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SURFACES WITH IDEMPOTENT SHAPE OPERATOR MATRIX DEFINED ALONG A SURFACE CURVE

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Abstract. In this study, we investigate surfaces in differential geometry whose shape operator is idempotent. Such an algebraic constraint on the operator imposes strict geometric restrictions on the surface, particularly on its curvature functions. We classify surfaces according to the values of their Gaussian curvature, mean curvature, and principal curvatures when the shape operator matrix defined along a surface curve satisfies $S^2 = S$.

The results show that the idempotency condition leads to three distinct geometric cases depending on whether the geodesic torsion vanishes. These classifications provide insight into the structure of flat, minimal, elliptic, and umbilical surfaces.

Keywords: Shape operator, Idempotency, Principal curvature, Gaussian curvature, Mean curvature.

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1. Introduction

Differential geometry studies the geometric and analytic properties of surfaces and curves. One of the central tools in this field is the shape operator, a symmetric linear transformation that encodes the bending behavior of a surface along different tangential directions.

A linear operator S is called idempotent if $S^2 = S$. Imposing this condition on the shape operator matrix defined along a surface curve produces strong geometric consequences and limits the possible curvature configurations of the surface. In this work, we classify surfaces based on the behavior of their curvature quantities under the idempotency condition.

The idempotency constraint $S^2 = S$ forces the normal curvatures and the geodesic torsion along the curve to take only specific values, which leads to three distinct geometric cases. As a result, we obtain a full local classification of such surfaces, when the geodesic torsion does not vanish, the surface must be flat with mean curvature $\frac{1}{2}$, while in the torsion-free case the surface is necessarily planar, spherical, or locally cylindrical depending on the pair

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of normal curvatures. These findings illustrate how an algebraic condition on the shape operator imposes strong geometric sharpness, providing insight into the interplay between extrinsic curvature and operator theory on surfaces.

In this paper, all notions considered on a surface are assumed to be differentiable. Furthermore, the ring of differentiable functions on a surface is assumed to be an integral domain.

2. Preliminaries

In this section, we recall the basic concepts and formulas from the differential geometry of curves and surfaces that will be used throughout the paper. Let M be a regular surface with local parametrization $X(u,v)$ The coefficients of the first fundamental form $Edu^2 + 2Fdudv + Gdv^2$ are defined by

$$E = \langle X_u, X_u \rangle, \qquad F = \langle X_u, X_v \rangle, \qquad G = \langle X_v, X_v \rangle.$$

The first fundamental form measures lengths and angles on the surface.[1]

Let U be the unit normal vector field on M. The coefficients of the second fundamental form $Ldu^2 + 2Mdudv + Ndv^2$ are

$$L = \langle X_{uu}, U \rangle, \qquad M = \langle X_{uv}, U \rangle, \qquad N = \langle X_{vv}, U \rangle.$$

The second fundamental form captures how the surface bends in $\mathbb{R}^3$. [1] The normal curvature of γ at a point is defined by

$$k_n = \frac{Ldu^2 + 2M\,dudv + N\,dv^2}{E\,du2 + 2F\,dudv + G\,dv^2},$$

where E, F, G are the coefficients of the first fundamental form and L, M, N are the coefficients of the second fundamental form of the surface. [3] The geodesic torsion of the curve with respect to the surface is defined by [4]

$$\tau_g = \frac{(EM - FL)\,(u')^2 + (EN - GL)\,u'v' + (FN - GM)\,(v')^2}{\sqrt{EG - F^2}},$$

Let M be a regular oriented surface in $\mathbb{R}^3$ with unit normal vector field U. For a point $P \in M$, the tangent plane $T_P(M)$ consists of all tangent vectors of surface curves passing through P.[2]

The *shape operator* is the linear transformation

$$S_P : T_P(M) \longrightarrow T_P(M), \qquad S_P(X) = -D_X U,$$

where $D_X U$ denotes the directional derivative of the unit normal field U along the tangent vector X. Let S_P be the shape operator of the surface M at a point $P \in M$. The *Gaussian curvature* and *mean curvature* of M at P are defined by

$$K(P) = \det(S_P), \qquad H(P) = \frac{1}{2}\mathrm{trace}(S_P).$$

The eigenvalues of S_P are called the *principal curvatures* of M at P. [3]

