get

$$\begin{aligned}-sinh^2\theta\breve{g}([\mathcal{Z},\mathcal{W}],\mathcal{X}_a) &= -\breve{g}(A_{SR\mathcal{W}}\mathcal{Z},\mathcal{P}\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{Z}}R\mathcal{W},\mathcal{P}\mathcal{X}_a)\\ &+ \breve{g}(A_{S\mathcal{W}}\mathcal{Z},\mathcal{X}_a) + \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a)\\ &+ \breve{g}(A_{SR\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a) + \breve{g}(\mathcal{G}_{\mathcal{W}}R\mathcal{Z},\mathcal{P}\mathcal{X}_a)\\ &- \breve{g}(A_{S\mathcal{Z}}\mathcal{W},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{W}}\mathcal{Z},\mathcal{X}_a).\end{aligned}$$

By using the symmetry property of the shape operator and interchanging $\mathcal{X}$ and $\mathcal{P}\mathcal{X}_a$ for any $\mathcal{X}_a \in \mathcal{D}^T$, we get

$$\begin{aligned}-sinh^2\theta\breve{g}_1([\mathcal{Z},\mathcal{W}],\mathcal{X}_a) &= -\breve{g}(A_{SR\mathcal{W}}\mathcal{X}_a,\mathcal{Z}) - \breve{g}(\mathcal{G}_{\mathcal{Z}}R\mathcal{W},\mathcal{X}_a)\\ &- \breve{g}(A_{S\mathcal{W}}\mathcal{P}\mathcal{X}_a,\mathcal{Z}) - \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a)\\ &+ \breve{g}(A_{SR\mathcal{Z}}\mathcal{X}_a,\mathcal{W}) + \breve{g}(\mathcal{G}_{\mathcal{W}}R\mathcal{Z},\mathcal{X}_a)\\ &+ \breve{g}(A_{S\mathcal{Z}}\mathcal{P}\mathcal{X}_a,\mathcal{W}) + \breve{g}(\mathcal{G}_{\mathcal{W}}\mathcal{Z},\mathcal{P}\mathcal{X}_a).\end{aligned} \tag{3.28}$$

Also, by using m_4 and m_3 (b), we find

$$\breve{g}(\mathcal{G}_{\mathcal{W}}\mathcal{Z},\mathcal{P}\mathcal{X}_a)-\breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a) = -\breve{g}(\mathcal{G}_{\mathcal{W}}R\mathcal{Z},\mathcal{X}_a)+\breve{g}(\mathcal{U}_{\mathcal{W}}\mathcal{X}_a,S\mathcal{Z}) + \breve{g}(\mathcal{G}_{\mathcal{Z}}R\mathcal{W},\mathcal{X}_a)-\breve{g}(\mathcal{U}_{\mathcal{Z}}\mathcal{X}_a,S\mathcal{W}). \quad (3.29)$$

By using (2.14) and from (3.28), (3.29), we get proof.
Also, for type-2 the proof is obtained using the same method. □

4. Geometry of Pointwise Semi-Slant Warped Product Submanifolds in Nearly Para-Kaehler Manifolds

Let $(\mathcal{L},\breve{g}_1)$ and $(\mathcal{E},\breve{g}_2)$ be two semi-Riemannian submanifold, $k:\mathcal{L}\to(0,\infty)$ and $q:\mathcal{L}\times\mathcal{E}\to\mathcal{L}$, $a:\mathcal{L}\times\mathcal{E}\to\mathcal{E}$ the projection maps obtained by $q(t,p)=t$, $a(t,p)=p$ for all $(t,p)\in\mathcal{L}\times\mathcal{E}$. The warped product $\mathcal{M}=\mathcal{L}\times_k\mathcal{E}$ is the manifold $\mathcal{L}\times\mathcal{E}$ with the semi-Riemannian constructure. In that case,

$$\breve{g}(\mathcal{X}_a,\mathcal{Y}_b)=\breve{g}_1(q_*\mathcal{X}_a,q_*\mathcal{Y}_b)+(k\circ q)^2\breve{g}_2(q_*\mathcal{X}_a,q_*\mathcal{Y}_b)$$

for every $\mathcal{X}_a$ and $\mathcal{Y}_b$ of $\mathcal{M}$ where $*$ denotes the tangent map [9]. The function k is called the warping function. Especially, if the warping function is constant, $\mathcal{M}$ is called to be trivial.

For $\mathcal{X}_a$, $\mathcal{Y}_b$ on $\mathcal{L}$ and $\mathcal{V}_c$, $\mathcal{W}_d$ vector fields on $\mathcal{E}$. Later, using Lemma 7.3 of [9], we obtain

$$\nabla_{\mathcal{X}_a}\mathcal{V}_c=\nabla_{\mathcal{V}_c}\mathcal{X}_a=\mathcal{V}_c(lnk) \quad (4.30)$$

where ∇ is the Levi-Civita connection on $\mathcal{K}$.

Theorem 4.1. *Let $\bar{\mathcal{M}}$ be a nearly para-Kaehler manifold. Then, there don't exist pointwise semi-slant non-trivial warped product type 1-2 submanifolds $\mathcal{M}=\mathcal{M}^\theta\times_k\mathcal{M}^T$ in nearly para-Kaehler manifold $\bar{\mathcal{M}}$.*

Proof. For type-1, using (3.22), (2.1) (2.2), (2.5), (2.6) and (2.7), we get

$$\begin{aligned}\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{Y}_b) &= \breve{g}(\nabla_{\mathcal{X}_a}\mathcal{V}_c,\mathcal{Y}_b)=\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{V}_c,\mathcal{Y}_b)\\ &= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{V}_c,\mathcal{P}\mathcal{Y}_b)\\ &= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}^2\mathcal{V}_c+\mathcal{S}\mathcal{R}\mathcal{V}_c,\mathcal{Y}_b)+\breve{g}(A_{\mathcal{S}\mathcal{V}_c}\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b).\end{aligned}$$

From (Theorem 3.3.) we obtain

$$\begin{aligned}\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{Y}_b) &= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\cosh^2\theta\mathcal{V}_c,\mathcal{Y}_b)+\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{R}\mathcal{V}_c,\mathcal{Y}_b)+\breve{g}(A_{\mathcal{S}\mathcal{V}_c}\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b)\\ &= \sinh 2\theta\mathcal{X}_a(\theta)\breve{g}(\mathcal{V}_c,\mathcal{Y}_b)\cosh^2\theta\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{V}_c,\mathcal{Y}_b)\\ &- \breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Y}_b),\mathcal{S}\mathcal{R}\mathcal{V}_c)+\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),\mathcal{S}\mathcal{V}_c).\end{aligned}$$

Since $\mathcal{D}^\theta$ and $\mathcal{D}^T$ are orthogonal, using (3.22), we get

$$-\sinh^2\theta\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{Y}_b)=-\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Y}_b),\mathcal{S}\mathcal{R}\mathcal{V}_c)+\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),\mathcal{S}\mathcal{V}_c).$$

In above equation interchanging $\mathcal{X}_a$ and $\mathcal{Y}_b$, we get

$$-\sinh^2\theta\mathcal{V}_c(lnk)\breve{g}(\mathcal{Y}_b,\mathcal{X}_a) = -\breve{g}(\breve{h}(\mathcal{Y}_b,\mathcal{X}_a),\mathcal{S}\mathcal{R}\mathcal{V}_c) + \breve{g}(\breve{h}(\mathcal{Y}_b,\mathcal{P}\mathcal{X}_a),\mathcal{S}\mathcal{V}_c).$$

If we substract the last two equations from each other, we have

$$\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),\mathcal{S}\mathcal{V}_c) = \breve{g}(\breve{h}(\mathcal{Y}_b,\mathcal{P}\mathcal{X}_a),\mathcal{S}\mathcal{V}_c) \tag{4.31}$$

$$\begin{aligned}\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),\mathcal{S}\mathcal{V}_c) &= \breve{g}(\bar{\nabla}_{\mathcal{X}a}\mathcal{P}\mathcal{Y}_b,\mathcal{S}\mathcal{V}_c)\\ &= \breve{g}(P\bar{\nabla}_{Xa}\mathcal{Y}_b,\mathcal{P}\mathcal{V}_c) - \breve{g}(P\bar{\nabla}_{Xa}\mathcal{Y}_b,\mathcal{R}\mathcal{V}_c)\\ &- \breve{g}(\nabla_{Xa}\mathcal{V}_c,\mathcal{Y}_b) + \breve{g}(\nabla_{Xa}\mathcal{R}\mathcal{V}_c,\mathcal{P}\mathcal{Y}_b).\end{aligned}$$

Using (3.22), we get

$$\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),\mathcal{S}\mathcal{V}_c) = \mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{Y}_b) + \mathcal{R}\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b). \tag{4.32}$$

Using (23), (1), Theorem 2.1 and for $\mathcal{V}_c = \mathcal{R}\mathcal{V}_c$, $\mathcal{X}_a = \mathcal{P}\mathcal{X}_a$ we have

$$\begin{aligned}0 &= \mathcal{R}\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b)\\ &= \mathcal{R}^2\mathcal{V}_c(lnk)\breve{g}(\mathcal{P}\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b)\\ &= -\cosh^2\theta\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{Y}_b).\end{aligned}$$

Because of $\mathcal{V}_c(lnk) = 0$, lnk is constant. Proof is complete. Also for type-2, we use in a similar way. □

Remark 4.1. *We express that Theorem (4.1) is a generalization of Theorem (3.1) in* [22] *and Theorem (4.1) in* [23].

It is clear from the above theorem that there don't exist pointwise semi slant non-trivial warped product submanifolds of the first form $\mathcal{M} = \mathcal{M}^\theta \times_k \mathcal{M}^T$ in nearly para-Kaehler manifolds. Conversely, we demonstrate that there exist of the second form $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ in this part.

Now we write an example with related to the second form $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$.

Let $\mathcal{M}$ be a semi-Riemannian submanifold of $\bar{K}^{20}_{10}$ described by the immersion $\psi : \mathcal{M} \to \bar{K}^{20}_{10}$ with the cartesian coordinates $(x_1, ..., x_{20})$ and the almost para-complex structure $\mathcal{P}(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_{j-2}}$ $j = (3, 4, 7, 8, 11, 12, 15, 16, 19, 20)$ and $\mathcal{P}(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{i+2}}$ $i = (1, 2, 5, 6, 9, 10, 13, 14, 17, 18)$. Let $\bar{K}^{20}_{10}$ be a semi-Riemannian space of signature $(+,+,-,-,+,+,-,-,+,+,-,-,+,-,-,+,+,-,+,-)$ with the canonical basis $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{20}})$.

Example 4.1. *$\mathcal{M}$ be defined by the immersion ψ as follows*

$$\begin{aligned}\psi(a,b,c,d) &= (a\sin c, a\cos c, b\sin c, b\cos c, a\sin d, a\cos d, b\sin d,\\ &\quad b\cos d, x, 2a, y, 2b, \sqrt{2}d, \sqrt{2}c, c, d, \sqrt{3}c, \sqrt{3}d, x, y)\end{aligned}$$

$$\psi_{\mathtt{a}} = \sin \mathtt{c}\frac{\partial}{\partial x_1} + \cos \mathtt{c}\frac{\partial}{\partial x_2} + \sin d\frac{\partial}{\partial x_5} + \cos d\frac{\partial}{\partial x_6} + 2\frac{\partial}{\partial x_{10}}$$

$$\psi_{\mathtt{b}} = \sin \mathtt{c}\frac{\partial}{\partial x_3} + \cos \mathtt{c}\frac{\partial}{\partial x_4} + \sin d\frac{\partial}{\partial x_7} + \cos d\frac{\partial}{\partial x_8} + 2\frac{\partial}{\partial x_{12}}$$

$$\psi_{\mathtt{c}} = \mathtt{a}\cos \mathtt{c}\frac{\partial}{\partial x_1} - \mathtt{a}\sin \mathtt{c}\frac{\partial}{\partial x_2} + b\cos \mathtt{c}\frac{\partial}{\partial x_3} - b\sin \mathtt{c}\frac{\partial}{\partial x_4} + \sqrt{2}\frac{\partial}{\partial x_{14}} + \frac{\partial}{\partial x_{15}} + \sqrt{3}\frac{\partial}{\partial x_{17}}$$

$$\psi_{d} = \mathtt{a}\cos d\frac{\partial}{\partial x_5} - \mathtt{a}\sin d\frac{\partial}{\partial x_6} + \mathtt{b}\cos d\frac{\partial}{\partial x_7} - \mathtt{b}\sin d\frac{\partial}{\partial x_8} + \sqrt{2}\frac{\partial}{\partial x_{13}} + \frac{\partial}{\partial x_{16}} + \sqrt{3}\frac{\partial}{\partial x_{18}}$$

defines a pointwise semi-slant submanifold $\mathcal{M}$ with type-1,2 in $(\bar{K}^{20}_{10}, \mathcal{P}, \breve{g})$ para-complex manifold with $\mu = \mathcal{R}^2 = \frac{8}{(\mathtt{a}^2-\mathtt{b}^2)(\mathtt{b}^2-\mathtt{a}^2+6))}$ Actually $D^\theta = span\{\psi_{\mathtt{c}}, \psi_{\mathtt{d}}\}$ is pointwise slant distribution and $\mathcal{D}^T = span\{\psi_a, \psi_{\mathtt{b}}\}$ is invariant distribution.
So, we get that $\mathcal{D}^T$ and D^θ distributions are integrable. The induced metric tensor $\breve{g}_{\mathcal{M}}$ on $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ is given by
$\breve{g}_{\mathcal{M}} = 6({d_a}^2 - {d_b}^2) + (\mathtt{a}^2 - \mathtt{b}^2)({d_c}^2 + {d_d}^2)$. *Thus,*
**) if $0 < (\mathtt{a}^2 - \mathtt{b}^2) < 2$ or $6 > (\mathtt{a}^2 - \mathtt{b}^2) > 4$, $\mathcal{M}$ is a pointwise semi-slant non-trivial warped product type-1 submanifold in nearly para-Kaehler manifold $\bar{K}^{20}_{10}$ with warping function $k = \sqrt{(\mathtt{a}^2 - \mathtt{b}^2)}$.*
**) if $2 < (\mathtt{a}^2 - \mathtt{b}^2) < 4$ $\mathcal{M}$ is a pointwise semi-slant non-trivial warped product type-2 submanifold in nearly para-Kaehler manifold $\bar{K}^{20}_{10}$ with warping function $k = \sqrt{(\mathtt{a}^2 - \mathtt{b}^2)}$*

We now give below lemmas for later use.

Lemma 4.1. *Let $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ be a pointwise semi-slant non-trivial warped product type-1,2 submanifold in nearly para-Kaehler manifold $\bar{\mathcal{M}}$. In that case, we get*

(i) $\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{V}_c) = 0$

(ii) $\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{Z}) = (\mathcal{X}_a lnk)cosh^2\theta||\mathcal{Z}||^2$ (for type-1)

$\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{Z}) = (\mathcal{X}_a lnk)cos^2\theta||\mathcal{Z}||^2$ (for type-2)

(iii) $\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{Z}) = -(\mathcal{P}\mathcal{X}_a lnk)cosh^2\theta||\mathcal{Z}||^2$ (for type-1)

$\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{Z}) = -(\mathcal{P}\mathcal{X}_a lnk)cos^2\theta||\mathcal{Z}||^2$ (for type-2)

for $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{D}^T)$ and $\mathcal{V}_c, \mathcal{Z} \in \Gamma(\mathcal{D}^\theta)$.

Proof. Using (2.7), (2.1) and (2.2) we get

$$\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{V}_c) = \breve{g}(\nabla_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \mathcal{V}_c) + \breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{R}\mathcal{V}_c).$$

From (30) and because of $\mathcal{P}\mathcal{Y}_b$ with $\mathcal{V}_c$ and $\mathcal{R}\mathcal{V}_c$ with $\mathcal{Y}_b$ orthogonality, we obtain

$$\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{V}_c) = \mathcal{X}_a(lnk)\breve{g}(\mathcal{P}\mathcal{Y}_b, \mathcal{V}_c) + \mathcal{X}_a(lnk)\breve{g}(\mathcal{Y}_b, \mathcal{R}\mathcal{V}_c) = 0.$$

Proof is complete and we get proof of equation (ii) and (iii) with smilar way.

If we interchange $\mathcal{Z}$ by $\mathcal{RZ}$ in (ii) and (iii), we obtain

$$\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a,\mathcal{RZ}),\mathcal{SRZ}) = (\mathcal{X}_a lnk)cosh^2\theta||\mathcal{Z}||^2(type-1), \tag{4.33}$$

$$\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a,\mathcal{RZ}),\mathcal{SRZ}) = (\mathcal{X}_a lnk)cos^2\theta||\mathcal{Z}||^2(type-2) \tag{4.34}$$

and

$$\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{RZ}),\mathcal{SRZ}) = -(\mathcal{P}\mathcal{X}_a lnk)cosh^2\theta||\mathcal{Z}||^2(type-1), \tag{4.35}$$

$$\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{RZ}),\mathcal{SRZ}) = -(\mathcal{P}\mathcal{X}_a lnk)cos^2\theta||\mathcal{Z}||^2(type-2). \tag{4.36}$$

□

Now, using the above lemma, we get the following results.

Corollary 4.1. *Let $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ be pointwise proper semi-slant warped product type-1,2 submanifold in nearly para-Kaehler manifold $\bar{\mathcal{M}}$. In that case, we obtain*

$$\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{RZ}),\mathcal{SZ}) = -\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Z}),\mathcal{SRZ}) = -\frac{1}{3}(\mathcal{X}_a lnk)cosh^2\theta||\mathcal{Z}||^2(type-1) \tag{4.37}$$

and

$$\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{RZ}),\mathcal{SZ}) = -\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Z}),\mathcal{SRZ}) = -\frac{1}{3}(\mathcal{X}_a lnk)cos^2\theta||\mathcal{Z}||^2(type-2) \tag{4.38}$$

for $\mathcal{X}_a \in \Gamma\mathcal{D}^T$ and $\mathcal{V}_c, \mathcal{Z} \in \Gamma\mathcal{D}^\theta$.

