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MULTIPLICATIVE LEBESGUE SPACES

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Abstract. This study introduces Lebesgue spaces of multiplicative functions defined from the set of real numbers to the set of geometric real numbers; it is demonstrated that these spaces constitute multiplicative normed spaces, and multiplicative versions of Minkowski's and Hölder's inequalities are established.

Keywords: Multiplicative Lebesgue spaces, Multiplicative Minkowski's inequality, Multiplicative Hölder's inequality.

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1. Introduction and Preliminaries

The operations of addition and subtraction serve as the foundation for classical calculus, which Newton and Leibniz established. In their fundamental *Non-Newtonian Calculus*, Grossman and Katz [9] showed that an unlimited number of unique calculi exist, each offering a self-consistent mathematical framework. They presented the notion of "generators" (γ-generators), which are bijective functions that alter the arithmetic of the real line, thus establishing novel operations for addition, subtraction, multiplication, and division. Through modifications to the foundational field operations, Grossman and Katz formulated an extended theory in which the classical derivative and integral are simply specific instances within a wider framework. Among the diverse calculi introduced by Grossman and Katz, the geometric calculus, also known as multiplicative calculus, has attracted the greatest scholarly interest in recent literature owing to its efficacy in addressing problems characterized by exponential growth and decay.

A generator is a bijective function whose domain is the set of real numbers $\mathbb{R}$ and whose codomain is a subset of $\mathbb{R}$. Take any generator γ with a range $B \subseteq \mathbb{R}$ and $u, v \in \mathbb{R}$. As to γ-arithmetic, we denote the arithmetic whose realm is B and whose operations are described as follows:

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$$\begin{aligned} \gamma\text{-addition} &: \quad u \oplus v = \gamma\left\{\gamma^{-1}(u) + \gamma^{-1}(v)\right\} \\ \gamma\text{-subtraction} &: \quad u \ominus v = \gamma\left\{\gamma^{-1}(u) - \gamma^{-1}(v)\right\} \\ \gamma\text{-multiplication} &: \quad u \odot v = \gamma\left\{\gamma^{-1}(u) \times \gamma^{-1}(v)\right\} \\ \gamma\text{-division} &: \quad u \oslash v = \gamma\left\{\gamma^{-1}(u) \div \gamma^{-1}(v)\right\} \end{aligned}$$

Specifically, if the γ-generator is given as the identity function, such that $\gamma(u) = u$ for all $u \in \mathbb{R}$, it follows that $\gamma^{-1}(u) = u$. Consequently, γ-arithmetic reduces to the classical arithmetic.

$$\begin{aligned} \gamma\text{-addition } &: u \oplus v = \gamma\{z + w\} = z + w && : \text{ classical addition} \\ \gamma\text{-subtraction } &: u \ominus v = \gamma\{u - v\} = u - v && : \text{ classical subtraction} \\ \gamma\text{-multiplication } &: u \odot v = \gamma\{u \cdot v\} = u \cdot v && : \text{ classic multiplication} \\ \gamma\text{-division } &: u \oslash v = \gamma\{u \div v\} = u \div v && : \text{ classical division} \end{aligned}$$

Employing the exponential function exp as the γ-generator, such that $\gamma(u) = \exp(u)$ for all $u \in \mathbb{R}$, it follows that $\gamma^{-1}(u) = \ln u$. Consequently, γ-arithmetic corresponds to the geometric arithmetic.

$$\begin{aligned} \gamma\text{-addition } &: u \oplus v = \exp(\ln u + \ln v) = u \cdot v && : \text{ geometric addition} \\ \gamma\text{-subtraction } &: z \ominus w = \exp(\ln u - \ln v) = u \div v && : \text{ geometric subtraction} \\ \gamma\text{-multiplication } &: z \odot w = \exp(\ln u \cdot \ln v) = u^{\ln v} && : \text{ geometric multiplication} \\ \gamma\text{-division } &: z \oslash w = \exp(\ln u \div \ln v) = u^{\frac{1}{\ln v}} && : \text{ geometric division} \end{aligned}$$

As a generator, we select the exponential function exp, which maps from the real numbers to the set $\mathbb{R}(G)$, where $\mathbb{R}(G)$ is defined as the set of positive real numbers. The set $\mathbb{R}(G)$ is indicated by

$$\mathbb{R}(G) := \{\exp(u) : u \in \mathbb{R}\} := \mathbb{R}^{+}.$$

In this context, the geometric zero is defined as $\exp(0) = 1$, whereas the geometric one is represented by $\exp(1) = e$. Also, the multiplicative absolute value of a geometric real number u is given as

$$|u|_M = \exp|\ln(u)|.$$

In geometric arithmetic, considering the monotonically increasing functions exp and ln, it is demonstrated that the fundamental order $<$ (or $\leq$) within the real numbers $\mathbb{R}$ is maintained within this arithmetic system.