Along a regular surface curve, the Darboux frame $\{T, V, U\}$ consists of:

$$T : \text{tangent vector}, \qquad U : \text{the unit normal vector of the surface}, \qquad V = U \times T.$$

The Darboux equations are

$$\begin{aligned} T' &= k_g V + k_n U, \\ V' &= -k_g T + \tau_g U, \\ U' &= -k_n T - \tau_g V, \end{aligned}$$

where k_g is the geodesic curvature and k_n is the normal curvature. [5]

Let M be a regular surface parametrized by $X(u,v)$ and $\beta(s) = X(u(s), v(s))$ a unit-speed curve lying on M. Let T, V, U denote the Darboux frame of β.

The *normal curvature* of M in the direction of V is

$$\kappa_n(V) = \frac{\lambda_1 (u')^2 + \lambda_2 u'v' + \lambda_3 (v')^2}{EG - F^2},$$

where along β,

$$\begin{aligned} \lambda_1 &= F(FL - EM) + E(EN - FM), \\ \lambda_2 &= F(GL - FM) + E(FN - GM), \\ \lambda_3 &= G(GL - FM) + F(FN - GM). \end{aligned}$$

Following the formulation given in [5], the matrix representation of S with respect to the Darboux frame $\{T, V\}$ along the surface curve is

$$S = \begin{pmatrix} k_n(T) & \tau_g \\ \tau_g & k_n(V) \end{pmatrix},$$

where $k_n(T)$ and $k_n(V)$ denote the normal curvatures and τ_g denotes the geodesic torsion. [4] The Gaussian curvature and mean curvature are given by

$$\begin{aligned} K &= \det(S) = k_n(T)k_n(V) - \tau_g^2, \\ H &= \frac{1}{2}\operatorname{trace}(S) = \frac{k_n(T) + k_n(V)}{2}. \end{aligned}$$

The principal curvatures are

$$k_{1,2} = \frac{1}{2}(k_n(T) + k_n(V) \pm \sqrt{(k_n(T) - k_n(V))^2 + 4\tau_g^2})$$

A matrix $A \in \mathbb{R}^{n\times n}$ is called *idempotent* if

$$A^2 = A,$$

as defined in [6]. Such matrices represent linear transformations whose second application has no further effect; that is, $A(Ax) = Ax$ for all $x \in \mathbb{R}^n$. Idempotent matrices arise naturally in many areas of mathematics, including linear algebra, statistics, and differential geometry [6, 7, 8]

Idempotent matrices possess several well–known structural properties:

- Their eigenvalues can only be 0 or 1.
- The trace equals the rank: $\operatorname{tr}(A) = \operatorname{rank}(A)$
- If A is idempotent, then $I - A$ is also idempotent, and $A(I - A) = 0$.
- The transpose of an idempotent matrix is again idempotent.

Geometrically, idempotent matrices describe projections onto subspaces. For instance, the matrix

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

represents the orthogonal projection onto the x–axis in $\mathbb{R}^2$ and satisfies $P^2 = P$. This interpretation is significant in our setting, since imposing the condition $S^2 = S$ on the shape operator forces the curvature directions to behave analogously to projection operators, which severely restricts the possible geometric configurations of the surface.

3. Main Results

In this section, we classify the local geometry of a surface along a curve under the assumption that the associated shape operator matrix is idempotent.

Theorem 3.1. *Let M be a surface and let S be the shape operator matrix defined along a surface curve with respect to the Darboux frame $\{T, V, U\}$. If S is idempotent, that is, $S^2 = S$, then the following hold:*

1) *If the geodesic torsion satisfies $\tau_g \neq 0$, then*

$$H = \tfrac{1}{2}, \qquad K = 0, \qquad k_1 = 1, \qquad k_2 = 0.$$

The surface is flat but not minimal, and it is non-umbilical.