If we replace $\mathcal{X}_a$ by $\mathcal{P}\mathcal{X}_a$ in (37) and (38), we get

$$\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a,\mathcal{RZ}),\mathcal{SZ}) = -\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a,\mathcal{Z}),\mathcal{SRZ}) = -\frac{1}{3}(\mathcal{P}\mathcal{X}_a lnk)cosh^2\theta||\mathcal{Z}||^2(type-1) \tag{4.39}$$

and

$$\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a,\mathcal{RZ}),\mathcal{SZ}) = -\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a,\mathcal{Z}),\mathcal{SRZ}) = -\frac{1}{3}(\mathcal{P}\mathcal{X}_a lnk)cos^2\theta||\mathcal{Z}||^2(type-2). \tag{4.40}$$

Theorem 4.2. *Let $\mathcal{M}$ be a pointwise semi-slant type-1,2 submanifold of nearly para-Kaehler manifold $\bar{\mathcal{M}}$. In that case, $\mathcal{M}$ is locally a non-trivial warped product submanifold $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$, such that, $\mathcal{M}^T$ is a holomorphic submanifold and $\mathcal{M}^\theta$ is a pointwise slant submanifold in $\bar{\mathcal{M}}$ If the following situation is satisfied*
for type-1

$$\mathcal{A}_{\mathcal{SRZ}}\mathcal{X}_a - \mathcal{A}_{\mathcal{SZ}}\mathcal{P}\mathcal{X}_a = (1-\frac{1}{3}\cosh^2\theta)\mathcal{X}_a(\gamma)\mathcal{Z} \tag{4.41}$$

for type-2

$$\mathcal{A}_{\mathcal{SRZ}}\mathcal{X}_a - \mathcal{A}_{\mathcal{SZ}}\mathcal{P}\mathcal{X}_a = (1-\frac{1}{3}\cos^2\theta)\mathcal{X}_a(\gamma)\mathcal{Z} \tag{4.42}$$

where $\gamma = lnk$ is a function on $\mathcal{M}$ so that $\mathcal{Z}(\gamma) = 0$ for any $\mathcal{X}_a \in \Gamma(\mathcal{D}^T)$, $\mathcal{Z} \in \Gamma(\mathcal{D}^\theta)$.

Proof. Let $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ be a proper pointwise semi-slant non-trivial warped product type-1 submanifold in nearly para-Kaehler manifolds $\bar{\mathcal{M}}$. In that case, from (2.1), (2.5), (2.7) and Lemma 4.4, we get

$$\breve{g}(\mathcal{A}_{\mathcal{SZ}}\mathcal{P}\mathcal{X}_a, \mathcal{Y}_b) = 0 \tag{4.43}$$

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{Z}}\mathcal{P}\mathcal{X}_a, \mathcal{Z}) = (\mathcal{X}_a\gamma)||\mathcal{Z}||^2 \tag{4.44}$$

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{Z}}\mathcal{X}_a, \mathcal{Z}) = \frac{1}{3}(\mathcal{X}_a\gamma)\cosh^2\theta||\mathcal{Z}||^2 \quad (type-1) \tag{4.45}$$

$\mathcal{V}_c, \mathcal{Z} \in \Gamma(\mathcal{D}^\theta)$ and $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{D}^T)$ which specifies that $\mathcal{A}_{\mathcal{S}\mathcal{Z}}\mathcal{P}\mathcal{X}_a$ with related to $\mathcal{D}^\theta$. On the contrary, accept that $\mathcal{M}$ is a pointwise semi-slant type-1 submanifold of nearly para-Kaehler manifold $\bar{\mathcal{M}}$ and using (44) and (45), we get

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{Z}}\mathcal{X}_a - \mathcal{A}_{\mathcal{S}\mathcal{Z}}\mathcal{P}\mathcal{X}_a) = (1 - \frac{1}{3}\cosh^2\theta)\mathcal{X}_a(\gamma)\mathcal{Z}. \tag{4.46}$$

So, we get (4.41). Then from Lemma 3.1 (2), $\mathcal{D}^\theta$ is integrable and from Lemma 3.2 (1), $\mathcal{D}^T$ is totally geodesic. Let $\mathcal{M}^\theta$ be the integral manifold of $\mathcal{D}^\theta$. Because of Weingarten operator $A_{\mathcal{N}}$ is self-adjoint and using (2.1),(2.2),(2.5) and (2.7) we have

$$\begin{aligned}
\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{V}_c}\mathcal{X}_a - \mathcal{A}_{\mathcal{S}\mathcal{V}_c}\mathcal{P}\mathcal{X}_a, \mathcal{Z}) &= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{R}\mathcal{V}_c, \mathcal{Z}) - \breve{g}(\bar{\nabla}_{\mathcal{P}\mathcal{X}_a}\mathcal{S}\mathcal{V}_c, \mathcal{Z}) \\
&+ \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{S}\mathcal{Z} \\
&= -\breve{g}(\mathcal{X}_a, \bar{\nabla}_{\mathcal{Z}}\mathcal{P}\mathcal{S}\mathcal{V}_c) \\
&= -\breve{g}(\mathcal{X}_a, \nabla_{\mathcal{Z}}\mathcal{R}^2\mathcal{V}_c) - \breve{g}(\mathcal{X}_a, \nabla_{\mathcal{Z}}\mathcal{V}_c).
\end{aligned}$$

Using (2.18) for type-1 we get

$$\begin{aligned}
\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{V}_c}\mathcal{X}_a - \mathcal{A}_{\mathcal{S}\mathcal{V}_c}\mathcal{P}\mathcal{X}_a, \mathcal{Z}) &= 2\cosh\theta\sinh\theta\mathcal{Z}(\theta)\breve{g}(\mathcal{X}_a, \mathcal{V}_c) \\
&+ (-1 + \cosh^2\theta)\breve{g}(\mathcal{X}_a, \nabla_{\mathcal{Z}}\mathcal{V}_c) \\
&= \sinh^2\theta\breve{g}(\mathcal{X}_a, \nabla_{\mathcal{Z}}\mathcal{V}_c).
\end{aligned}$$

Using (2.2) we get

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{V}_c}\mathcal{X}_a - \mathcal{A}_{\mathcal{S}\mathcal{V}_c}\mathcal{P}\mathcal{X}_a, \mathcal{Z}) = \sinh^2\theta(\mathcal{X}_a, \breve{h}_\theta(\mathcal{V}_c, \mathcal{Z})). \tag{4.47}$$

Then (4.46) indicate that

$$\breve{h}_\theta(\mathcal{V}_c, \mathcal{Z}) = (\frac{1}{3} + \frac{2}{3}cosech^2\theta)\nabla_\gamma\breve{g}(\mathcal{V}_c, \mathcal{Z})$$

which indicate that $\mathcal{M}^\theta$ is a totaly umbilical submanifold in $\mathcal{M}$ with the mean curvature vector field $(\frac{1}{3} + \frac{2}{3}cosech^2\theta)\nabla_\gamma$, where ∇_γ is the gradient of γ.
Conversely, by direct calculations, we have

$$\begin{aligned}
\breve{g}(\nabla_{\mathcal{V}_c}\nabla_\gamma, \mathcal{X}_a) &= [\mathcal{V}_c\breve{g}(\nabla_\gamma, \mathcal{X}_{\mathtt{a}}) - \breve{g}(\nabla_\gamma, \nabla_{\mathcal{V}_c}\mathcal{X}_a)] \\
&= [\mathcal{V}_c(X_a(\gamma)) - [\mathcal{V}_c, X_{\mathtt{a}}]\gamma - \breve{g}(\nabla_\gamma, \nabla_{\mathcal{X}_a}\mathcal{V}_c)] \\
&= [\mathcal{V}_c, \mathcal{X}_a]\gamma + \mathcal{X}_a(\mathcal{V}_c(\gamma))[\mathcal{V}_c(X_a(\gamma)) - [\mathcal{V}_c, X_a]\gamma - \breve{g}(\nabla_\gamma, \nabla_{\mathcal{X}_a}\mathcal{V}_c)] \\
&= [\mathcal{X}_a(\mathcal{V}_c(\gamma))[\mathcal{V}_c(\mathcal{X}_a(\gamma)) - \breve{g}(\nabla_\gamma, \nabla_{\mathcal{X}_a}\mathcal{V}_c)].
\end{aligned}$$

Because of $\mathcal{V}_c(\gamma) = 0$, we get

$$\breve{g}(\nabla_{\mathcal{V}_c}\nabla_\gamma, \mathcal{X}_a) = \breve{g}(\nabla_\gamma, \nabla_{\mathcal{X}_a}\mathcal{V}_c).$$

Conversely, since $\nabla_\gamma \in \Gamma(T\mathcal{M}^T)$ and $\mathcal{M}^T$ is totally geodesic in $\mathcal{M}$, it shows that $\nabla_{\mathcal{X}_a}\mathcal{V}_c \in \Gamma(T\mathcal{M}^\theta)$ for $\mathcal{V}_c \in \Gamma(\mathcal{D}^\theta)$, $\mathcal{X}_a \in \Gamma(\mathcal{D}^T)$. So, $\breve{g}(\nabla_{\mathcal{V}_c}\nabla_\gamma, \mathcal{X}_a) = \mathtt{0}$. Then the sphrecial situation is also accomplished, that is $\mathcal{M}^\theta$ is an extrinsic sphere in $\mathcal{M}$. So, proof is complete. Using a similar way, the result is also obtained for type-2. □

5. An Optimal Inequality

We first indicate an orthonormal frame. Let $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ be a $(m + \mathtt{n})$ dimensional pointwise semi-slant warped product submanifold a $(\mathtt{m} + 2n)$-dimensional $\bar{\mathcal{M}}$ nearly para-Kaehler manifold. We give orthonormal frames according to type-1 and type-2. Firstly for type-1, we indicate the orthonormal frames respectively;
$\{\mathtt{E}_1, .., \mathtt{E}_m, \bar{\mathtt{E}}_1, ..., \bar{\mathtt{E}}_n, \mathtt{E}^*_1, ..., \mathtt{E}^*_n\}$ of $\bar{\mathcal{M}}$ so that, restricted to $\mathcal{M}$, $\{\mathtt{E}_1, .., E_{\mathtt{m}}, \bar{E}_1, ..., \bar{\mathtt{E}}_{\mathtt{n}}\}$ are tangent to $\mathcal{M}$. So $\{E_1, .., \mathtt{E}_{\mathtt{m}}, \bar{\mathtt{E}}_1, ..., \bar{E}_{\mathtt{n}}\}$ form an orthonormal frame of $\mathcal{M}$. We can indicate $\{E_1, .., \mathtt{E}_{\mathtt{m}}, \bar{\mathtt{E}}_1, ..., \bar{\mathtt{E}}_{\mathtt{m}}\}$ in such a way that $\{E_1, .., \mathtt{E}_{\mathtt{m}}\}$ form an orthonormal frame of $\mathcal{D}^T$ and $\{\bar{E}_1, ..., \bar{E}_n\}$ form an orthonormal frame of $\mathcal{D}^\theta$, where $dim(\mathcal{D}^T) = m$ and $dim(\mathcal{D}^\theta) = n$. We can indicate $\{\mathtt{E}^*_1, ..., \mathtt{E}^*_{\mathtt{n}}\}$ as an orthonormal frame of $\mathcal{S}(\mathcal{D}^\theta)$. In that case, $n = 2p$ and orthonormal frames are $\{\bar{E}_1, ..., \bar{\mathtt{E}}_{2\mathtt{p}}\}$ of $(\mathcal{D}^\theta)$ and $\{E^*_1, ..., E^*_{2p}\}$ of $\mathcal{S}(\mathcal{D}^\theta)$.

$$\bar{\mathtt{E}}_2 = sech\theta\mathcal{R}\bar{\mathtt{E}}_1, ..., \bar{\mathtt{E}}_{2p} = sech\theta\mathcal{R}\bar{\mathtt{E}}_{2p-1}, \quad (type-1)$$

$$\bar{\mathtt{E}}^*_1 = csch\theta\mathcal{S}\bar{\mathtt{E}}_1, ..., \bar{\mathtt{E}}^*_{2p} = csch\theta\mathcal{S}\bar{\mathtt{E}}_{2p}, \quad (type-1)$$

We assume that
* on $\mathcal{D}^T$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,m}$, where $m = dim(\mathcal{D}^T)$; also, supposed that $\breve{g}(\mathtt{E}_v, \mathtt{E}_v) = 1$.
* on $\mathcal{D}^\theta$: orthonormal basis $\{\mathtt{E}^*_w\}_{w=1,...,n}$. $n = dim(\mathcal{D}^\theta)$ also $\breve{g}(\mathtt{E}^*_w, \mathtt{E}^*_w) = \mp 1$.
* on $\mathcal{P}\mathcal{D}^T$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,m}$, where $d_1 = dim\mathcal{P}(\mathcal{D}^T)$ also $\breve{g}(\mathcal{P}\mathtt{E}_v, \mathcal{P}\mathtt{E}_v) = -1$.
* on $\mathcal{S}\mathcal{D}^\theta$: orthonormal basis $\{\mathtt{E}^*_w\}_{w=1,...,n}$, where $n = dim\mathcal{S}(\mathcal{D}^\theta)$ also $\breve{g}(\mathtt{E}^*_w, \mathtt{E}^*_w) = \mp 1$.

Theorem 5.1. *Let $\mathcal{M}$ be a $(\mathtt{m} + \mathtt{n})$ dimensional pointwise semi-slant type-1 warped product submanifold $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ in nearly para-Kaehler manifold $\bar{\mathcal{M}}^{m+2n}$, where $\mathcal{M}^\theta$ is a proper pointwise slant submanifold and $\mathcal{M}^T$ is a invariant submanifold of $\mathcal{M}$. Assume that $\mathcal{M}^T$ is spacelike. So, we get*
1)

$$||\breve{h}||^2 \leq 4n(csch^2\theta + \frac{1}{9}coth^2\theta)||grad(lnk)||^2, \qquad dim(\mathcal{K}^\theta) = n \tag{5.48}$$

where $grad(lnk)$ is the gradient of lnk.
2) *If the equality sign of (5.48) holds the same way, $\mathcal{M}^\theta$ is totally umbilical and $\mathcal{M}^T$ is totally geodesic in $\bar{\mathcal{M}}$. Also, $\mathcal{M}$ is minimal submanifold of $\bar{\mathcal{M}}$.*

Proof. From description $||\breve{h}_1||^2 = ||\breve{h}_1(D^\theta, D^\theta)||^2 + 2||\breve{h}_1(\mathcal{D}^\theta, \mathcal{D}^T)||^2 + ||\breve{h}_1(\mathcal{D}^T, \mathcal{D}^T)||^2$. We get

$$\begin{aligned} ||\breve{h}||^2 &= \sum_{\mathtt{k}=1}^{\mathtt{m}+2\mathtt{p}} \sum_{v,w=1}^{m} \breve{g}(\breve{h}(\mathtt{E}_v, \mathtt{E}_w), \mathtt{E}^*_{\mathtt{k}})^2 + \sum_{k=1}^{m+2\mathtt{p}} \sum_{r,s=1}^{2\mathtt{p}} \breve{g}(\breve{h}(\bar{\mathtt{E}}_r, \bar{\mathtt{E}}_s), E^*_{\mathtt{k}})^2 \\ &+ 2\sum_{k=1}^{\mathtt{m}+2p} \sum_{r=1}^{2p} \sum_{v=1}^{\mathtt{m}} \breve{g}(\breve{h}(\mathtt{E}_v, \bar{\mathtt{E}}_r), E^*_k)^2 \end{aligned}$$

where $\{E^{*}_{\mathtt{k}}\}$ is an orthonormal basis of $\mathcal{TM}^{\perp}$. Accepted the adapted frame, we will indicate above equation as

$$\begin{aligned} ||\breve{h}||^2 &= \sum_{a=1}^{2p}\sum_{v,w=1}^{2p}\breve{g}(\breve{h}(\mathtt{E}_v,\mathtt{E}_w), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2 + \sum_{a,r,s=1}^{2p}\breve{g}(\breve{h}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_s), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2 \\ &+ 2\sum_{v=1}^{m}\sum_{a,r=1}^{2p}\breve{g}(\breve{h}(\bar{\mathtt{E}}_r,\mathtt{E}_v), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2. \end{aligned}$$

By using((37) and (39), we obtain

$$\begin{aligned} ||\breve{h}||^2 &= \sum_{a,r,s=1}^{2p}\breve{g}(\breve{h}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_s), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2 + 2\sum_{v=1}^{m}\sum_{a,r=1}^{2p}(csch\theta)^2[(\mathcal{P}\mathtt{E}_v(lnk))\breve{g}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_a))^2 \\ &+ 2\mathcal{P}\mathtt{E}_v(lnk)\breve{g}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_a)\mathtt{E}_v(lnk)(\bar{\mathtt{E}}_r,\mathcal{R}\bar{\mathtt{E}}_a) + (\mathtt{E}_v(lnk)\breve{g}(\bar{\mathtt{E}}_r,\mathcal{R}\bar{\mathtt{E}}_a))^2]. \end{aligned}$$

Thus

$$\begin{aligned} &\sum_{v=1}^{m}\sum_{a,r=1}^{2p}\mathcal{P}\mathtt{E}_v(lnk)\breve{g}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_a)\mathtt{E}_v(lnk)\breve{g}(\bar{\mathtt{E}}_r,\mathcal{R}\bar{\mathtt{E}}_a) \\ &= \sum_{v=1}^{m}\sum_{a,r=1}^{2p}\breve{g}(grad(lnk),\mathcal{P}\mathtt{E}_v)\breve{g}(grad(lnk),\mathtt{E}_v)\breve{g}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_a)\breve{g}(\bar{\mathtt{E}}_r,\mathcal{R}\bar{\mathtt{E}}_a) \\ &= -\sum_{a,r=1}^{2p}[\sum_{v=1}^{m}\breve{g}(\breve{g}(grad(lnk),\mathtt{E}_v)\mathtt{E}_v,\mathcal{P}grad(lnk))]\breve{g}_1(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_a)\breve{g}(\bar{\mathtt{E}}_r,\mathcal{R}\bar{\mathtt{E}}_a) = 0. \end{aligned}$$

By using (33), (35) and lemma 4.4, the above equation will be simlified as

$$||\breve{h}_1||^2 = \sum_{a,\mathrm{r},s=1}^{2\mathrm{p}}\breve{g}(\breve{h}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_s), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2 + 4n||grad(lnk)||^2[csch^2\theta + \frac{1}{9}coth^2\theta].$$

So, we get the inequality (48). Also the equality sign of (48) gives, we get

$$\sum_{a=1}^{2p}\sum_{r,s=1}^{2p}\breve{g}(\breve{h}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_s), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2 = 0. \tag{5.49}$$

Since $\mathcal{M}^T$ is a totally geodesic in $\mathcal{M}$, (4.46) equation specifies that $\mathcal{M}^T$ is totally geodesic in $\bar{\mathcal{M}}$. Also, (5.49) equation specifies that $\breve{h}$ vanishes on $\mathcal{D}^\theta$. Because of $\mathcal{D}^\theta$ is a spherical distribution in $\mathcal{M}$, we reach that $\mathcal{M}^\theta$ is a totally umbilicial submanifold of $\bar{\mathcal{M}}$. Also, using (4.46) and (5.49), we reach that $\mathcal{M}$ is minimal in $\bar{\mathcal{M}}$. □

Remark 5.1. *If the manifold $\mathcal{M}^\theta$ in the above theorem is timelike, in that case, (48) should be changed by*

$$||\breve{h}||^2 \geq 4n(csch^2\theta + \frac{1}{9}coth^2\theta)||grad(lnk)||^2, \qquad dim(\mathcal{K}^\theta) = n \tag{5.50}$$

where $grad(lnk)$ is the gradient of lnk.
Similarly, if pointwise slant submanifold $\mathcal{M}^{\theta}$ is type-2 , we achieve.