One can give the definition of the multiplicative metric space as it is presented in the work by [22]. Consider U is a non-empty set and d_M is a function from $U \times U$ to $\mathbb{R}^{+}(G) \cup \{1\}$, where the following axioms hold for all $u, v, w \in U$:

i. $d_M\ (u, v) = 1$ if and only if $u = v$,
ii. $d_M\ (u, v) = d_M\ (v, u)$,
iii. $d_M\ (u, w) \leq d_M\ (u, v)\, d_M\ (v, w)$.

The function d_M is defined as a multiplicative metric function, and also (U, d_M) is said to be a multiplicative metric space. Let U be a vector space in the multiplicative sense over the field $\mathbb{R}(G)$. Assume that $\|.\|_M$ is a function from U to $\mathbb{R}(G)$, where the following axioms hold for all $u, v \in U$ and $c \in \mathbb{R}(G)$:

(1) $\|u\|_M = 1$ if and only if $u = 0_U$,
(2) $\|cu\|_M = |c|_M \|u\|_M$,
(3) $\|uv\|_M \leq \|u\|_M \|v\|_M$.

Hence, $(U, \|.\|_M)$ is said a multiplicative normed space [22]. Also, this multiplicative norm $\|.\|_M$ on U determines a multiplicative metric d_M on U such that

$$d_M\ (u, v) = \left\| \frac{u}{v} \right\|_M$$

for all $u, v \in U$. This metric is described as the multiplicative metric induced by the multiplicative norm.

Furthermore, the collection of equivalence classes of measurable functions h mapping real numbers to real numbers that satisfy the condition

$$\int\limits_{\mathbb{R}} |h\,(x)|^p \, d\mu\,(x) < \infty$$

is denoted by the symbol $L^p_{\mathbb{R}}\,(\mathbb{R})$.

In non-Newtonian calculus, the arithmetic of the domain and value sets of functions is crucial. Special instances of non-Newtonian calculus are classified based on these arithmetics. When the generating function, which is in the domain, is defined as the identity function I and the generating function, which is in the range, is represented by the exponential function exp, geometric calculus is established. In geometric calculus, classical arithmetic is employed in the domain region of a function, whereas geometric arithmetic is utilized in its range region. In certain studies the term "multiplicative calculus" is employed in the literature in place of "geometric calculus." The term "multiplicative" is meaningful due to the fact that multiplication in geometric calculus accomplishes the role of addition in classical calculus. Stanley [21] further advanced the pedagogical comprehension of this calculus by contrasting the "arithmetic" character of classical calculus with the "geometric" nature of multiplicative operations. The essential characteristics of multiplicative calculus, including the derivative and integral for positive real-valued functions, are comprehensively analyzed in [2]. The multiplicative calculus for complex-valued functions is examined at length in references [24, 1, 3]. Afterward, the paper [22] addressed complex sequence spaces from the perspective of geometric calculus. Although most of the multiplicative calculus works mentioned above do not explicitly mention generator functions, it is evident that the derivative and integral definitions provided in these works are derivative and integral definitions for functions for which classical arithmetic is employed in the domain region and geometric arithmetic is used in the range region.

In the article [7], the Lebesgue measure of real numbers is considered in a non-Newtonian context. The investigation of non-Newtonian Lebesgue measure and non-Newtonian measurable sets has been extensively examined in references [16, 17, 18, 19]. Subsequently, the basic features of non-Newtonian Lebesgue spaces are examined in the paper [8]. The studies [14, 15], examine measurable sets of positive real numbers within the framework of multiplicative calculus.

The research conducted by [5] and [23] is cited for investigations pertaining to non-Newtonian sequence spaces. Specifically, non-Newtonian Lebesgue sequence spaces and their geometric characteristics are examined in [10]. Based on recent research, relevant publications, including [20, 6], address multiplicative Lebesgue sequence spaces and their characteristics.

Furthermore, the current literature includes investigations into integral operators within the framework of non-Newtonian calculus and its particular instance, multiplicative calculus [25, 11, 12, 4].

Finally, in accordance with the last section of [13] that is about abstract measure integration in the multiplicative sense, it is observed that certain results related to are established.