2) *If $\tau_g = 0$, then the normal curvatures satisfy $k_n(T), k_n(V) \in \{0, 1\}$ and the following subcases arise:*
   - A) $k_n(T) = k_n(V) = 0$
   - B) $k_n(T) = k_n(V) = 1$
   - C) $(k_n(T), k_n(V)) \in \{(1,0), (0,1)\}$*:*

Proof. Let the shape operator matrix along the surface curve be

$$S = \begin{pmatrix} k_n(T) & \tau_g \\ \tau_g & k_n(V) \end{pmatrix}.$$

The idempotency condition $S^2 = S$ yields

$$\begin{pmatrix} k_n(T)^2 + \tau_g^2 & k_n(T)\tau_g + \tau_g k_n(V) \\ \tau_g k_n(T) + k_n(V)\tau_g & \tau_g^2 + k_n(V)^2 \end{pmatrix} = \begin{pmatrix} k_n(T) & \tau_g \\ \tau_g & k_n(V) \end{pmatrix},$$

and therefore we have these equations

$$k_n(T)^2 + \tau_g^2 = k_n(T), \tag{3.1}$$

$$k_n(V)^2 + \tau_g^2 = k_n(V), \tag{3.2}$$

$$\tau_g \left(k_n(T) + k_n(V)\right) = \tau_g. \tag{3.3}$$

Case 1: $\tau_g \neq 0$ From (3.3) we obtain

$$k_n(T) + k_n(V) = 1.$$

Squaring this identity

$$\begin{aligned}\big(k_n(T)+k_n(V)\big)^2 &= k_n(T)^2+2k_n(T)k_n(V)+k_n(V)^2\\ &= 1 \quad\Rightarrow\quad k_n(T)^2+k_n(V)^2+2k_n(T)k_n(V)=1. \end{aligned}\tag{3.4}$$

Substituting $k_n(T)^2$ and $k_n(V)^2$ from (3.1)–(3.2) we obtain

$$\begin{aligned}(k_n(T)-\tau_g^2)+(k_n(V)-\tau_g^2)+2k_n(T)k_n(V)&=1\\ \Rightarrow\quad 2k_n(T)k_n(V)&=2\tau_g^2\\ \Rightarrow\quad k_n(T)k_n(V)&=\tau_g^2.\end{aligned}$$

Hence, the Gaussian curvature is

$$K=\det(S)=k_n(T)k_n(V)-\tau_g^2=0, \tag{3.5}$$

and the mean curvature is

$$H=\frac{k_n(T)+k_n(V)}{2}=\frac{1}{2}. \tag{3.6}$$

The principal curvatures are

$$k_1=1,\quad k_2=0. \tag{3.7}$$

Case 2: $\tau_g=0$. In this case, equations (3.1)–(3.2) become

$$k_n(T)^2-k_n(T)=0,\qquad k_n(V)^2-k_n(V)=0,$$

which imply

$$k_n(T),\,k_n(V)\in\{0,1\}.$$

Therefore, the possible geometric configurations are:

$k_n(T)=0,\ k_n(V)=0:$ $K=0,\ H=0;$ the surface is a plane or a planar patch,

$k_n(T)=1,\ k_n(V)=1:$ $K=1,\ H=1;$ the surface is elliptic and umbilical (e.g., unit sphere),

$k_n(T)\neq k_n(V):$ $K=0,\ H=\frac{1}{2};$ the surface is locally flat, non-minimal, and non-umbilical.

This completes the proof. □

4. EXAMPLES

In this section, we illustrate the cases of Theorem 3.1 by examining classical surfaces whose shape operator matrix becomes idempotent at certain points or along certain curves.

Example 4.1. ***Circular Cylinder.***

Consider the circular cylinder of radius r*. Its principal curvatures are*

$$k_1 = \frac{1}{r}, \qquad k_2 = 0.$$

The shape operator is idempotent if and only if its eigenvalues belong to $\{0, 1\}$*, which requires*

$$\frac{1}{r} = 1 \quad \Rightarrow \quad r = 1.$$

Thus, only the unit cylinder satisfies $S^2 = S$*.*

Along a general surface curve not aligned with principal directions, the geodesic torsion satisfies $\tau_g \neq 0$*. Therefore, by Theorem 3.1 (Case 1),*

$$K = 0, \qquad H = \frac{1}{2}, \qquad k_1 = 1, \qquad k_2 = 0.$$

Hence, the unit circular cylinder is a natural example of a surface whose idempotent shape operator corresponds to the nonzero–torsion case.