Theorem 5.2. *Let $\mathcal{M}$ be a $(m+\mathtt{n})$-dimensional pointwise semi-slant warped product submanifold $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^{\theta}$ in nearly para-Kaehler manifold $\bar{\mathcal{M}}^{m+2n}$. Assume that, $\mathcal{M}^{\theta}$ is spacelike and timelike, respectively. In that case, (for type-2)*

$$||\breve{h}||^2 \leq 4n(\frac{1}{9}cot^2\theta + csc^2\theta)||grad(lnk)||^2 \tag{5.51}$$

$$(respectively, \quad ||\breve{h}||^2 \geq 4n(\frac{1}{9}cot^2\theta + csc^2\theta)||grad(lnk)||^2) \tag{5.52}$$

where $grad(lnk)$ is the gradient of lnk.

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

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CALCULATION
*Volume **1**, Issue **1**, 2025, Pages:**19-34***
E-ISSN: 3062-2107
www.simadp.com/calculation

TRIANGULAR NUMBERS AND CENTERED SQUARE NUMBERS HIDDEN IN PYTHAGOREAN RUNS

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Abstract.

The Pythagorean theorem, which asserts that in a right triangle, the sum of the squares of the legs is equal to the square of the hypotenuse and is mathematically expressed as $a^2 + b^2 + c^2$ can be generalized to equations with 5, 7, or more variables. If we seek to find t consecutive numbers that satisfy such equations, which can be extended infinitely by increasing the number of variables, and observe the equality of sums of squares for each case, we encounter what are known as Pythagorean runs. In this study, it was observed that within Pythagorean runs, which can become increasingly complex as we increase the number of variables, there exists a strikingly unique solution set when we restrict ourselves to finding consecutive integers.

By examining the consecutive integers that form these Pythagorean runs, new findings have emerged. Specifically, Pythagorean runs were analyzed using triangular numbers and centered square numbers. A hypothesis was formulated, positing that there is a unique solution involving consecutive integers for Pythagorean runs with figurate numbers. This hypothesis has been proven using both inductive and geometric proof methods.

Keywords: Consecutive Numbers, Pythagorean Runs, Triangular Numbers, Centered Square Numbers.
2020 Mathematics Subject Classification: 11Dxx.

1. Introduction

In a right triangle, the Pythagorean theorem states that the square of the hypotenuse (the longest side) is always equal to the sum of the squares of the other two sides. This fundamental principle can be algebraically expressed as $a^2 + b^2 = c^2$. By extending this principle with more variables, we can formulate new equations involving sums of squares with a structure similar to the Pythagorean Theorem:

$a^2 + b^2 = c^2$, involving 3 variables.
$a^2 + b^2 + c^2 = d^2 + e^2$, involving 5 variables.
$a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2$, involving 7 variables.
$a^2 + b^2 + c^2 + d^2 + e^2 = f^2 + g^2 + h^2 + j^2$, involving 9 variables, and so forth.

Received: 2024.11.14 *Revised: 2024.12.10* *Accepted: 2024.12.13*
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An infinite number of integers satisfy these equations, which initially appear as sums of squares. But what if we restrict these integers to be consecutive?

For the Pythagorean Theorem, which is the first of these equations, many different integer combinations like (7, 24, 25), (8, 15, 17), and (9, 40, 41) come to mind. However, when seeking consecutive integers that satisfy the Pythagorean Theorem, there is only one solution: (3, 4, 5). This can be observed as follows: $3^2 + 4^2 = 5^2$.

In such equations, regardless of the number of variables added, only one set of consecutive integers forms a solution, just as with the Pythagorean Theorem. For example:

For the equation $a^2 + b^2 + c^2 = d^2 + e^2$, the consecutive numbers are: $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.

For the equation $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2$, the consecutive numbers are: $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$.

The sequences of consecutive integers that satisfy such equations extend as follows:

$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$

$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$

$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$

According to the literature, these perfect sequences can continue indefinitely, and for any equation with one term, a unique solution of consecutive integers exists. These sequences, which appear as sums of squares of consecutive integers, are known in the literature as "Pythagorean runs", and various intriguing studies have explored their properties.

This study began by gathering the proofs for special cases of equations with three, five and seven variables, with the aim of discovering new insights into consecutive integers that satisfy equations with more variables, and to compare these findings with existing literature.

The discovery of the relationship between triangular numbers and centered square numbers, which are well-known for their fascinating properties in number theory, and their connection to consecutive integers forming solutions to the studied equations, constitutes the original aspect of this research.

At the conclusion of the study, an original algorithm was developed to extend these perfect sequences indefinitely using figurate numbers.

2. Material and Method

2.1. **The Pythagorean Theorem.** The Pythagorean Theorem, which has been proven and widely known for centuries, from ancient Egypt to the present day, has captivated the attention of many, including renowned mathematicians such as Euclid, Archimedes and Sabit bin Qurra, as well as the 20th president of the United States, James A. Garfield, who famously provided a simple proof using a trapezoid. The Pythagorean Theorem, which describes the geometric relationship in right triangles, states that in a right triangle, the square of the hypotenuse (the side opposite to the right angle) is equal to the sum of the squares of the other two legs (the sides adjacent to the right angle). This relationship, expressed algebraically as $a^2 + b^2 = c^2$, where a and b are the legs of the right triangle, is often treated as a fundamental mathematical exercise [1]. Kindly refer to Figure 1 for further reference.

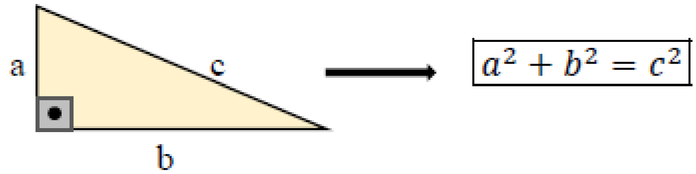

FIGURE 1. Pythagorean Theorem

Definition 2.1.1: A triple (a, b, c) is called a Pythagorean triple if it satisfies the equation $a^2 + b^2 = c^2$, ,where a, b and c positive integers [2, p. 4].

The right triangle with side lengths 3, 4, and 5 provides the first integer solution to the Pythagorean Theorem, forming what is known as the Pythagorean triple (3, 4, 5), as it satisfies the equation $3^2 + 4^2 = 5^2$. Pythagorean triples can be extended to other integer combinations, such as (5, 12, 13), where $5^2 + 12^2 = 13^2$. These examples demonstrate the recurring relationship between integer side lengths in right triangles governed by the Pythagorean theorem.

Definition 2.1.2: A Pythagorean triple (a, b, c) is called a primitive Pythagorean triple if it satisfies the equation gcd(a, b, c) = 1 [2, p. 4].

Many primitive Pythagorean triples can be derived, such as (3, 4, 5), (16, 63 ve 65), (21, 20 ve 29), (55, 48 ve 73), (65, 72 ve 97),(1155, 1292 ve 1733), (20737, 23184 ve 31105) [2].

These triples consist of positive integers that satisfy the equation $a^2 + b^2 = c^2$ and have no common divisor greater than 1, thereby representing primitive solutions to the Pythagorean theorem.

Proposition 2.1.1: Given that a, b and $c \in Z^+$, the only primitive Pythagorean triple consisting of consecutive integers that satisfy the equation $a^2 + b^2 = c^2$ is (3, 4, 5).

Proof 2.1.1: Let us express a and c in the form of b such that they are consecutive integers. Thus, we have $c = b + 1$ and $a = b - 1$. Now let us substitute these into the equation and solve for equality:

$$a^2 + b^2 = c^2 \tag{1}$$

$$(b-1)^2 + b^2 = (b+1)^2$$

$$(b-1)^2 + b^2 - (b+1)^2 = 0$$

$$(b-1+b+1).(b-1-b-1) + b^2 = 0$$

$$2b.(-2) + b^2 = 0$$

$$b^2 - 4b = 0$$

$$b.(b-4) = 0$$

From here, we find $b = 0$ or $b = 4$. However, since $b \in Z^+$, b cannot be 0. Therefore b must be 4. In this case, $a = 3, c = 5$. Substituting these results into the equation $a^2 + b^2 = c^2$, we obtain the equality $3^2 + 4^2 = 5^2$. As can be understood from this direct proof, the numbers 3, 4 and 5 have a distinct significance as the only consecutive integers that solve the equation $a^2 + b^2 = c^2$.

Let us examine Figure 2 for the visual geometric proof found in literature. The proof by Michael Boardman, which resperents $3^2 + 4^2 = 5^2$ by diving it into squares, is illustrated in Figure 2:

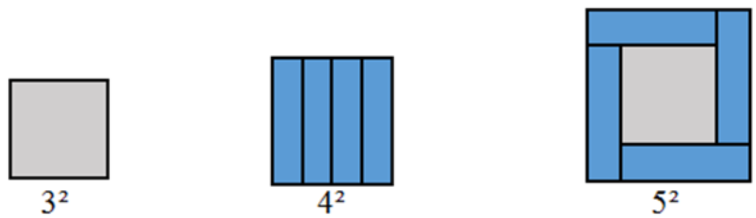

FIGURE 2. Boardman's dissection of squares for $3^2 + 4^2 = 5^2$ [3]

Boardman sliced the 4-square into four congruent rectangles that fit against the four sides of the 3-square to produce the 5-square [3, p.323]. As illustrated in Figure 2, this transformation demonstrates how a square can be restructured into a larger square by breaking it down and incorporating it with smaller square. By dividing the 4-square into four equal parts and placing them around the sides of 3-square, as shown in the figure, we produce a 5-square. This geometric process leads to the vertification of the equation $3^2 + 4^2 = 5^2$.

2.2. **Extending the Pythagorean Equation With Additional Variables While Preserving Its Fundamental Principle.**

The problem is to determine, if possible, two consecutive integers the sum of whose squares equals the sum of the squares of three conse- cutive integers; three consecutive integers, the sum of whose squares equals the sum of the squares of four consecutive integers; and so on [4, p.155].

The problem Alfred mentioned is to expand the equation $a^2 + b^2 = c^2$ by adding more variables while preserving its fundamental principle, thereby creating new equations that maintain a structure similar to the Pythagorean Theorem:

$a^2 + b^2 = c^2$, with 3 variables,

$a^2 + b^2 + c^2 = d^2 + e^2$, with 5 variables,

$a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2$, with 7 variables,

$a^2 + b^2 + c^2 + d^2 + e^2 = f^2 + g^2 + h^2 + j^2$, with 9 variables,

$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + = g^2 + h^2 + j^2 + k^2 + m^2$, with 11 variables, and so on.

Let us proceed by analyzing the equation with five variables and attempt to find consecutive numbers that satisfy the equation $a^2 + b^2 + c^2 = d^2 + e^2$.

For example; $13^2 + 14^2 = 10^2 + 11^2 + 12^2$

$133^2 + 134^2 = 108^2 + 109^2 + 110^2$

$1321^2 + 1322^2 = 1078^2 + 1079^2 + 1080^2$ " [4, p.155].

When we examine the individual examples in this study, we observe that only the first example consists entirely of consecutive numbers. This observation supports the second proposition that we aim to prove in our research.

Proposition 2.2.1: The only solution set composed of consecutive integers for the five-variable equation $a^2 + b^2 + c^2 = d^2 + e^2$, where $a, b, c, d, e \in Z^+, is (10, 11, 12, 13, 14)$.

Proof 2.2.1: The fact that the numbers a, b, c, d and e in the equation $a^2+b^2+c^2 = d^2+e^2$ are consecutive allows us to express these numbers in terms of a single variable. Let the middle number be $c = x$. In this case $a = x - 2, b = x - 1, d = x + 1$ and $e = x + 2$. Now, substituting these expressions into the equation, we can proceed to solve for equality.

$$(x - 2)^2 + (x - 1)^2 + x^2 = (x + 1)^2 + (x + 2)^2$$
$$x^2 - 4x + 4 + x^2 - 2x + 1 + x^2 = x^2 + 2x + 1 + x^2 + 4x + 4$$
$$3x^2 - 6x + 5 = 2x^2 + 6x + 5$$
$$x^2 - 12x = 0$$
$$x.(x - 12) = 0$$

Thus, x = 0 or x = 12 are the solutions.

Since $c \in Z^+$, c cannot be 0 $(c \neq 0)$, so we conclude that $c = 12$. Therefore $a = 10, b = 11, d = 12, e = 13$. Substituting these values into the equation $a^2 + b^2 + c^2 = d^2 + e^2$ we obtain: $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.

Let us examine Figure 3 for the visual geometric proof referenced in the literature. The proof by Michael Boardman, which decomposes $10^2 + 11^2 + 12^2 = 13^2 + 14^2$ into squares, is illustrated in Figure 3:

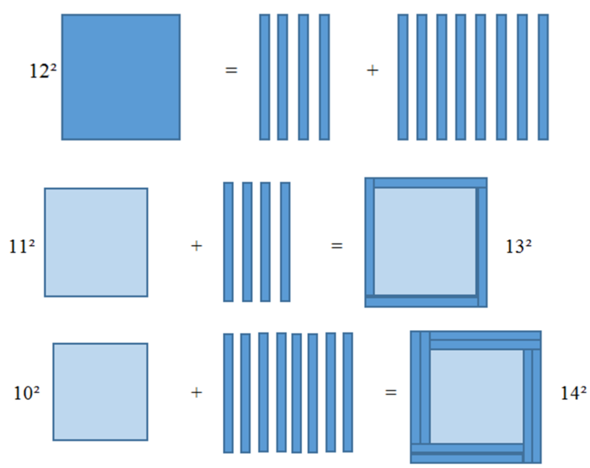

FIGURE 3. Boardman's dissection of squares for $10^2 + 11^2 + 12^2 = 13^2 + 14^2$ [5]

The geometric proof in Figure 3 demonstrates that if we divide a square with side length 12 into 12 parts and distribute them evenly along the edges of the 11-unit and 10-unit squares, we can transform these squares into 13-unit and 14-unit squares. This confirms the equation $10^2 + 11^2 + 12^2 = 13^2 + 14^2$ [5]. From this, it can be inferred that the middle number, 12, plays a crucial role in the geometric proof of the equation.

Let us now find the consecutive numbers that satisfy the seven-variable equation $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2$. Refer to Figure 4 for the geometric proof found in the literature. The proof by Michael Boardman, which divides the equation $21^2+22^2+23^2+24^2 = 25^2+26^2+27^2$ into squares, is illustrated in Figure 4:

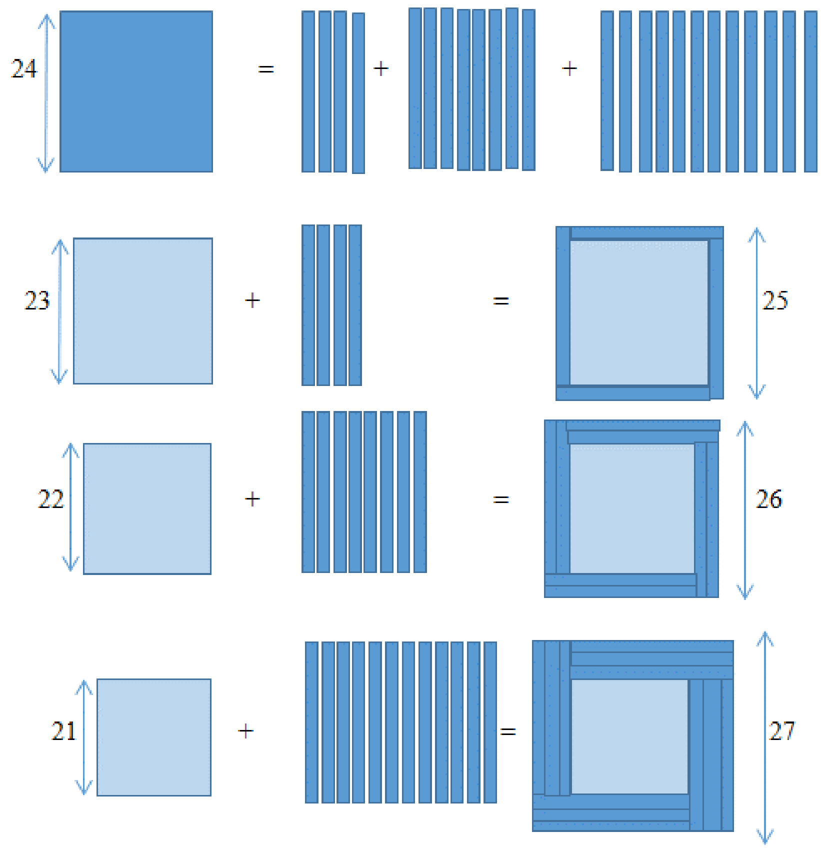

FIGURE 4. Boardman's dissection of squares for $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$ [6]

In Figure 4, a 24-unit square is divided into 24 equal parts, which are subsequently grouped as $24 = 4 + 8 + 12$. By arranging these groups neatly along the edges of the squares with side lengths of 23,22 and 21 units, respectively, we can transform them into squares with side lengths of 25, 26 and 27 units [6]. The geometric proof presented in Figure 4 illustrates that the middle number,24, is essential in establishing the equation $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$.

At this juncture, let us enumerate the equations for the sum of consecutive squares that we have examined in detail, involving three, five, and seven variable:

$$3^2 + 4^2 = 5^2$$

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

Boardman termed these identities Pythagorean runs, because they involve consecutive positive integers, just like $3^2 + 4^2 = 5^2$, the simplest of the Pythagorean triples [5, p.21] .