2. Lebesgue Spaces of Geometric Real-Valued Functions

This section introduces multiplicative Lebesgue spaces, a specific case of non-Newtonian Lebesgue spaces as defined in [8]. In [8], non-Newtonian calculus is applied to the domain and range of the functions within the specified space. In geometric calculus, classical arithmetic applies to the domain of the specified functions, whereas geometric arithmetic pertains to the range. Consequently, the standard Lebesgue measurable space $(\mathbb{R}, \Sigma, \mu)$ is utilized in the domain while defining multiplicative Lebesgue spaces. Furthermore, in order to define the concept of a function being multiplicative-measurable, we will use the generator function as the exp function in the definition provided in [8], as geometric arithmetic is employed in the range of the functions specified in multiplicative calculus.

To begin, we will define the multiplicative-characteristic function and the multiplicative-simple function by replacing the generator function with the exp function in the general definitions provided in [8].

Definition 2.1. *Let A be any subset of $\mathbb{R}$. Then, a multiplicative characteristic function χ_A^M is defined by*

$$\chi_A^M(x) = \begin{cases} e, & x \in A \\ 1, & x \notin A \end{cases}.$$

Also, a function ρ from $\mathbb{R}$ to $\mathbb{R}(G)$ is called multiplicative simple function that is denoted by

$$\rho = \exp\left(\sum_{j=1}^{k} \ln(a_j) \ln\left(\chi_{B_j}^M\right)\right)$$

for some $k \in \mathbb{N}$, such that $a_j \in \mathbb{R}(G)$ and $B_j \in \Sigma$ for all $j = 1, 2, \ldots k$. Therefore, the definition of the multiplicative integral of a multiplicative simple function ρ on A is as follows:

$$\int_{\mathbb{R}}^{M} \rho^{d\mu} = \exp\left(\sum_{j=1}^{k} \ln(a_j)\, \mu(B_j)\right).$$

Definition 2.2. *A function $g : \mathbb{R} \longrightarrow \mathbb{R}(G) \cup \{+\infty\}$ is said to be multiplicative measurable if,*

$$\{x \in \mathbb{R} |\ \ln(g(x)) > \ln(\gamma)\} \in \Sigma$$

for all $\gamma \in \mathbb{R}(G)$.

Lemma 2.1. *A function $g : \mathbb{R} \longrightarrow \mathbb{R}(G)$ is multiplicative measurable if and only if $\ln g$ is measurable.*

Proof. Let us take a multiplicative measurable function g. Then, we write

$$\{x \in \mathbb{R}| \ \ln(g(x)) > \ln(\gamma)\} \in \Sigma,$$

for all $\gamma \in \mathbb{R}(G)$. There exists a $\beta \in \mathbb{R}$ such that $\gamma = \exp(\beta)$, for each $\gamma \in \mathbb{R}(G)$. Then, we obtain

$$\{x \in \mathbb{R}| \ \ln(g(x)) > \ln(\gamma)\} = \{x \in \mathbb{R}| \ \ln(g(x)) > \beta\} \in \Sigma,$$

for all $\beta \in \mathbb{R}$. This means that the function $\ln g$ is measurable.

Now, let us take a measurable function $\ln g$. Therefore, we have

$$\{x \in \mathbb{R}| \ \ln(g(x)) > \beta\} \in \Sigma,$$

for all $\beta \in \mathbb{R}$. Similarly, given that for each $\beta \in \mathbb{R}$ there exists a $\gamma \in \mathbb{R}(G)$ such that $\ln(\gamma) = \beta$, the statement

$$\{x \in \mathbb{R}| \ \ln(g(x)) > \beta\} = \{x \in \mathbb{R}| \ \ln(g(x)) > \ln(\gamma)\} \in \Sigma$$

is stated as desired. This indicates that the function g possesses multiplicative measurability. □

Proposition 2.1. *Let g and h be multiplicative measurable functions and $c \in \mathbb{R}(G)$. Then the functions gh and $c^{\ln g} = g^{\ln c}$ are also multiplicative measurable functions.*

Proof. Let g and h be multiplicative measurable functions and $c \in \mathbb{R}(G)$. By Lemma 2.1, the functions $\ln g$ and $\ln h$ are measurable functions. Given that the sum of two measurable functions is measurable, the function

$$\ln g + \ln h = \ln(gh)$$

is, as a result, measurable. Applying Lemma 2.1 once more, we state that the function gh is a multiplicative measurable function.