Example 4.2. ***Plane.*** *For a plane, all coefficients of the second fundamental form vanish:*

$$L = M = N = 0.$$

Thus, we have

$$k_n(T) = 0, \qquad k_n(V) = 0, \qquad \tau_g = 0.$$

The shape operator is the zero matrix, which trivially satisfies $S^2 = S$*. This corresponds to Theorem 3.1 (Case 2A), giving*

$$K = 0, \qquad H = 0, \qquad k_1 = k_2 = 0.$$

Example 4.3. ***Sphere.*** *For a sphere of radius* R*, both principal curvatures equal* $1/R$*:*

$$k_1 = k_2 = \frac{1}{R}.$$

Idempotency of the shape operator requires the eigenvalues to be 1*, hence*

$$\frac{1}{R} = 1 \quad \Rightarrow \quad R = 1.$$

Thus, the only sphere satisfying $S^2 = S$ *is the one with a unit radius. Since all normal directions are principial, we have*

$$k_n(T) = k_n(V) = 1, \qquad \tau_g = 0.$$

This corresponds to Theorem 3.1 (Case 2B), giving

$$K = 1, \qquad H = 1, \qquad k_1 = k_2 = 1.$$

Example 4.4. ***Parabolic Cylinder.***

Consider the parabolic cylinder $z = \frac{1}{2}x^2$*. At the point* $x = 0$*, the principal curvatures of the parabolic cylinder satisfy* $k_1 = 1$ *and* $k_2 = 0$*. Hence, the eigenvalues of the shape operator belong to* $\{0, 1\}$ *at the point, and the idempotency condition requires the eigenvalues The idempotency condition* $S^2 = S$ *is satisfied only along the curve* $\{x = 0\}$ *The idempotency condition requires the eigenvalues of* S *to lie in* $\{0, 1\}$*. Thus,*

$$\kappa(x) \in \{0, 1\}.$$

Since $\kappa(x) = 1$ *only at* $x = 0$*, the idempotency condition is satisfied along the curve* $\{x = 0\}$ *on the surface.*

At such a point we have

$$k_n(T) = 1, \qquad k_n(V) = 0, \qquad \tau_g = 0.$$

This corresponds to Theorem 3.1 (Case 2C), yielding

$$K = 0, \qquad H = \frac{1}{2}, \qquad k_1 = 1, \qquad k_2 = 0.$$

Thus, the parabolic cylinder provides an example of a surface for which the normal curvatures take different values in $\{0, 1\}$*, and the shape operator is idempotent only along a specific curve.*

5. Conclusion

In this work, we classified surfaces whose shape operator matrix defined along a surface curve is idempotent. The condition $S^2 = S$ imposes strong algebraic restrictions on the normal curvatures and geodesic torsion, resulting in three distinct geometric cases. These results connect an algebraic operator constraint with intrinsic and extrinsic geometry and contribute to the broader understanding of curvature-based surface classifications.

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References

[1] Şahin, B. (2022). *Diferansiyel geometri.* Palme Yayınevi.

[2] Şahin, B. (2012). *Manifoldların diferansiyel geometrisi.* Nobel Akademik Yayıncılık.

[3] Hacısalihoğlu, H. H. (1998). *Diferansiyel geometri.* Hacısalihoğlu Yayıncılık.

[4] Uyar Düldül, B., & Düldül, M. (2019). Shape operator via Darboux frame curvatures and its applications. *International Electronic Journal of Geometry, 12*(2), 182–187.

[5] Uyar Düldül, B., & Düldül, M. (2019). Extended Darboux frame curvatures of Frenet curves on parametric 3-surfaces. *Kuwait Journal of Science, 46*(1), 15–23.

[6] Nicholson, W. K. (1986). *Elementary linear algebra* (3rd ed.). Prindle, Weber & Schmidt.

[7] Harville, D. A. (1997). *Matrix algebra from a statistician's perspective.* Springer.

[8] Abadir, K. M., & Magnus, J. R. (2005). *Matrix algebra.* Cambridge University Press.

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