If $T_n = 1 + 2 + ... + n, (4T_n - n)^2 + ... + (4T_n)^2 = (4T_n + 1)^2 + ... + (4T_n + n)^2$ [6].

Thanks to Boardman, we can now readily construct any equation that satisfies the conditions we are seeking:

For $n = 3$, with $T_3 = 6$, we have $(4.6 - 3)^2 + ... + (4.6)^2 = (4.6 + 1)^2 + ... + (4.6 + 3)^2$.

This results in the equation: $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$

For $n = 4$, with $T_4 = 10$, we have $36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$.

For $n = 5$, with $T_5 = 15$, we have $55^2+56^2+57^2+58^2+59^2+60^2 = 61^2+62^2+63^2+64^2+65^2$.

Boardman's research demostrates that if we establish the appropriate equations for each $n \geq 1$ according to the equality $(4T_n - n)^2 + ... + (4T_n)^2 = (4T_n + 1)^2 + ... + (4T_n + n)^2$, where $T_n = 1 + 2 + ... + n$, there exists a specific rhythmic arrangement that is evident not only horizontally but also vertically.

Refer to Figure 5 for the Pythagorean runs composed of consecutive squares:

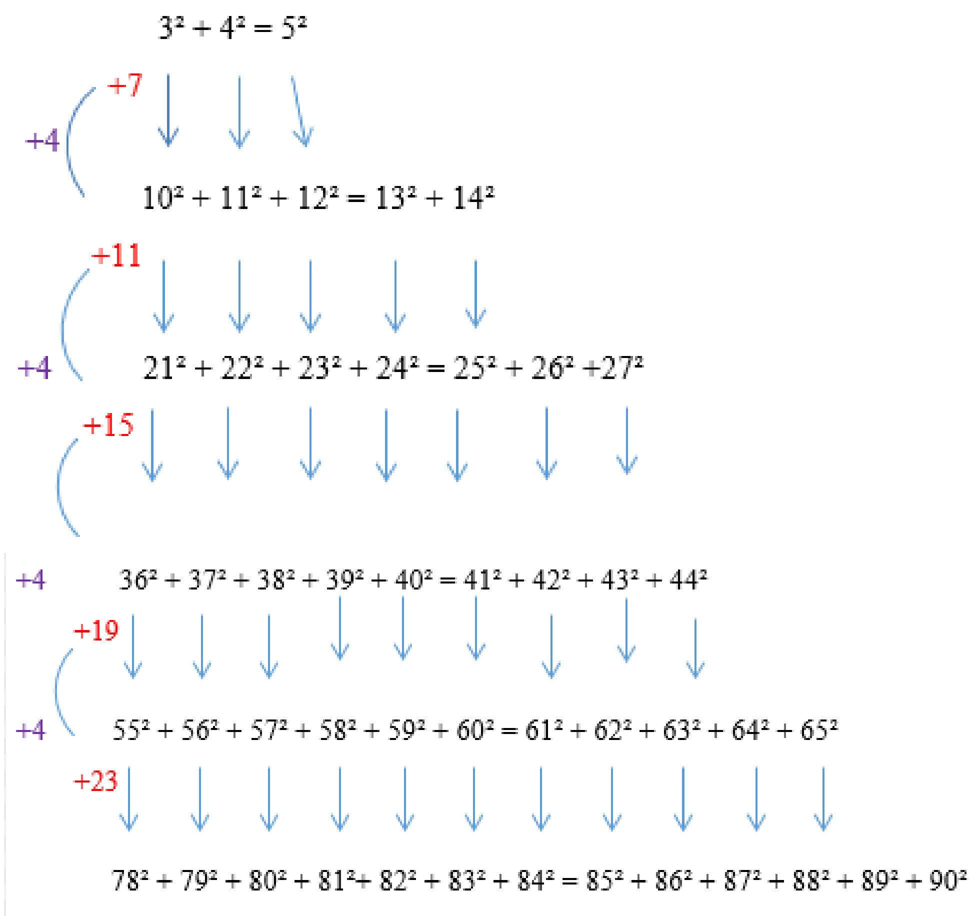

FIGURE 5. Pythagorean Runs of Consecutive Numbers

These Pythagorean sequences, with their astonishing and captivating mathematical essence, represent merely the tip of the iceberg. The technique of determining the identities of such equations by consecutive sums of squares was first discovered by Georges Dostor. [7, p. 44]. "Are there other relationships hidden within Pythagorean runs waiting to be uncovered?" has engaged mathematicians for quite some time and continues to be a pertinent question among them. The findings, proofs, and relationships we have compiled thus far are of considerable significance to our research, guiding us to explore how these non-coincidental equations can be solved with greater ease, under what conditions they hold validity, and concentrating on their mathematical properties to unveil the profound mathematical structure inherent in these equations.

2.3. **Figured Numbers Hidden in Consecutive Pythagorean Runs.**

Upon a meticulous examination of the equations presented in Figure 5, it becomes evident that there exist certain critical prerequisites within the unique solutions that facilitate the indefinite continuation of the equations found in the Pythagorean runs of consecutive squares.

Condition 2.3.1: The initial terms in all Pythagorean runs are triangular numbers.

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$
$$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$$
$$105^2+106^2+107^2+108^2+109^2+110^2+111^2+112^2 = 113^2+114^2+115^2+116^2+117^2+118^2+119^2$$

The numbers $3, 10, 21, 36, 55, 78, 105$ are known as triangular numbers. Consequently, it has been found that triangular numbers are embedded within the solution sets of the equations analyzed in our research.

These numbers are termed triangular numbers because they can be represented by arranging equal-diameter spheres in the shape of an equilateral triangle. Each triangular number is generated by adding an additional row to the preceding triangular number, meaning that successive row contains one more unit than the previous one. Therefore, when a series of equal-diameter spheres is organized in the configuration of an equilateral triangle, triangular numbers are produced. [8, p. 9]. To further elucidate triangular numbers, let us consider Figure 6:

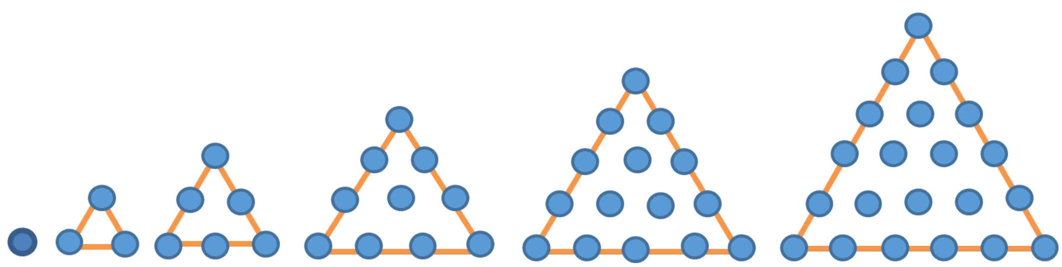

FIGURE 6. The representation of triangular numbers in an equilateral triangle figure

Starting from a point, add to it two points, so that to obtain an equilateral triangle. Six-points equilateral triangle can be obtained from three-points triangle by adding to it three points; adding to it four points gives ten-points triangle, etc. So, by adding to a point two, three, four etc. points, then organizing the points in the form of an equilateral triangle and counting the number of points in each such triangle, one can obtain the numbers1,3,6,10,15,21,28,36,45,55 which are called triangular number [9, p. 58]. Based on this, we can extend the sequence of triangular numbers indefinitely as follows: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 405, 435, and so on.

If we closely examine the initial terms of the Pythagorean runs, we observe the following equations:

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$
$$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$$

From the triangular numbers, a discernible pattern emerges, wherein the numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, and so forth alternate between being present and absent. Specifically, among the triangular numbers, we find the following sequence: 1 is absent,3 is present, 6 is absent, 10 is present, 15 is absent, 21 is present, 28 is absent, 36 is present, 45 is absent, 55 is present, 66 is absent, 78 is present, 91 is absent, and so on. This conclusion indicates that within the triangular number sequence 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 405, 435, 465, the numbers highlighted in red are particularly significant. Consequently, these red triangular numbers also constitute the initial terms of consecutive Pythagorean runs. The pattern observed with triangular numbers is indeed intriguing and may lead to valuable mathematical insights.

Condition 2.3.2: In the Pythagorean runs, the initial terms on the right side of the equations are centered square numbers.

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$
$$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$$

Thus, within the solution sets of the Pythagorean runs we have examined, it has been revealed that the centered square numbers, which can be arranged into a centered square configuration, are present.

Centered Square Number is a centered polygonal number consisting of a central dot with four dots around it, and then additional dots in the gaps between adjacent dots [10]. To enhance our understanding of centered square numbers, let us consider Figure 7:

If we enumerate the centered square numbers, 1, 5, 13, 25, 41, 61, 85, 113, 145, 181, 221, 265, 313, 365, 421, 481, 545, 613, 685, 761, 841, 925... [11].

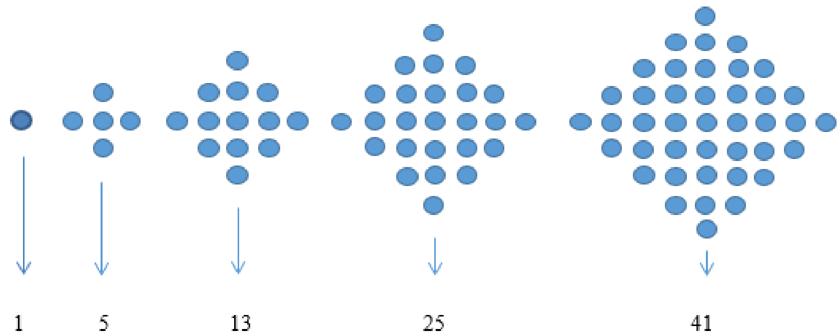

FIGURE 7. Arranging centered square numbers in a centered square figure

Centered square numbers are the sum of two consecutive square numbers and are congruent to 1 (mod 4) and the general term is $[n^2 + (n + 1)^2]$ [10].

Examples of obtaining some centered square numbers include:

For $n = 1, 1^2 + 2^2 = 5$

For $n = 2, 2^2 + 3^2 = 13$

For $n = 3, 3^2 + 4^2 = 25$

For $n = 4, 4^2 + 5^2 = 41$

For $n = 5, 5^2 + 6^2 = 61$

As can be observed; all centered square numbers, with the exception of 1, constitute the first terms on the right side of the equations in the Pythagorean runs, respectively. However, some triangular numbers are included in the Pythagorean runs.

Building upon this, the original Proposition 2.3.1 is developed. Figure 8 was discovered as the original non-verbal proof to validate this proposition. In Figure 8, the objective is to derive the triangular numbers relevant to the Pythagorean runs from the figures of centered square numbers.

Proposition 2.3.1: In the equations of Pythagorean runs, for $\forall\ n \in N^+$, the first term on the right side of the equation is a centered square number, expressible as $[n^2+(n+1)^2]$. The triangular number derived from these square numbers, which can be represented using the formula $[(n + n + 1).n]$, yields the first term on the left side of the equation.

Proof 2.3.1: The innovative approach we seek to explore involves deriving the triangular configuration illustrated in Figure 6 from the centered square numbers depicted in Figure 7, utilizing the geometric proof method. To facilitate comprehension, let us examine Figure 8:

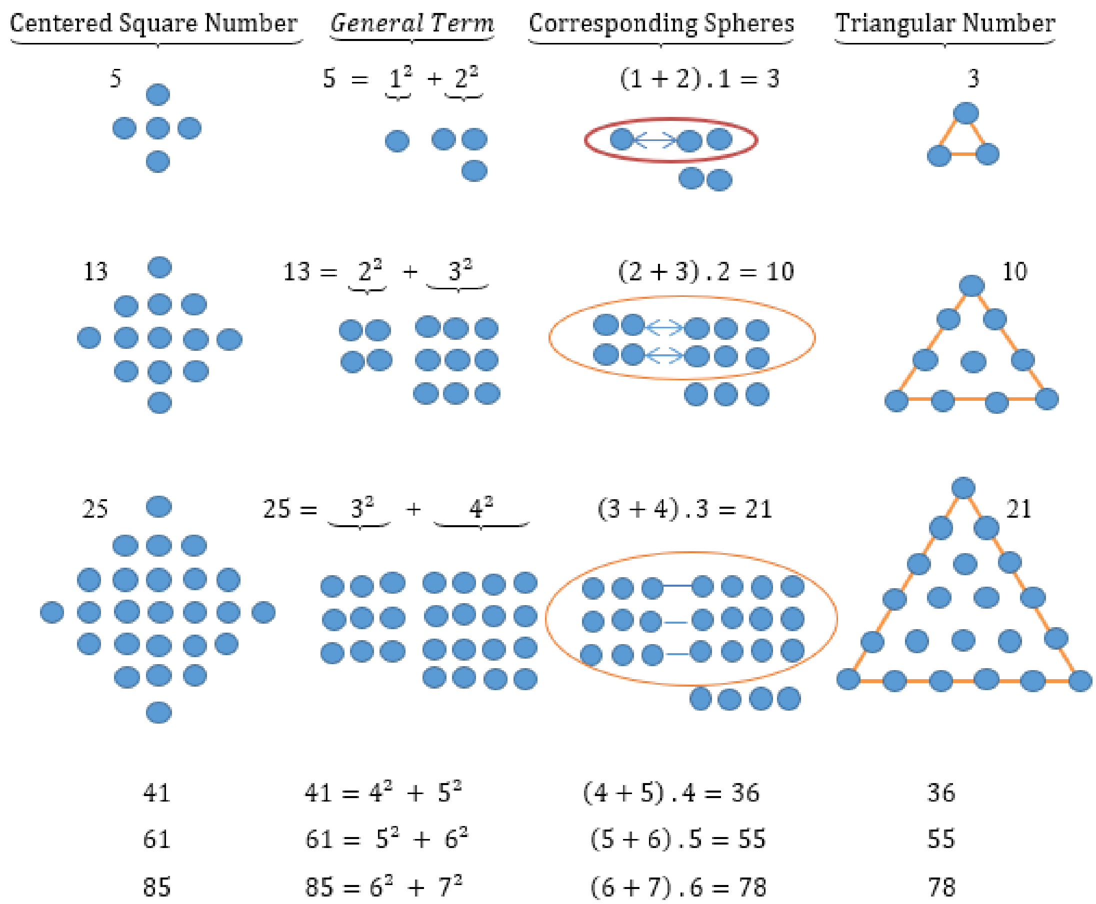

FIGURE 8. Geometric proof. Derivation of triangular numbers belonging to Pythagorean runs from centered square numbers

The original geometric proof presented in Figure 8 illustrates that certain triangular numbers can be derived from all centered square numbers expressible by the general term $[n^2 + (n+1)^2]$ within Pythagorean runs. Furthermore, these triangular numbers can be represented by the specific term $[(n + n + 1).n]$. Rearranging the equations yields;

For the centered square numbers, $n^2 + (n+1)^2 = 2n^2 + 2n + 1$,

For the triangular numbers, $(n + n + 1).n = 2n^2 + n$.

Condition 2.3.3: In equations associated with Pythagorean runs, for $\forall\ n \in N^+$, the first term on the left side of the equation is a triangular number expressible as $(2n^2 + n)$, while the first term on the right side is a centered square number expressible as $(2n^2 + 2n + 1)$.

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$
$$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$$

Conditions 2.3.1, 2.3.2, and 2.3.3, along with the original geometric proof presented in Figure 8, collectively provide highly significant validations for Pythagorean runs that can be extended indefinitely. No studies have been identified in the literature that establish a connection between the Pythagorean runs and centered square numbers or triangular numbers, highlighting the originality of this research. As a result of the investigation, a novel model has been developed encompassing all centered square numbers and triangular numbers hidden within the Pythagorean runs. Proposition 2.3.2, an original proposition pertaining to Pythagorean runs, was formulated using Figure 8 and Condition 2.3.3, and was subsequently proven through the method of induction, which is one of the recognized mathematical proof techniques.

Proposition 2.3.2: The Pythagorean runs, for $\forall\ n \in N^+$, are represented by consecutive terms expressed in the form $(2n^2+n)^2+...=(2n^2+2n+1)^2+...$, where $(2n^2+2n+1)$ denotes a centered square number and $(2n^2+n)$ signifies a triangular number. The total number of consecutive terms in the equation is $(2n+1)$, with the left side containing one additional term compared to the right side.

Proof 2.3.2: For $n=1$, the first equation S_1 is : $(2.1^2+1)^2+...=(2.1^2+2.1+1)^2+...$ and the total number of terms is $2.1+1=3$. Therefore, the three-term equation with consecutive terms is $3^2+4^2=5^2$.

For $n=2$, the second equation S_2 is : $(2.2^2+2)^2+...=(2.2^2+2.2+1)^2+...$ and the total number of terms is $2.2+1=5$.

$$10^2+...=13^2+...$$

The equation $10^2+11^2+12^2=13^2+14^2$ consists of five consecutive terms.

Let the k-th equation S_k be true for $n=k$.

Assume S_k has consecutive terms in the form $(2k^2+k)^2+...=(2k^2+2k+1)^2+...$ and let the total number of consecutive terms in the equation be $(2k+1)$ with the left side containing one additional term compared to the right side. Thus, we can express S_k as follows:

$S_k:(2k^2+k)^2+...=(2k^2+2k+1)^2+...$

To explicitly illustrate the consecutive terms of the proposition S_k , we have:

$S_k:(2k^2+k)^2+(2k^2+k+1)^2+(2k^2+k+2)^2+...+(2k^2+2k)^2=(2k^2+2k+1)^2+(2k^2+2k+2)^2+...+(2k^2+3k)^2$

To express the sum of the consecutive terms on both sides of this equation using the summation symbol Σ we introduce the variable z:

Thus, S_k can be rewritten as:

$$S_k:\Sigma_{z=0}^{k}(2k^2+k+z)^2=\Sigma_{z=1}^{k}(2k^2+2k+z)^2$$
$$S_k:\Sigma_{z=0}^{k}(2k^2+k+z)^2=\Sigma_{z=1}^{k}(2k^2+2k+z)^2$$
$$(2k^2+k)^2+\Sigma_{z=1}^{k}(2k^2+2k+z)^2=\Sigma_{z=1}^{k}(2k^2+k+z)^2$$
$$(2k^2+k)^2=\Sigma_{z=1}^{k}(2k^2+2k+z)^2-\Sigma_{z=1}^{k}(2k^2+k+z)^2$$

To utilize the difference of squares identity $a^2-b^2=(a-b).(a+b)$ we will arrange similar terms side by side as follows:

$$(2k^2+k)^2 = \underbrace{(2k^2+2k+1)^2-(2k^2+k+1)^2}_{for\ z=1} + \underbrace{(2k^2+2k+2)^2-(2k^2+k+2)^2}_{for\ z=2} + \ldots$$

$$\ldots + \underbrace{(2k^2+3k)^2-(2k^2+2k)^2}_{for\ z=k}.$$

Utilizing the difference of squares identity $a^2 - b^2 = (a-b).(a+b)$, we can derive the following:

$$(2k^2+k)^2 = k.(4k^2+3k+2) + k.(4k^2+3k+4) + \ldots + k.(4k^2+3k+2k)$$

$$(2k^2+k)^2 = k.[4k^2+3k+2 + 4k^2+3k+4 + \ldots + 4k^2+3k+2k]$$

$$(2k^2+k)^2 = k.[\underbrace{4k^2 + \ldots + 4k^2}_{\text{There are } k \text{ terms}} + \underbrace{3k + \cdots + 3k}_{\text{There are } k \text{ terms}} + \underbrace{2+4+\ldots+2k}_{\text{There are } k \text{ terms}}]$$

$$(2k^2+k)^2 = k.[k.4k^2 + k.3k + \underbrace{2+4+\ldots+2k}_{\sum_{z=1}^{k} 2z}]$$

$$k^2.(2k+1)^2 = k.[4k^3 + 3k^2 + \textstyle\sum_{z=1}^{k} 2z]$$

$$k^2.(2k+1)^2 = k.[4k^3 + 3k^2 + \textstyle\sum_{z=1}^{k} 2z],\ k \neq 0$$

$$k.(2k+1)^2 = 4k^3 + 3k^2 + \textstyle\sum_{z=1}^{k} 2z$$

$$k.(4k^2+4k+1) = 4k^3 + 3k^2 + \textstyle\sum_{z=1}^{k} 2z$$

$$4k^3+4k^2+k = 4k^3 + 3k^2 + \textstyle\sum_{z=1}^{k} 2z$$

$$4k^3+4k^2+k = 4k^3 + 3k^2 + \textstyle\sum_{z=1}^{k} 2z$$

$$4k^2 + k = 3k^2 + \textstyle\sum_{z=1}^{k} 2z$$

$$4k^2 + k - 3k^2 = \textstyle\sum_{z=1}^{k} 2z$$

The equality $k^2 + k = \Sigma_{z=1}^{k} 2z$ ...(1) is established. This equality holds true and is a consequence of the equation accepted as valid for $n = k$.

Now, we need to verify whether the proposition S_{k+1} for $n = k+1$ is correct. Specifically, we examine

$S_{k+1} : \Sigma_{z=0}^{k+1}(2(k+1)^2+k+1+z)^2 = \Sigma_{z=1}^{k+1}(2(k+1)^2+k+1+z)^2$, Is this statement valid?

$$\underbrace{[2(k+1)^2 + k+1]^2}_{for\ z=0} + \textstyle\sum_{z=1}^{k+1}(2(k+1)^2+k+1+z)^2 = \sum_{z=1}^{k+1}(2(k+1)^2+2(k+1)+z)^2$$

$$[2(k+1)^2+k+1]^2] = \Sigma_{z=1}^{k+1}(2(k+1)^2+2(k+1)+z)^2 - \Sigma_{z=1}^{k+1}(2(k+1)^2+k+1+z)^2$$

Utilizing the difference of squares identity $a^2 - b^2 = (a-b).(a+b)$, we can derive the following:

$$[(k+1).(2(k+1)+1)]^2 = \underbrace{[2(k+1)^2 + 2(k+1)+1]^2 - [2(k+1)^2 + (k+1)+1]^2}_{for\ z=1} +$$

$$+\underbrace{[2(k+1)^2 + 2(k+1)+2]^2 - [2(k+1)^2 + (k+1)+2]^2}_{for\ z=2} + \ldots$$

$$\ldots + \underbrace{[2(k+1)^2 + 3(k+1)]^2 - [2(k+1)^2 + 2(k+1)]^2}_{for\ z=k+1}$$

$$(k+1)^2.\ [(2(k+1)+1)]^2 = (k+1).\ [4(k+1)^2 + 3(k+1) + 2 + 4(k+1)^2 + 3(k+1) + 4 + \ldots$$

$$\ldots + 4(k+1)^2 + 3(k+1) + 2(k+1)]$$

$$(k+1)^2.\ (2k+3)^2 = (k+1).\ [(k+1).4(k+1)^2 + (k+1).3(k+1) + \underbrace{\underbrace{2+4+\cdots+2k}_{\sum_{z=1}^{k} 2z \ = \ k^2+k \ \ldots \ [1]} + 2(k+1)}_{k^2 + k + 2(k+1)}]$$

$$k^2 + k + 2(k+1)$$

$$k(k+1) + 2(k+1)$$

$$(k+1).(k+2)$$

$$(k+1)^2.(2k+3)^2 = (k+1).[(k+1).4(k+1)^2 + (k+1).3(k+1) + (k+1).(k+2)]$$

$$(k+1)^2.(2k+3)^2 = (k+1).(k+1).[4(k+1)^2 + 3(k+1) + k + 2]$$

$$(k+1)^2.(2k+3)^2 = (k+1)^2.[4(k+1)^2 + 3(k+1) + k + 2]$$

$$\cancel{(k+1)^2}.(2k+3)^2 = \cancel{(k+1)^2}.[4(k+1)^2 + 3(k+1) + k + 2],\ k \neq -1$$

$$(2k+3)^2 = 4(k+1)^2 + 3(k+1) + k + 2$$

$$(2k+3)^2 = 4(k^2+2k+1) + 3(k+1) + k + 2$$

$$(2k+3)^2 = 4k^2 + 8k + 4 + 3k + 3 + k + 2$$

$$(2k+3)^2 = 4k^2 + 12k + 9$$

As seen, according to the method of induction, the result (1) obtained from the assumption S_k, which is accepted to be true for $n = k$, has been substituted into the equation for $n = k+1$ to prove that the proposition S_{k+1} is also valid. In all the equations related to the Pythagorean runs, for $\forall\ n \in N^+$, the first term on the left side of the equation is a triangular number that can be expressed as $(2n^2+n)$, and the first term on the right side of the equation is a centered square number expressible as $(2n^2+2n+1)$. The relationship between the centered square numbers and triangular numbers, which are embedded in the

Pythagorean runs, was first discovered in this study and was formulated in Proposition 2.3.1, with a visual proof provided in Figure 8. Building on this, Proposition 2.3.2, developed to further address such equations, was proven using the method of mathematical induction.

Table 2.1 demonstrates the mathematical relationship between the triangular numbers, which appear as the first terms on the left side of the Pythagorean sequence equations, and the centered square numbers, which serve as the first terms on the right side, for the first ten natural numbers

n. Pythagorean runs	$2n^2+n$, Triangular number	$2n^2+2n+1$, Centered square number	$2n+1$, Number of consecutive terms	Equations related to Pythagorean sequences
1	3	5	3	$3^2+4^2=5^2$
2	10	13	5	$10^2+11^2+12^2=13^2+14^2$
3	21	25	7	$21^2+22^2+23^2+24^2=25^2+26^2+27^2$
4	36	41	9	$36^2+37^2+38^2+39^2+40^2=41^2+42^2+43^2+44^2$
5	55	61	11	$55^2+56^2+\cdots+60^2=61^2+\cdots+65^2$
6	78	85	13	$78^2+79^2+\cdots+84^2=85^2+\cdots+90^2$
7	105	113	15	$105^2+106^2+\cdots+112^2=113^2+\cdots+119^2$
8	136	145	17	$136^2+137^2+\cdots+144^2=145^2+\cdots+152^2$
9	171	181	19	$171^2+172^2+\cdots+180^2=181^2+\cdots+189^2$
10	210	221	21	$210^2+211^2+\cdots+220^2=221^2+\cdots+230^2$

Table 2.1. Pythagorean runs and related equations.

3. Results and Discussion

As a result of the research, a creative and original Proposition 2.3.1, establishing a significant connection between figurate numbers specifically triangular numbers and centered square numbers and Pythagorean sequences, was developed and proven through the visual proof in Figure 8. Building upon this original visual proof, Proposition 2.3.2 was formulated and subsequently proven using the method of mathematical induction.

4. Conclusion and Suggestions

By investigating the applications of Pythagorean sequences in fields such as physics, engineering, computer science, and others, further research could explore the contributions and practical applications that the developed proposition might offer in areas like wave theory, optics, sound analysis, or artificial intelligence. In this way, it may become possible to uncover real-world applications of the Pythagorean Theorem and similar mathematical equalities.

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CALCULATION
*Volume **1**, Issue **1**, 2025, Pages:**35-43***
E-ISSN: 3062-2107
www.simadp.com/calculation

ALMOST POTENT MANIFOLDS

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Abstract. In this paper we introduce a new manifold, namely potent manifolds. We give examples and investigate the integrability conditions. We also check the curvature relations of Kaehler potent manifolds and show that such manifolds are flat when it has constant sectional curvature. Then we introduce potent sectional curvature and obtain a spacial form of the curvature tensor field when its potent sectional curvature is a constant.

Keywords: Almost potent structure, almost potent manifold, Hermitian potent manifold, Kaehler potent,sectional curvature, holomorphic sectional curvature, real space form, potent sectional curvature, potent space form

2020 Mathematics Subject Classification: 53C15.

1. Introduction

Manifolds on which extra structures are defined present a geometrically rich structure. In this sense, the first structures are complex structures [29] and contact structures [2]. In the process, many new structures have been defined on manifolds and have received new names accordingly. These manifolds can be listed as product manifolds [29], quaternion manifolds [13], biproduct complex manifolds [6], para-contact manifolds [25], para-quaternionic manifolds [9], [11], [26], hyper Kaehler manifolds [3], metric mixed 3-structures [14], [27], almost tangent manifolds, [20], [28] and the like. Recently, with the definition of Golden manifolds [5] in the literature, new manifold classes such as metallic manifolds [10], poly-Norden manifolds [23], meta-Golden manifolds [21], meta-Metallic manifolds [7], bi-tangent quaternion manifolds [16], [18] have been introduced and their geometric properties have been examined.

In this paper, inspired by the concept of idempotent transformation, we introduce almost potent manifolds and investigate the geometric properties of such manifolds. We also introduce Hermitian potent manifolds, Kaehler potent manifolds and potent space forms.

The paper is organized as follows. First, the definition of an almost potent manifold is presented and examples are given. In addition, the Nijenhuis tensor field of an almost potent manifold is calculated and the integrability condition is given accordingly. In this

Received: 2024.10.31 *Accepted: 2024.12.23*

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case, the concept of a Kaehler potent manifold is given. In the third section, the constant sectional curvature of a Kaehler potent manifold is considered and it is shown that it is flat if it has constant sectional curvature. For this reason, holomorphic-like sectional curvature and holomorphic bisectional-like curvatures are investigated, but it turns out that these are also zero. Therefore, the notion of potent sectional curvature of a Kaehler potent manifold, is given. In the case that the potent sectional curvature is constant, a special case of the curvature tensor field of a Kaehler potent manifold is obtained.

2. Almost Potent Manifolds

In this section we introduce almost potent manifolds and almost Hermitian potent manifolds. We then investigate the integrability of almost potent manifolds and define Kaehler potent manifolds.

Definition 2.1. *Let M be a differentiable manifold and F an endomorphism on M. If F is idempotent, i.e.*

$$F^2X = FX \tag{2.1}$$

for all $X \in \chi(M)$, then F is called an almost potent structure on M. In this case (M, F) is called an almost potent manifold.

Example 2.1. *Consider $\mathbb{R}^2$ with the map $A(x, y) = (\frac{1}{2}(x + y), \frac{1}{2}(x + y))$. Then $(\mathbb{R}^2, A)$ is an almost potent manifold.*

Example 2.2. *Consider the right circular cylinder M given by*

$$X(u, v) = (\cos u, \sin u, v).$$

Then the matrix of the shape operator of M is

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.2}$$

Then it is easy to see that $S^2 = S$. Thus (M, S) is an almost potent manifold.

Example 2.3. *Consider the Clifford algebra $Cl(2, 0)$ with the basis e_1, e_2 such that $e_1^2 = e_2^2 = 1$. $Cl(2, 0)$ can be represented by the algebra $M(2, \mathbb{R})$ of all real matrices by taking*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now consider an idempotent matrix B and define F as

$$F = Be_i$$

Then F is also an idempotent matrix and $Cl(2, 0), F)$ is an almost potent manifold.

Example 2.4. *Let M be a Riemannian manifold and $\{e_1, e_2, e_3\}$ an orthonormal frame of M. Assume that there is an endomorphism F on M such that $Fe_1 = e_1$, $Fe_2 = e_2$ and $Fe_3 = 0$. Then (M, F) is an almost potent manifold.*

We now define Hermitian potent manifold.

Definition 2.2. *Let (M, F) be an almost potent manifold. If there is a Riemannian metric g on M such that*

$$g(FX, Y) = g(X, FY) \tag{2.3}$$

for $X, Y \in \chi(M)$, then (M, F, g) is called almost Hermitian potent manifold. In this case we have

$$g(FX, FY) = g(FX, Y) = g(X, FY). \tag{2.4}$$

Example 2.5. *Right circular cylinder given in Example 2.2 with the shape operator is an example of an almost Hermitian potent manifold.*

We now investigate the integrability of almost potent structure F. We first note that the Nijenhuis tensor field of an endomorphism or $(1, 1)$ tensor field is given by

$$N_F(X, Y) = [FX, FY] + F^2[X, Y] - F[FX, Y] - F[X, FY] \tag{2.5}$$

for $X, Y \in \chi(M)$. Using (2.1) and (2.5) we obtain the following

$$N_F(X, Y) = [FX, FY] + F[X, Y] - F[FX, Y] - F[X, FY]. \tag{2.6}$$

From (2.6), by direct computation, we have the following lemma.

Lemma 2.1. *Let (M, F, g) be almost Hermitian potent manifold. Then we have*

$$N_F(X, Y) = (\nabla_{FX}F)Y - F(\nabla_X F)Y - (\nabla_{FY}F)X + F(\nabla_Y F)X \tag{2.7}$$

for $X, Y \in \chi(M)$.

Thus from (2.7), we have the following result.

Proposition 2.1. *Let (M, F, g) be almost Hermitian potent manifold. Then an almost potent structure is integrable if it is parallel.*

Thus we give the following definition.

Definition 2.3. *Let (M, F, g) be almost Hermitian potent manifold. If F is parallel, i.e.*

$$(\nabla_X F)Y = 0 \tag{2.8}$$

for $X, Y \in \chi(M)$, then (M, F, g) is called Kaehler potent manifold.

Example 2.6. *Right circular cylinder given in Example 2.2 with the shape operator S is an example of a Kaehler potent manifold.*

We note the following properties from linear algebra [19] of inner product spaces with an idempotent map. For any idempotent operator F, its only possible eigenvalues are 0 and 1. An idempotent operator on almost Hermitian potent manifold is typically a projection operator. It projects vector fields onto a subspace of the space. If F is an idempotent operator, then the image $Im(F)$ is a subspace of $\chi(M)$. Thus we have

$$\chi(M) = Im(F) \oplus Ker(F) \tag{2.9}$$

where $KerF$ is the kernel of F. Moreover if $F = F^*$ (where F^* is the adjoint of F, $Im(F)$ and $Ker(F)$ are orthogonal complement to each other. From now on, we will denote the vector fields belonging to the image space of the endomorphism F on almost Hermitian potent manifold M by $\chi(ImF(M))$.

3. Curvature relations of a Kaehler potent manifold

In this section, we are going to investigate the curvature tensor fields of a Kaehler potent manifold and show that it is a flat when it is a real space form. We also check holomorphic-like and holomorphic bi-sectional like curvature. We first give the following algebraic properties of curvature tensor fields.

Lemma 3.1. *Let (M, F, g) a Kaehler potent manifold and R the curvature tensor field of M. Then we have*

$$R(X,Y)FZ = FR(X,Y)Z \tag{3.10}$$
$$g(R(X,Y)FZ, FW) = g(R(X,Y)Z, FW) \tag{3.11}$$
$$R(FX,FY) = R(X,FY) = R(FX,Y) \tag{3.12}$$

for $X, Y, Z, W \in \chi(ImF(M))$.

Proof. (3.10) is clear from (2.8). For (3.11), using (3.10) we have

$$g(R(X,Y)FZ, FW) = g(FR(X,Y)Z, FW).$$

Then from (2.4) we get (3.11).For (3.12) we first have

$$g(R(FX,FY)Z,W) = g(R(Z,W)FX,FY).$$

Using (3.11) we obtain

$$g(R(FX,FY)Z,W) = g(R(Z,W)X,FY) = g(R(X,FY)Z,W)$$

which gives the first part of (3.11). Second part follows from (2.4). □

If the Kaehler potent manifold has constant sectional curvature, the following situation occurs.

Proposition 3.1. *Let M be an $n-$ dimensional Kaehler potent manifold. If M has constant sectional curvature c at every point $p \in M$, then M is flat provided $n \geq 2$.*

Proof. Since M has constant sectional curvature c, the curvature tensor field of the manifold is

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}$$

for $X, Y, Z \in \chi(M)$. Using (3.10) we get

$$c\{g(Y,FZ)X - g(X,FZ)Y\} = c\{g(Y,Z)FX - g(X,Z)FY\}$$

for $X, Y, Z \in \chi(ImF(M))$. Taking $Y = FX$ and using (2.3) we arrive at

$$c\{g(FX,Z)X - g(X,FZ)FX\} = c\{g(FX,Z)FX - g(X,Z)FX\}.$$

Using (2.1) we have

$$c\{g(FX,Z)FX - g(X,FZ)FX\} = c\{g(FX,Z)FX - g(X,Z)FX\}.$$

Hence using (2.3) we derive

$$0 = c\{g(FX,Z)FX - g(X,Z)FX\}.$$

Thus taking $Z \perp X$ such that FX and Z are not ortgonal to each other, we get

$$0 = cg(FX, Z)FX.$$

This implies that $c = 0$. □

Remark 3.1. *The choice of vector fields used in the proof is very crucial. It is easy to see that this is valid in* $(\mathbb{E}^2 = (\mathbb{R}^2, <, >)$. *For example, if* $X = (1, 1)$, $Z = (-1, 1)$ *are chosen and the idempotent matrix is chosen as* $F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, *it is seen that this situation is valid.*

As in complex geometry, for a Kaehler potent manifold, the holomorphic-like sectional curvature and the holomorphic-like bi-sectional curvature can be defined as

$$H(X) = g(R(X, FX)FX, X)$$

and

$$H(X, Y) = g(R(X, FX)FY, Y)$$

for $X, Y \in \chi(ImF(M))$. However, the following propositions will show us that these notions do not work for Kaehler potent manifolds.

Proposition 3.2. *Every Kaehler potent manifold has zero holomorphic-like sectional curvature.*

Proof. From (3.10) we have

$$g(R(X, FX)FX, X) = g(FR(X, FX)X, X).$$

Then (2.3) gives

$$g(R(X, FX)FX, X) = g(R(X, FX)X, FX).$$

Hence $g(R(X, FX)FX, X) = -g(R(X, FX)FX, X)$ which completes proof. □

Proposition 3.3. *Every Kaehler potent manifold has zero holomorphic-like bi-sectional curvature.*

Proof. From (3.10) and (2.3) we have

$$\begin{aligned} H(X, Y) &= g(R(X, FX)FY, Y) \\ &= g(F(R(X, FX)Y, Y) \\ &= g(R(X, FX)Y, FY) \end{aligned}$$

for $X, Y \in \chi(ImF(M))$. Hence we derive

$$\begin{aligned} H(X, Y) &= g(R(X, FX)FY, Y) \\ &= -g(R(X, FX)FY, Y). \end{aligned}$$

Thus $H(X, Y) = 0$. □

4. Potent sectional curvature and potent space forms

From the previous section, it was seen that the Kaehler potent manifold with constant sectional curvature is flat and the concepts of holomorphic-like sectional curvature and holomorphic bi-sectional curvature do not work. Therefore, in this section, the notion of potent sectional curvature is introduced, an example is given, and a special expression is given for the curvature tensor field of Kaehler potent manifolds with constant potent section curvature. We present the following definition.

Definition 4.1. *Let (M,F,g) be a Kaehler potent manifold. For $X,Y \in \chi(ImF(M))$ the potent sectional curvature is defined by*

$$K(X \wedge FY) = \frac{g(R(X,FY)FY,X)}{g(X,X)g(FY,FY) - g(X,FY)^2}. \tag{4.13}$$

If the potent sectional curvature is constant for arbitrary vector fields X and Y and arbitrary point $p \in M$, then the Kaehler potent manifold is said to have constant sectional curvature, or simply a potent space form.

Example 4.1. *Consider half space $H = \{(x^1,x^2,x^3) \in \mathbb{R}^3 \mid x^1 > 0\}$ endowed with the Riemannian metric*

$$g = \frac{1}{K}\Big(\frac{dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3}{(x^1)^2}\Big) \tag{4.14}$$

for $K \in \mathbb{R}$. We define an idempotent map by $Fe_1 = e_1$, $Fe_2 = e_2$ and $Fe_3 = 0$ where $e_i = \frac{\partial}{\partial x_i}$, $i = 1,2,3$. Since $[e_i,e_j] = 0$ we have

$$\nabla_{e_1}e_1 = -\frac{1}{x^1}e_1,\ \nabla_{e_2}e_2 = \frac{1}{x^1}e_1, \nabla_{e_1}e_2 = -\frac{1}{x^1}e_2, \nabla_{e_2}e_1 = -\frac{1}{x^1}e_2.$$

Thus we get

$$g(R(e_1,Pe_2)Pe_2,e_1) = -\frac{1}{K}\frac{1}{(x^1)^4}, g(R(e_2,Pe_1)Pe_1,e_2) = -\frac{1}{K}\frac{1}{(x^1)^4}.$$

Hence we obtain

$$K(e_1 \wedge Pe_2) = K(e_2 \wedge Pe_1) = -K.$$

Thus (H,g,F) is a potent space form.

We now obtain a special expression of the curvature tensor field of a potent space form.

Theorem 4.1. *Let $(M(c),g,F)$ be a potent space form. Then we have*

$$R(X,FY)FZ = c\{g(Y,FZ)FX - g(FX,Z)FY\} \tag{4.15}$$

$X,Y,Z \in \chi(ImF(M))$.

Proof. Since M is a potent space form, we have

$$g(R(X,FY)FY,X) = c\{g(X,X)g(Y,FY) - g(X,FY)^2\}. \tag{4.16}$$

Replacing Y by $Y+Z$ in (4.16), we get

$$g(R(X,FY)X,FZ) = c\{-g(X,X)g(Y,FZ) + g(X,FY)g(X,FZ)\} \tag{4.17}$$

Using (2.3) and (3.11) we derive

$$R(X,FY)FX = c\{-g(X,X)FY + g(X,FY)FX\}. \tag{4.18}$$

Substituting X by $X+Z$ in (4.18) we obtain

$$\begin{aligned} R(X,FY)FZ + R(Z,FY)FX &= c\{-2g(X,Z)FY + g(X,FY)FZ \\ &\quad + g(Z,FY)FX\}. \end{aligned} \tag{4.19}$$

Replacing X by $X+Z$ in (4.16) we get

$$g(R(X,FY)FY,Z) = c\{g(X,Z)g(Y,FY) - g(X,FY)g(Z,FY)\}. \tag{4.20}$$

Hence we have

$$R(X,FY)FY = c\{g(Y,FY)X - g(X,FY)FY\}. \tag{4.21}$$

If $Y+Z$ is written instead of Y in (4.21) we arrive at

$$\begin{aligned} 2R(X,FY)FZ - R(Z,FY)X &= c\{2g(Y,FZ)X - g(X,FY)FZ \\ &\quad - g(X,FZ)FY\}. \end{aligned} \tag{4.22}$$

Substituting FX instead of X in (4.22) and using (2.1), (2.3) and (3.12) we get

$$\begin{aligned} 2R(X,FY)FZ - R(Z,FY)FX &= c\{2g(Y,FZ)FX - g(X,FY)FZ \\ &\quad - g(X,FZ)FY\}. \end{aligned} \tag{4.23}$$

Thus from (4.21) and (4.23) we conclude that

$$3R(X,FY)FZ = c\{3g(Y,FZ)FX - g(X,FZ)FY - 2g(X,Z)FY\}. \tag{4.24}$$

Finally substituting FX instead of X in (4.24), and using (3.12) and (2.4) we obtain

$$R(X,FY)FZ = c\{g(Y,FZ)FX - g(FX,Z)FY\}$$

which is (4.16). □

5. Concluding Remarks

This paper presents a new class of manifolds. This manifold class is quite different from the manifold classes in the literature. In the paper, examples of the existence of such manifolds are presented, their properties are examined and their sectional curvatures are investigated. When it is seen that the classical sectional curvatures do not work, a new sectional curvature notion is introduced, an example is given and accordingly the curvature tensor field of the manifold is specifically expressed. As can be seen, this presented manifold class shows that it will have a rich geometry. Therefore, we invite researchers to explore this manifold class.

In the first stage, we especially propose the following research problems.

Open Problem 1. As it is well known, submanifolds of manifolds endowed with extra structures on them offer a very rich research [4]. Therefore, the investigation of submanifolds (such as invariant, anti-invariant, semi-invariant, slant) that will be defined depending on the potent structure of an almost potent manifold may produce interesting results.

Open Problem 2. Harmonic maps between manifolds are one of the most important research topics in differential geometry. For example, it is well known that holomorphic maps between two Kaehler manifolds are harmonic [1]. It would be an interesting research problem to investigate the harmonicity of the map defined between a Kaehler potent manifold (potent maps may be defined similarly to holomorphic maps).

Open Problem 3. Riemannian submersions defined on manifolds with a special structure have interesting geometric properties [8], [24]. Therefore, studying the geometric properties of a Riemannian submersion defined on a Kaehler potent manifold will produce rich research results.

Open Problem 4. Special curves on a manifold begin with Nomizu and Yano's definition of the notion of a circle on a manifold [17]. After this concept, helices and similar concepts were also defined [12],[15], [22]. The geometry of special curves on a Kaehler potent manifold will produce interesting geometric results.

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CALCULATION
*Volume **1**, Issue **1**, 2025, Pages:**44-51***
E-ISSN: 3062-2107
www.simadp.com/calculation

MOTION OF GALILEAN PARTICLES WITH CURVATURE AND TORSION

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Abstract. This paper examines the motion of particles governed by an action that depends on the curvature and torsion of their trajectories in the Galilean 3-space G_3. We derive the Euler-Lagrange equation corresponding to the action $H(\gamma) = \int_\gamma f(\kappa, \tau)ds$ in G_3. We present examples to clarify the solutions of the system, clearly explaining their properties and relevance. With examples specifically focusing on the natural Hamiltonian problem derived from the Frenet frame of the curve and a generalization of these natural Hamiltonians, we aim to illustrate their key features and underlying principles.

Keywords: Curvature, Euler-Lagrange equations, Galilean geometry, Torsion, Variational calculus
2020 Mathematics Subject Classification: 53A04, 53A35, 37K25

1. Introduction

The study of particle motion governed by geometric properties such as curvature and torsion provides an understanding of underlying physical and mathematical principles. The interplay between curvature and torsion in defining particle trajectories has long been a subject of interest in both classical mechanics and modern theoretical studies. These geometric properties not only characterize the shape and behavior of curves in space but also play a critical role in variational principles, where the goal is often to identify extremal trajectories that satisfy specific physical constraints. Such analyses have applications across disciplines, including physics, where they model the dynamics of systems, and mathematics, where they enrich the theory of differential geometry and the calculus of variations (see, [1, 4, 5, 6, 9, 10, 11, 12, 15, 17], etc.).

Building on these foundations, variational problems emerge as a central framework for analyzing particle motion and other systems. Deeply rooted in the calculus of variations, they hold an essential role in mathematical analysis and find extensive applications across disciplines such as physics and engineering. These problems aim to identify extrema (minimum or maximum values) of functionals, which are mappings from a space of functions to real

Received: 2024.12.04 *Revised: 2024.12.25* *Accepted: 2024.12.31*

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numbers. The objective of a typical variational problem is to find a function that optimizes a specific quantity, often expressed as an integral. For instance, the Hamiltonian of a space curve, defined as $H = \int f(\kappa, \tau, \kappa', \tau', ...)ds$, where f is a real arbitrary function, depends on the curvature (κ) and torsion (τ) of the curve. A simplified form of this Hamiltonian, expressed as $\int f(\kappa)ds$, depends solely on curvature, offering a more tractable framework for analyzing specific classes of variational problems. This reduction highlights the significance of curvature in determining the geometry and behavior of particle trajectories, particularly in cases where torsion does not contribute to the dynamics. Such models are widely used in applications ranging from the study of elastic rods and thin filaments to the analysis of geodesics and other naturally occurring curves. By focusing exclusively on curvature, these simplified Hamiltonians allow for deeper insights into the fundamental principles governing the shape and stability of particle paths. Extending this concept, a more general form of the Hamiltonian, $\int f(\kappa, \tau)ds$, incorporates both curvature and torsion as central variables. This generalization captures a broader range of geometric and physical phenomena, enabling the study of more complex particle trajectories. The inclusion of torsion reflects the three-dimensional twisting of the trajectory, adding a critical layer of complexity that is essential for understanding systems where both bending and twisting motions play a role. This formulation is particularly useful in problems involving helical structures, dynamical systems, and energy-minimizing configurations in elastic and physical systems. By considering both curvature and torsion, $\int f(\kappa, \tau)ds$ provides a comprehensive framework for exploring the interplay between these geometric properties in shaping particle motion. Capovilla et al. [4] examined the equilibrium conditions of space curves under local energy penalties associated with their curvature and torsion. They derived the Euler–Lagrange equations using the Frenet–Serret frame and exploited Noether's theorem to identify conservation laws tied to Euclidean invariance. The study highlighted specific integrable cases of the Hamiltonian $H = \int f(\kappa, \tau)ds$ connecting the results to physical applications like polymer stiffness and elastic properties of DNA. Following this, Ferrández et al. [9] explored the motion of relativistic particles in 3D pseudo-Riemannian spaces, governed by a Lagrangian as a general function of curvature and torsion. They derived Euler-Lagrange equations, identified dynamical invariants using Killing vector fields, and provided moduli spaces of solutions through integrable critical curves, extending the study of geometrically constrained motions. On the other hand, Tükel [19] contributed to this field by adopting a variational approach to determine critical points of the total squared torsion functional for curves in Euclidean and Minkowski 3-space, further enriching the understanding of these intricate geometric structures.

Shifting focus to Galilean geometry, we enter a framework where space is treated as a rigid, three-dimensional entity and time flows uniformly for all observers, independent of motion. Unlike the relativistic interplay of space and time, the Galilean model offers a simpler, classical foundation for understanding motion and forces, making it a natural setting to explore the mechanics of particles influenced by curvature and torsion. Bilir et al. [7] study investigates the classical variational problem of elastic curves in the Galilean plane, deriving Euler-Lagrange equations, determining the curvature of arc-length parameterized curves, and providing explicit examples. Tükel and Turhan [20] examined elastic curves in Galilean 3$-$space G_3, deriving the Euler-Lagrange equations for the bending energy functional under boundary conditions. They solved the resulting differential equations and provided explicit examples to characterize elastic curves within this geometric framework. They examined natural Hamiltonians derived from the derivatives of the principal normal and binormal vectors of Frenet curves in Galilean and pseudo Galilean 3-space, solving the variational problem for the total squared torsion functional and identifying critical points characterized by constant torsion or curvature [16, 21]. Turhan, in his latest work [18], investigated hyperelastic curves in G_3, focusing on their characterization as extremals of a curvature energy functional, which is also a specific case of a natural Hamiltonian functional. By deriving Euler-Lagrange equations, he provided insights into the geometric behavior of these curves under the Galilean metric structure and illustrated their applications through detailed examples.

This paper focuses on the variational problem defined by the functional $\int f(\kappa,\tau)ds$ representing the curvature and torsion of curves in G_3. By employing the principles of the calculus of variations, we derive the associated Euler-Lagrange equations to characterize critical points of this functional under specific boundary conditions. We provide examples to highlight their key properties and potential applications. We center our analysis on natural Hamiltonian systems derived from the Frenet frame of curves and extend it to a generalization of these Hamiltonians.

2. Preliminaries

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ vectors in G_3. So, the Galilean scalar product of vectors is given as

$$< x, y >_{G_3} = \begin{cases} x_1 y_1, \text{ if } x_1 \neq 0 \text{ or } y_1 \neq 0, \\ x_2 y_2 + x_3 y_3, \text{ if } x_1 = 0 \text{ and } y_1 = 0. \end{cases}$$

The vectors x and y are said to be perpendicular in the Galilean sense if $< x, y >= 0$. The vector $x = (x_1, x_2, x_3)$ is known as isotropic (non-isotropic) if $x_1 = 0$ ($x_1 \neq 0$). Any unit non-isotropic vector has the form $x = (1, x_2, x_3)$. For the vector x, the Galilean norm is written as

$$\|x\|_{G_3} = \begin{cases} |x_1|, \text{ if } x_1 \neq 0, \\ \sqrt{x_2^2 + x_3^2}, \text{ if } x_1 = 0, \end{cases}$$

[22].

A curve $\alpha : I \subset R \to G_3$ is called as admissible if it has no inflection points and no isotropic tangents (see, [3, 8, 23]). For a unit speed admissible curve $\alpha(s)$ parametrized by

$$\alpha(s) = (s, \alpha_2(s), \alpha_3(s)),$$

where s is the arclength parameter of α, we can give the curvature $\kappa(s)$ and the torsion $\tau(s)$ as follows

$$\kappa(s) = \sqrt{\alpha_2''(s) + \alpha_3''(s)}$$

and

$$\tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)}.$$

On the other hand, the Frenet frame of the curve $\alpha(s)$ in G_3 is given by

$$\begin{aligned} T(s) &= \alpha'(s), \\ N(s) &= \frac{1}{\kappa(s)}\alpha''(s), \\ B(s) &= \frac{1}{\kappa(s)}(0, -\alpha_3''(s), \alpha_2''(s)), \end{aligned}$$

where T, N, and B are respectively known as the tangent vector, principal normal vector and binormal vectors of $\alpha(s)$. So, the Frenet equations of $\alpha(s)$ are written in matrix form as

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ 0 & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{2.1}$$

[8, 20].

3. Particles with Galilean Curvature and Torsion

Let G_3 be the Galilean 3-space and γ be an admissible curve with speed $\nu = \|\gamma'(t)\|$, curvature κ, torsion τ and Frenet frame $\{T, N, B\}$. This section concerns the model, whose action is given by the functional $H(\gamma) = \int_\gamma f(\kappa, \tau) ds$ in G_3. Let $\Gamma = \Gamma(t, r)$ be a variation of $\gamma : [0, \ell] \to G_3$ with $\Gamma(t, 0) = \gamma(t)$. Associated with Γ, we consider the variation vector field W along $\gamma(t)$. The vector fields $V(t, r)$, $W(t, r)$ can be defined, where $V(0, t) = \gamma'(t)$ and $W(t) = W(0, t)$ is a variational vector field along $\gamma(t)$ (see, [2, 9, 18]). If s denotes the arclength parameter, then $\gamma(s, r)$, $\kappa^2(s, r)$, $V(s, r)$, etc. can be written for the corresponding reparametrizations, where $s \in [0, \ell]$ and ℓ is arc length of γ.

We arrive at the following Lemma from the Frenet equations in (2.1).

Lemma 3.1 ([21])**.** *Let $\gamma(t, r)$ be a variation of curve $\gamma \in G_3$. Then the following formulas are satified:*

i) $W(\nu) =< W', T > \nu$,
ii) $W(\kappa) =< W'', N > -2\kappa < W', T >$,
iii) $W(\tau) = \left(\frac{1}{\kappa}(< W'', B >)\right)' - < W', \tau T >$.

Now we assume that γ is a stationary point of the functional $H(\gamma)$. Then, we have

$$\partial H(W) = \left.\frac{\partial H(\gamma + \varepsilon W)}{\partial \varepsilon}\right|_{\varepsilon=0} = 0.$$

Thus, we obtain

$$\left.\frac{\partial H}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_0^\ell \left(f_\kappa W(\kappa) + f_\tau W(\tau) + < W', T >\right) ds.$$

Taking into consideration Lemma 3.1, we find

$$\left.\frac{\partial H}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_0^\ell \begin{pmatrix} < W'', f_\kappa N > + < W', -2\kappa f_\kappa T > + f_\tau < W''', \frac{1}{\kappa} B > \\ + < W'', f_\tau (\frac{1}{\kappa} B)' > + < W', -f_\tau \tau T > + < W', f T > \end{pmatrix} ds.$$

Then, by using standard arguments involving the above formulas and integration by parts, the first variation of $H(\gamma)$ along γ in the direction of W is given by

$$\frac{\partial H}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \int_0^\ell < E, W > ds + B[W,\gamma]|_0^\ell,$$

where

$$E = \left((-f + \tau f_\tau + 2\kappa f_\kappa)\, T + (f_\kappa N)' - \left(\frac{f_\tau'}{\kappa} B\right)'\right)'$$

and

$$\begin{aligned} B[W,\gamma]|_0^\ell &= < W'', \frac{f_\tau}{\kappa} B > \bigg|_0^\ell + < W', f_\kappa N - \frac{f_\tau'}{\kappa} B > \bigg|_0^\ell \\ &\quad + < W, (f - \tau f_\tau - 2\kappa f_\kappa)\, T - (f_\kappa N)' + \left(\frac{f_\tau'}{\kappa} B\right)' > \bigg|_0^\ell . \end{aligned}$$

Point out that, we used f_κ and f_τ to denote the partial derivatives of f with respect to κ and τ, respectively. Also, we restrict ourselves to variations with fixed endpoints having the same Frenet frames on them. Then, the boundary term $B[W,\gamma]|_0^l = 0$, so that the critical curves are characterized by the vanishing of the Euler–Lagrange operator $E' = 0$. If the necessary calculations are made and the Frenet equations are used, we have

$$(-f + \tau f_\tau + 2\kappa f_\kappa)' = 0, \tag{3.2}$$

$$-f\kappa + \tau\kappa f_\tau + 2\kappa^2 f_\kappa + f_\kappa'' - \tau^2 f_\kappa + (\frac{f_\tau'}{\kappa})'\tau + (\frac{f_\tau'\tau}{\kappa})' = 0 \tag{3.3}$$

and

$$f_\kappa'\tau + (f_\kappa\tau)' - (\frac{f_\tau'}{\kappa})'' + \frac{f_\tau'}{\kappa}\tau^2 = 0. \tag{3.4}$$

From (3.2), we obtain

$$f = \tau f_\tau + 2\kappa f_\kappa + A, \tag{3.5}$$

where A constant. Substituting (3.5) into (3.3), we obtain

$$-\kappa A + f_\kappa'' - \tau^2 f_\kappa + (\frac{f_\tau'}{\kappa})'\tau + (\frac{f_\tau'\tau}{\kappa})' = 0. \tag{3.6}$$

Theorem 3.1. *The motion equations of the action, defined by the functional* $H(\gamma) = \int_\gamma f(\kappa,\tau)ds$ *in* G_3 *are characterized by the Euler-Lagrange equations (3.4) and (3.6).*

To further illustrate this theory, we now provide examples that demonstrate the application of the Euler-Lagrange equations and highlight the geometric and physical implications of the solutions.

In the Galilean 3-space G_3, curves can be classified based on their curvature and torsion values, as outlined in [14]. This classification includes special cases such as straight lines, plane curves, circular helices, generalized helices, Salkowski curves, and anti-Salkowski curves, each defined by specific geometric properties. In the following examples, we will refer to this classification as a basis for analyzing critical curves

Example 3.1. *Let* γ *is an admissible curve in* G_3. *We examine whether* γ *is critical for* H *based on its curvature and torsion values, as outlined in the cases below to illustrate its geometric properties:*

i) Straight line. A straight line γ in G_3 is a critical point of the functional H.

ii) Planar curve. Let γ be a planar curve in G_3. If γ is is a critical point of the functional H, then, γ satisfies the equation $-\kappa A + f''_\kappa = 0$.

iii) Circular helices (W-curves). Any circular helix or $W-$curve in G_3 is a solution of the functional H when the function $f(\kappa, \tau) = 0$.

iv) Salkowski and anti Salkowski curves. A Salkowski curve in G_3 is critical for the functional H if it satisfies the following Euler-Lagrange equation

$$-\kappa^2 A + f''_\tau \tau + (\frac{f'''_\tau}{\tau})' = 0.$$

Moreover, it is evident from the Euler-Lagrange equations (3.4) and (3.6), that there is no anti-Salkowski curve in G_3 that is critical for H.

Example 3.2. *Let $\gamma(s) = (s, \dfrac{(s - \sin s \cos s)}{4}, \dfrac{(\sin s^2 - s^2)}{4})$ be a regular curve in G_3 with $\kappa(s) = \sin s$ and $\tau(s) = 1$* [14]. *If the values of curvature and torsion are incorporated into the Euler-Lagrange equations, it can be observed that the curve in question serves as an example for the derived results for $s = \frac{\pi}{2} + k\pi$, $k \in Z$.*

We know that a simple model for the Hamiltonian is in the form $\int f(\kappa)ds$ which depends on the curvature. Especially, a natural Hamiltonian $\int \kappa^2 ds$ generated by $< T', T' >$ is known as a bending energy functional and critical points of this functional under suitable condition are called as elastic curves [13]. Elastic curves and its generalization under given first order boundary data have been worked and developed by many authors up to now. In his study, Turhan characterized hyperelastic curves in G_3 by addressing a generalization of the functional formed by the inner product of first derivative of the tangent vector of the curve, which is known as a natural Hamiltonian [18]. Upon examining the derived Euler-Lagrange equation, it is an undeniable fact that the results serve as an example for the problem addressed in this study. Beyond this, another obvious question arises: how can the critical points of the natural Hamiltonian functional formed by other frame elements (produced by $< N', N' >$ and $< B', B' >$) of the Frenet frame be determined? In another example, the critical points of the natural Hamiltonian constructed using the binormal vector field of the curve are examined.

Example 3.3. *We consider an admissible curve γ with Galilean Frenet frame $\{T, N, B\}$ and curvature κ and torsion τ in G_3. A generalization of the natural Hamiltonian is considered as the functional $\int < B', B' >^{n/2} ds$. This functional is a generalized torsion energy action given by $\int \tau^n ds$. The critical points of this functional are characterized by the following Euler-Lagrange equations*

$$n(n-1)\tau^{n-1}\tau' = 0, \tag{3.7}$$

$$(n-1)\tau^n \kappa + \left(\frac{n(n-1)\tau^{n-2}\tau'}{\kappa}\right)' \tau + \left(\frac{n(n-1)\tau^{n-1}\tau'}{\kappa}\right)' = 0 \tag{3.8}$$

and

$$\frac{n(n-1)\tau^n\tau'}{\kappa} - \left(\frac{n(n-1)\tau^{n-2}\tau'}{\kappa}\right)'' = 0. \tag{3.9}$$

From (3.7), we get τ is a constant value. If τ is zero, then Eqs. (3.8) and (3.9) are satisfied for any value of κ. If $\tau \neq 0$, then τ is constant and from (3.8), we obtain κ is zero.

4. Acknowledgments

While preparing this work, the authors utilized an artificial intelligence tool to enhance the language and clarity of the abstract and introduction sections. Following this, the author thoroughly reviewed and made necessary edits, assuming full responsibility for the content of the publication.

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CALCULATION
*Volume **1**, Issue **1**, 2025, Pages:**52-62***
E-ISSN: 3062-2107
www.simadp.com/calculation

A STUDY ON THE TOPOLOGY OF GRAPH COMPLEXES

ALPER ÜLKER *

Abstract. In this paper, we consider the homotopy types of independence complexes of some graphs. Moreover, we study the homotopy types of graphs which are expanded from a given graph via certain operations. For any graph whose independence complex is contractible, we calculate the homotopy type of clique complex of its central graph. In addition to these, we build a complex from a bipartite graph to calculate homotopy types of some complexes.

Keywords: Independence complex, Clique complex, Homotopy type, Central graph.
2020 Mathematics Subject Classification: Primary 55P10, Secondary 05E40, 05C69.

1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V(G)$ is called *independent set* if any two vertices x and y in S are non-adjacent in G. The *independence complex* of a simple undirected graph G is the simplicial complex whose simplices are the independent sets of G. It is denoted by $\mathrm{Ind}(G)$. If any two edges in $M \subseteq E(E)$ are non-adjacent, then M is called a *matching*. The *matching complex* $\mathrm{M}(G)$ of a graph G is a simplicial complex with vertices are the edges of G and faces are the matchings of G. The *clique complex* $\Delta(G)$ of a graph G is the simplicial complex whose vertex set is the vertices of G and faces are the cliques of G. The independence complex of a graph G is the clique complex of its complement. Also, the independence complex of the line graph of G is the matching complex of G. These arguments justify the study of independence complexes of graphs in relation to their clique and matching complexes. Numerous papers have been written on the topic of independence complexes of graphs from an algebraic perspective [8, 9, 16], and topological perspective [1, 5, 6, 7, 10, 11]. Main studies about the complexes arising from the graphs are to determine the its homotopy types.

In [13], D. Kozlov calculated homotopy types of independence complexes of cycle and path graphs. Engström studied the homotopy types of claw-free graphs [6]. In [11], Kawamura investigated homotopy types of independence complexes of chordal graphs. Ehrenborg and Hetyei studied the independence complexes of forests [5]. In [4], Csorba investigated the

Received: 2024.11.28 *Revised: 2024.12.24* *Accepted: 2025.01.13*
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homotopy types of independence complexes of graphs whose edges are subdivided. Jonsson proved that the independence complexes of bipartite graphs have the same homotopy type as those of the suspensions of simplicial complexes [10]. Also, in [1] Barmak introduced a notion called *star cluster* and provided a novel tool to study the topology of independence complexes. Many of our proofs are based on this notion. In addition to these, many authors studied the face enumeration of independence complexes [9, 12].

In this paper, we deal with graph complexes such as independence, clique and matching complexes. The paper is organized as follows: Section 2 focuses on fundamental definitions and previously established results that will serve as the foundation for the remainder of the paper. In Section 3, we study the topology of Lozin transformation of a graph. Also, we computed the homotopy types of the clique complex of central graphs for any contractible graph. In the last section, we introduced a complex arising from a bipartite graph and calculated homotopy types of some complexes arising from bipartite graphs.

2. Preliminaries

Given a graph $G = (V, E)$, the set $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ is called the *open neighborhood* of u in G and $N_G[u] = N_G(u) \cup \{u\}$ is called the *closed neighborhood* of u. The induced subgraph of G on $S \subseteq V(G)$ is the graph consists of vertex set $V(S)$ and two vertices in S are adjacent if and only if they are adjacent in G, this subgraph is denoted by $G[S]$. The subgraph $G\backslash U$ is obtained by deleting the vertices of U and remove all the edges connecting the vertices of U. A *complete graph* K_n on n vertices is a graph in which for every vertices u and v, there is an edge uv in K_n.

A finite *(abstract) simplicial complex* Δ on the vertex set $V(\Delta)$ is a collection of subsets of $V(\Delta)$ which satisfies; $\sigma \in \Delta$ and $\gamma \subseteq \sigma$, then $\gamma \in \Delta$. The elements of Δ are called *faces* or *simplices*. The maximal faces of Δ with respect to inclusion are called *facets*. The dimension of a face σ of Δ is defined by $\dim(\sigma) = |\sigma| - 1$.

The k-skeleton of a simplcial complex Δ is a simplicial complex which consisting of i-simplices of Δ with $i \leq k$.

A subcomplex Δ' of a complex Δ is called an *induced subcomplex* of Δ; if $\sigma \in \Delta$ and $\sigma \subseteq V(\Delta')$, then $\sigma \in \Delta'$. The induced subcomplex of Δ on $U \subseteq V(\Delta)$ is denoted by $\Delta[U]$. Let Δ and Γ are simplicial complexes with disjoint vertex sets. Then the *join* of Δ and Γ is the simplicial complex whose faces consists of $\sigma \cup \tau$ such that $\sigma \in \Delta$ and $\tau \in \Gamma$. The join of Δ and Γ is denoted by $\Delta * \Gamma$.

Next we give definitions of star, link and deletion of a face of a simplicial complex.

Definition 2.1. *If σ is a face of Δ, then the link, deletion and star of σ are defined as follows:*

$$\mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cup \tau \in \Delta, \sigma \cap \tau = \emptyset\} \text{ is called link of } \sigma,$$
$$\mathrm{del}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cap \tau = \emptyset\} \text{ is called deletion of } \sigma,$$
$$\mathrm{st}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cup \tau \in \Delta\} \text{ is called star of } \sigma.$$

The definitions of link and deletion of a vertex x of a graph G can be translated for independence complexes as follows:

$$\mathrm{lk}_{\mathrm{Ind}(G)}(x) = \mathrm{Ind}(G\backslash N_G[x]) \text{ and } \mathrm{del}_{\mathrm{Ind}(G)}(x) = \mathrm{Ind}(G\backslash x).$$

In the following, the definitions of cone and suspension of a simplicial complex are given.

Definition 2.2. *The cone* $\mathrm{C}(\Delta)$ *of a simplicial complex* Δ *with apex* $v \in V(\Delta)$ *is* $v * \Delta$.

Definition 2.3. *The suspension* $\Sigma(\Delta)$ *of a simplicial complex* Δ *is* $\{u, v\} * \Delta$.

In the following, the definitions of independence complex, clique complex and matching complex will be given.

Definition 2.4. *Let* $G = (V, E)$ *be a graph. The independence complex of* G *is the simplicial complex on* V *whose faces are the independent sets of* G *and denoted by* $\mathrm{Ind}(G)$.

Definition 2.5. *Let* $G = (V, E)$ *be a graph. The clique complex of* G *is the simplicial complex on* V *whose faces are the cliques of* G *and denoted by* $\Delta(G)$.

Definition 2.6. *Let* $G = (V, E)$ *be a graph. The matching complex of* G *is the simplicial complex on* V *whose vertices are the edges and faces are the matchings of* G*. This complex is denoted by* $\mathrm{M}(G)$.

Definition 2.7. *Let* Δ *be a simplicial complex.* Δ *is said to be a flag complex if and only if every missing face of* Δ *is of size two.*

From the above definitions, the following proposition can be given.

Proposition 2.1. *Let* G *be a simple undirected graph. Then the complexes* $\mathrm{Ind}(G)$, $\Delta(G)$ *and* $\mathrm{M}(G)$ *are flag complexes.*

Star cluster of a face is defined by Barmak in [1], we recall its definition in the following:

Definition 2.8. [1] *Let* σ *be a simplex of a simplicial complex* Δ*. The star cluster of* σ *in* Δ *as the subcomplex* $\mathrm{SC}_{\Delta}(\sigma) = \bigcup_{v \in \sigma} \mathrm{st}_{\Delta}(v)$.

The following lemma is very important for the rest of the paper and proved in [1].

Lemma 2.1. *(Theorem 3.6,* [1]*) Let* G *be a graph and let* $v \in G$ *be a non-isolated vertex which is contained in no triangle i.e. no two vertices of* $N_G(v)$ *are adjacent. Then* $N_G(v)$ *is a simplex of* $\mathrm{Ind}(G)$ *and* $\mathrm{Ind}(G) \simeq \Sigma(\mathrm{st}_{\mathrm{Ind}(G)}(v) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(v)))$.

The next theorem is about the homotopy types of triangle-free graphs.

Theorem 2.1. *(Theorem 3.5,*[1]*) Let* G *be a graph such that there exits a vertex* $v \in G$ *which is contained in no triangle. Then the independence complex of* G *has the homotopy type of a suspension.*

Remark 2.1. *Let* x *be a vertex of a graph* G *which is contained in no triangle. Then the faces of the simplicial complex* $\mathrm{st}_{\mathrm{Ind}(G)}(x) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(x))$ *consist of the independent sets* σ *such that* $\sigma \cup \{x\}$ *and* $\sigma \cup \{y\}$ *are independent for* $y \in N_G(x)$*. If the independent set* σ *consists of only vertices in* $N_G(y) \setminus \{x\}$ *for* $y \in N_G(x)$*, then* σ *is the boundary of* $(\mid \sigma - 1 \mid)$*-simplex in* $\mathrm{st}_{\mathrm{Ind}(G)}(x) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(x))$.

Definition 2.9. *A space* X *is called contractible, if* X *is homotopy equivalent to a point.*

The next lemma is about homotopy types of the suspension of a contractible space.

Lemma 2.2. [5] *If Δ is a contractible complex, then $\Sigma(\Delta)$ is a contractible complex.*

Proposition 2.2. [5, 15] *The homotopy types of the suspension of wedge sum of spheres are as follows;*

$$\Sigma\left(\mathbb{S}^{k_1} \vee \mathbb{S}^{k_2} \vee ... \vee \mathbb{S}^{k_i}\right) \simeq \mathbb{S}^{k_1+1} \vee \mathbb{S}^{k_2+1} \vee ... \vee \mathbb{S}^{k_i+1}.$$

3. Graph operations and its topology

In this section we study the topology of complexes of graphs expand from a graph via a particular operation. The Lozin transformation of a graph is an operation when applied it increases graphs induced matching number and introduced in [14]. The homotopy type of independence complex of Lozin transformed graph studied by the authors in [2]. We study this by the notion star cluster.

Definition 3.1. *Let G be a graph and x be a vertex of G. The Lozin's transformation $\mathcal{L}_x(G)$ of G with respect to x is defined as follows:*

(i) Partition the neighborhood $N_G(x)$ of the vertex x into two subsets Y and Z in arbitrary way,

(ii) add a $P_4 = (\{y, a, b, z\}, \{ya, ab, bz\})$ to the rest of the graph,

(iii) connect vertex y of the P_4 to each vertex in Y, and connect z to each vertex in Z.

The following figure shows Lozin transformation of G with respect to vertex x. The partition was done with respect to vertex x, the edge uv forms the partition Y and vertices p and w form the partition Z.

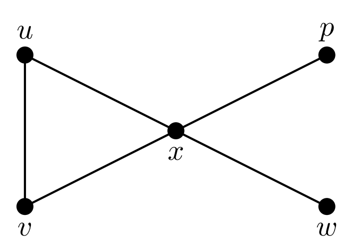

Figure 1. A graph G

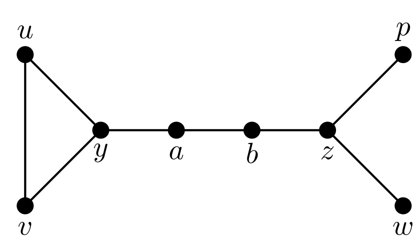

Figure 2. $\mathcal{L}_x(G)$

The Lozin transtormation expands a graph from one vertex. In the following theorem, we give the homotopy type of the Lozin transformed graph by means of the original graph.

Theorem 3.1. *([2]) Let G be a graph and $\mathcal{L}_x(G)$ be its Lozin transformation with respect to x. Then $\mathrm{Ind}(\mathcal{L}_x(G))$ is homotopy equivalent to $\Sigma(\mathrm{Ind}(G))$.*

Proof. Let $\mathcal{L}_x(G)$ be the Lozin transformation of G with respect to the vertex x. If we partition $N_G(x)$ into two subsets namely Y and Z i.e. $N_G(x) = Y \cup Z$. Then the vertices a and b are contained in no triangle, since $y - a - b - z$ is a path. We prove this for a and by symmetry it is similar for b. The complex $\mathrm{Ind}_{\mathcal{L}_x(G)}(N(a))$ is the 1-simplex yb. It is enough to show that $\mathrm{st}_{\mathrm{Ind}(\mathcal{L}_x(G))}(a) \cap \mathrm{SC}_{\mathrm{Ind}(\mathcal{L}_x(G))}(N_{\mathcal{L}_x(G)}(a)) = \mathrm{Ind}(G)$. Let $\sigma \in \mathrm{st}_{\mathrm{Ind}(\mathcal{L}_x(G))}(a) \cap \mathrm{SC}_{\mathrm{Ind}_{\mathcal{L}_x(G)}}(N_{\mathcal{L}_x(G)}(a))$ be a maximal face. Then σ is a maximal independent set of $\mathcal{L}_x(G)$ which can be extended to a and also can be extended to y or b. If $\sigma \cup \{a\}$ and $\sigma \cup \{y\}$ are independent sets and $\sigma \cup \{b\}$ is not independent. Then $\sigma \cup \{z\}$ is an independent set. Thus $\sigma \cap Y = \emptyset$ and $\sigma \cap Z = \emptyset$. If $\sigma \cup \{a\}$ and $\sigma \cup \{b\}$ are independent sets and $\sigma \cup \{y\}$ and $\sigma \cup \{z\}$ are not independent. Then $\sigma \cap Y \neq \emptyset$ or $\sigma \cap Z \neq \emptyset$ or both. Thus one can conclude that σ is a maximum independent set of $\mathrm{Ind}(G)$. Conversely, let $\sigma \in \mathrm{Ind}(G)$ be a maximum independent set. If $x \in \sigma$ then $\sigma \cap N_G(x) = \emptyset$. So σ can be extended to a and y or σ can be extended to a and b in $\mathcal{L}_x(G)$. Thus $\sigma \in \mathrm{st}_{\mathrm{Ind}(\mathcal{L}_x(G))}(a) \cap \mathrm{SC}_{\mathrm{Ind}(\mathcal{L}_x(G))}(N_{\mathcal{L}_x(G)}(a))$. □

If a graph contains an induced path P_4, then we can contract endpoints of P_4 into one vertex. In other words, we can reverse the Lozin operation of a graph if the graph has P_4.

Corollary 3.1. *Let G be a graph. If G has a subgraph P_4 whose internal vertices are of degree two and end vertices are not adjacent. Then $\mathrm{Ind}(H)$ is homotopy equivalent to $\Sigma(\mathrm{Ind}(G))$ where H is formed by contracting the end vertices of P_4 into one vertex.*

Proof. The contraction of a P_4 from end vertices into one vertex in a graph is clearly reversing the Lozin operation. □

When applying the Lozin transformation to a graph G with respect to the vertex x, we partition the vertex set of $N_G(x)$ into two disjoint sets Y and Z. However, the next theorem states that if there exists a vertex $t \in Y \cap Z$, then the Corrollary 3.1 still applies.

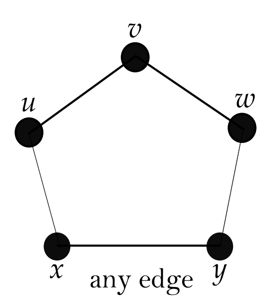

FIGURE 3. Adding P_3 to any edge of a graph

Theorem 3.2. *Let G be a graph and G' is obtained by attaching a P_3 by adding edges from endpoints of P_3 to endpoints of any edge from end vertices to make a C_5. Then $\mathrm{Ind}(G') \simeq \Sigma(\mathrm{Ind}(G))$.*

Proof. Let $u-v-w$ be a path attached to the edge xy of a graph G. Then $x-u-v-w-y$ is a C_5. So v is in no-triangle and $N_G(v)$ is a 1-simplex uw. So Theorem 3.1 implies that $\text{Ind}(G') \simeq \Sigma(\text{st}_{\text{Ind}(G)}(v) \cap \text{SC}_{\text{Ind}(G)}(N_G(v)))$. Now we show that $\text{st}_{\text{Ind}(G)}(v) \cap \text{SC}_{\text{Ind}(G)}(N_G(v)) = \text{Ind}(G)$. Let $\sigma \in \text{st}_{\text{Ind}(G)}(v) \cap \text{SC}_{\text{Ind}(G)}(N_G(v))$ be a face. Then σ is an independent set which can be extended to independent set $\sigma \cup \{v\}$. Also σ can be extended to $\sigma \cup \{u\}$ or $\sigma \cup \{w\}$ or both. Thus $\sigma \in \text{Ind}(G)$.

Now assume that $\sigma \in \text{Ind}(G)$ is an independent set. If $x \in \sigma$ and $y \notin \sigma$ then $\sigma \cup \{v\}$ and $\sigma \cup \{w\}$ is independent. If $x \notin \sigma$ and $y \in \sigma$ then $\sigma \cup \{v\}$ and $\sigma \cup \{u\}$ is independent. Suppose that both x and y are not in σ. Then $\sigma \cup \{v\}$ and $\sigma \cup \{u, w\}$ are independent sets of G'. Thus $\sigma \in \text{st}_{\text{Ind}(G)}(v) \cap \text{SC}_{\text{Ind}(G)}(N_G(v))$. Therefore $\text{st}_{\text{Ind}(G)}(v) \cap \text{SC}_{\text{Ind}(G)}(N_G(v)) = \text{Ind}(G)$, this completes proof. ☐

Example 3.1. *Let G_n be the graph constructed by gluing cycles C_5 as described in Figure 4. Then the homotopy type of* $\text{Ind}(G_n)$ *can be calculated by Theorem 3.2. Thus* $\text{Ind}(G_n) \simeq \Sigma^n(\mathbb{S}^0) \simeq \mathbb{S}^n$.

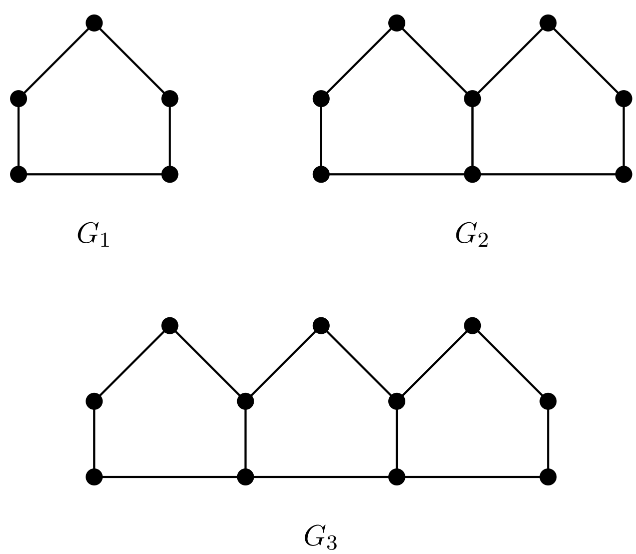

Figure 4. $\text{Ind}(G_n)$

In the following we give the definition of central graph of an undirected simple graph. This operation increases the number of vertices of the original graph.

Definition 3.2. *Let G be a simple and undirected graph and let $V(G)$ and $E(G)$ are vertex and edge sets of G, respectively. The central graph of G, denoted by $C(G)$, is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.*

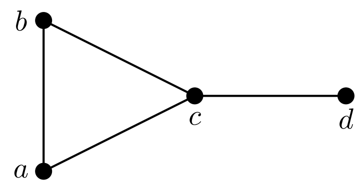

FIGURE 5. A graph G

Example 3.2. *The graph depicted in Figure 6 is the central graph of G which is depicted in Figure 5.*

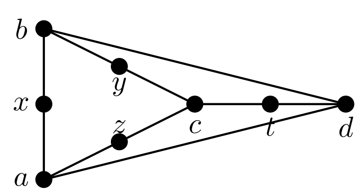

FIGURE 6. Central graph of G

Theorem 3.3. [3] *Let Δ be a simplicial complex with vertex V. If there exits a subset $A \subseteq V$ such that $\dim(\Delta[A]) = 0$ and $\Delta[V \backslash A]$ is contractible, then $\Delta \simeq \bigvee_{x \in A} \Sigma(\mathrm{lk}_\Delta(x))$.*

Proposition 3.1. *Let $G = (V, E)$ be a graph and $C(G)$ be its central graph. If $\mathrm{Ind}(G)$ is contractible, then $\Delta(C(G)) \simeq \bigvee_{|E(G)|} \mathbb{S}^1$.*

Proof. Assume that S is the set of vertices added to $V(G)$ when subdividing the edges. It is clear that the complex $\Delta(C(G)[V \backslash S])$ is $\mathrm{Ind}(G)$. So it is contractible by assumption. Since $\Delta(C(G)[S])$ is a discrete set of vertices, this implies that $\dim \Delta(C(G)[S]) = 0$. Then, by Theorem 3.3, one can decude that $\Delta(C(G)) \simeq \bigvee_{x \in S} \Sigma(\mathrm{lk}_{\Delta(C(G))}(x))$. In $\Delta(C(G))$, for every vertex $x \in S$, the complex $\mathrm{lk}_{\Delta C(G)}(x)$ consists of two disjoint vertices, since they are adjacent to vertices which form an edge in G. Thus $\mathrm{lk}_{\Delta(C(G))}(x) \simeq \mathbb{S}^0$ for every $x \in S$. Therefore we have $\Delta_{C(G)} \simeq \bigvee_{|E(G)|} \mathbb{S}^1$ by Proposition 2.2. ☐

In [11], the author stated that for each wedge $\bigvee \mathbb{S}^{k_t}$ of finitely many spheres, there exists a chordal graph G such that $\mathrm{Ind}(G)$ is homotopy equivalent to $\bigvee \mathbb{S}^{k_t}$ for $k_t \geq \gamma(G) - 1$, where $\gamma(G)$ is the domination number of G. In the following theorem, spheres are 1-dimensional and graph is arbitrary.

Theorem 3.4. *For each $\bigvee \mathbb{S}^1$ of finitely many 1-spheres, there exists a graph G such that $\mathrm{Ind}(G)$ is homotopy equivalent to $\bigvee \mathbb{S}^1$.*

Proof. Our proof is based on the construction of a complex which has homotopy type $\bigvee \mathbb{S}^1$ and homotopy equivalent to an independence complex of a graph. Assume that the number of 1-spheres in the product $\mathbb{S}^1 \vee \mathbb{S}^1 \cdots \vee \mathbb{S}^1$ equals to m. Let H be a graph consists of two connected components; a vertex x and a path P_{m+1}. Since $\mathrm{Ind}(H)$ is a contractible graph with m edges, then the clique complex of its central graph $\Delta(C(H))$ is homotopy equivalent to $\bigvee_m \mathbb{S}^1$ by Proposition 3.1. Therefore $\mathrm{Ind}(G) = \Delta(C(H)) \cong \bigvee_m \mathbb{S}^1$ with $G = \overline{C(H)}$. ☐

Example 3.3. *Given a graph P_4 which is contractible, its central graph and complement shown in Figure 7 and 8. So $\mathrm{Ind}(\overline{C(P_4)}) = \Delta(C(P_4))$ is homotopy equivalent to $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1$.*

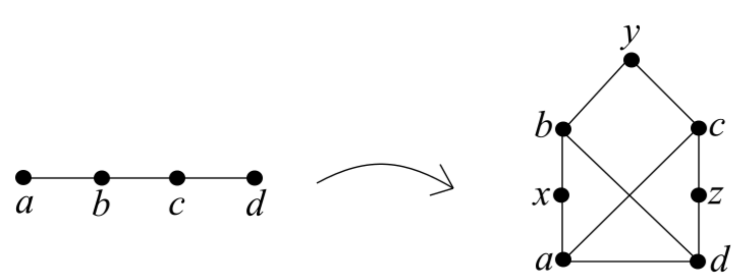

FIGURE 7. Graph P_4 and its central graph $C(P_4)$

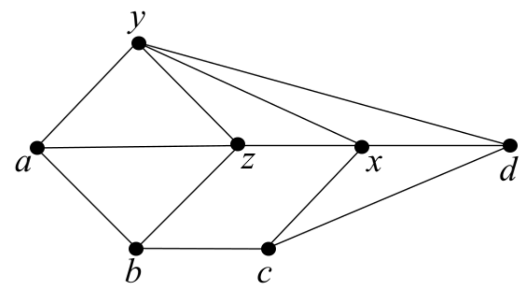

FIGURE 8. Complement of $C(P_4)$.

4. Building a complex from a bipartite graph

A complex is homeomorphic to its barycentric subdivision and this subdivision is an independence complex of a graph. Thus it is a well-known fact that every simplicial complex is homotopy equivalent to an independence complex $\text{Ind}(G)$ of a graph G.

In [10], Jonsson defined a complex $\Gamma_{G,V} \subseteq 2^V$ as follows:
Let $G = V \cup W$ be a bipartite graph. A set $\sigma \subseteq V$ belongs to $\Gamma_{G,V}$ if and only if there is a vertex $w \in W$ such that $\sigma \cup \{w\}$ is an independent set in G.

The following theorem states that the suspension of this complex is homotopy equivalent to independence complex of bipartite graph G.

Theorem 4.1. [10] *Let G be a bipartite graph with nonempty parts V and W. Then* $\text{Ind}(G) \simeq \Sigma(\Gamma_{G,V})$.

From above arguments, we can define a simplicial complex on a bipartite graph B as follows:

Definition 4.1. *Let B be a bipartite graph with bipartition U and W. Then the simplicial complex associated to this graph B is a complex whose vertex set is U and faces are the subsets $\sigma \subseteq U$ such that $\sigma \cup \{v\}$ is an independent set of B for some $u \in W$. This complex is denoted by Δ_B.*

Remark 4.1. *The complex* $\text{Ind}(B)$ *is homotopy equivalent to* $\Sigma(\Delta_B)$.

Proposition 4.1. *If $B = U \cup W$ is a complete bipartite graph $K_{m,n}$, then Δ_B is an empty complex. Moreover,* $\text{Ind}(B) \simeq \Sigma(\Delta_B) \simeq S^0$.

Proof. Since for any $u \in U$ there exits no $\sigma \subseteq W$ such that $\sigma \cup \{u\}$ is an independent set of B, this implies that Δ_B is an empty complex. From Proposition 2.17, one can conclude that $\text{Ind}(B) \simeq S^0$. □

Proposition 4.2. *If the graph B consists of n disjoint edges, then Δ_B is the boundary complex of a $(n-1)$-simplex and* $\text{Ind}(G) \simeq \mathbb{S}^{n-1}$.

Proof. Since, for any x_i, the set $\{x_1, x_2, ..., \hat{x}_i, ..., x_n\}$, obtained by omitting, forms a facet of Δ_B, it follows that Δ_B is the boundary complex of a $(n-1)$-simplex. Then by Proposition 2.17, one can therefore conclude that $\text{Ind}(G) \simeq \Sigma(\Delta_B) \simeq \mathbb{S}^{n-1}$. □

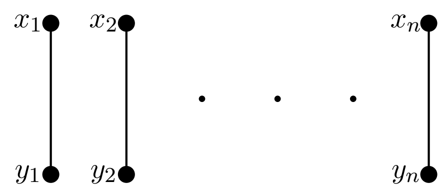

FIGURE 9. n disjoint edges

Definition 4.2. *Let Δ be a simplicial complex on vertex set $V(\Delta)$. The Alexander dual of Δ is the simplicial complex $\Delta^* = \{A \subset V(\Delta) : V(\Delta)\backslash A \notin \Delta\}$.*

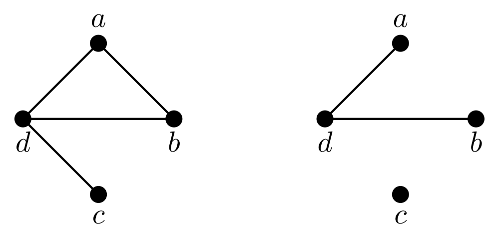

FIGURE 10. A simplicial complex and its Alexander dual

In [4], Csorba showed that the independence complex of a graph whose edges are subdivided exactly once is homotopy equivalent to the suspension of Alexander dual of the independence complex of that graph. In the next theorem, we will build a complex and give its homotopy type. It provides a different method.

Theorem 4.2. *([4], Theorem 5) Let G_2 be the edge subdivision of a graph G. Then the independence complex of G_2 has the same homotopy type of $\Sigma(\mathrm{Ind}(G)^*)$.*

Proof. Let $V(G) = X = \{x_1, ..., x_m\}$. If the set $\{y_1, y_2, ..., y_m\}$ consists of the vertices added to G when subdividing the edges, then we have $V(G_2) = V(G) \cup \{y_1, y_2, ..., y_m\}$. If we set $Y = \{y_1, y_2, ..., y_m\}$, then G_2 is a bipartite graph with bipartition $G_2 = X \cup Y$. Thus $\Delta_{G_2} = \{F_i \subset X : F_i \cup \{y_i\}$ is independent set for some $y_i \in Y\}$. Since $V\backslash F_i$ is an edge of G and not a face of $\mathrm{Ind}(G)$. Therefore we have $\mathrm{Ind}(G_2) \simeq \Sigma(\mathrm{Ind}(G)^*)$. □

5. Conclusion

We use the star cluster notion to determine the homotopy types of complexes arising from triangle-free graphs. We construct a complex from a bipartite graph which is a triangle-free graph. Further studies may be a concern to construct new methodologies for other graph complexes such as matching and clique complexes.

6. Acknowledgment

The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

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VOLUME 1 ISSUE 1 January 2025
E-ISSN: 3062-2107 www.simadp.com/calculation

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