Considering Lemma 2.1, in the case where the function g is a multiplicative measurable function, the function $\ln g$ is a measurable function. Given that the product of a measurable function and a constant is still measurable, the function obtained by multiplying the measurable function $\ln g$ by a constant $\ln c$, denoted as

$$\ln c \ln g = \ln g^{\ln c}$$

is likewise measurable. Employing Lemma 2.1 again, we can establish that the function $g^{\ln c}$ is a multiplicative measurable function. □

Definition 2.3. *One can define the multiplicative absolute value and the positive and negative components of a multiplicative function. Let g be a multiplicative measurable function from $\mathbb{R}$ to $\mathbb{R}(G)$. Therefore, the following can be expressed:*

$$|g(x)|_M = \exp|\ln(g(x))|,$$

$$g^+(x) = \max_M(g(x), 1) = \exp(\max(\ln g(x), \ln 1)) = \exp(\max(\ln g(x), 0))$$

and

$$\begin{aligned} g^{-}(x) &= \max_{M}\left(\frac{1}{g(x)},1\right)=\exp\left(\max\left(\ln\frac{1}{g(x)},\ln 1\right)\right) \\ &= \exp\left(\max\left(-\ln g(x),0\right)\right), \end{aligned}$$

for all $x\in\mathbb{R}$. *Consider a set* $B\subset\mathbb{R}(G)$. *Hence, the multiplicative supremum of this set is as follows:*

$$\sup_{M} B := \exp\left(\sup_{b\in\ B}(\ln b)\right).$$

Assume that h *is a multiplicative measurable function from* $\mathbb{R}$ *to* $\mathbb{R}^{+}(G)\cup\{1\}$, *where* $\mathbb{R}^{+}(G)=\{u\in\mathbb{R}(G)|\ \ u>1\}$. *As a result, multiplicative integral of the function* h *is given to be*

$$\int_{\mathbb{R}}^{M} h^{d\mu}=\sup_{M}\left\{\int_{\mathbb{R}}^{M}\rho^{d\mu}\middle|\ \rho \text{ is multiplicative simple and } 1\leq\rho\leq h\right\}.$$

Proposition 2.2. *Let* g *be a multiplicative measurable function. Then, one can write*

$$g(x)=\frac{g^{+}(x)}{g^{-}(x)} \text{ and } |g(x)|_{M}=g^{+}(x)\,g^{-}(x),$$

for all $x\in\mathbb{R}$. *Also, the functions* g^{+}, g^{-} *and* $|g|_{M}$ *are multiplicative measurable functions.*

Proof. Let g be a multiplicative measurable function. Thus, we have

$$g(x)=\exp(h(x)),$$

where h is a function from $\mathbb{R}$ to $\mathbb{R}$. Therefore, we determine

$$g=\exp(h)=\exp\left(h^{+}-h^{-}\right)$$

such that h^{+} and h^{-} are the positive and negative parts of the function h, respectively. Hence, we obtain

$$\begin{aligned} g(x) &= \exp\left(h^{+}(x)-h^{-}(x)\right) \\ &= \exp\left(\max(h(x),0)-\max(-h(x),0)\right) \\ &= \exp\left(\max(\ln g(x),0)-\max(-\ln g(x),0)\right) \\ &= \frac{\exp\left(\max(\ln g(x),0)\right)}{\exp\left(\max(-\ln g(x),0)\right)}=\frac{g^{+}(x)}{g^{-}(x)}, \end{aligned}$$

for all $x\in\mathbb{R}$. Initially, let us consider the positive and negative components of the function $\ln g$ as $(\ln g)^{+}$and $(\ln g)^{-}$, respectively. In this context, employing the relation

$$|\ln(g)|=(\ln g)^{+}+(\ln g)^{-}$$

results in

$$\begin{aligned} |g(x)|_{M} &= \exp|\ln(g(x))| \\ &= \exp\left((\ln g)^{+}(x)+(\ln g)^{-}(x)\right) \\ &= \exp\left((\ln g)^{+}(x)\right)\exp\left((\ln g)^{-}(x)\right) \\ &= \exp\left(\max(\ln g(x),0)\right)\exp\left(\max(-\ln g(x),0)\right) \\ &= g^{+}(x)\,g^{-}(x), \end{aligned} \tag{2.1}$$

for all $x \in \mathbb{R}$. Let us now examine the multiplicative measurability of these functions. We have

$$g^{+}(x) = \exp(\max(\ln g(x), 0)) = \exp\left((\ln g)^{+}(x)\right),$$

and

$$g^{-}(x) = \exp(\max(-\ln g(x), 0)) = \exp\left((\ln g)^{-}(x)\right),$$

for all $x \in \mathbb{R}$. By Lemma 2.1, it is known that the function $\ln g$ is a measurable function. Given that the positive and negative components of a measurable function $\ln g$ are also measurable, the functions $(\ln g)^{+}$ and $(\ln g)^{-}$ are measurable. Consequently, by Lemma 2.1, the functions

$$g^{+} = \exp\left((\ln g)^{+}\right)$$

and

$$g^{-} = \exp\left((\ln g)^{-}\right)$$

are multiplicatively measurable. If the product of two multiplicative measurable functions, as delineated in Proposition 2.1, is likewise multiplicative measurable, then the function $|g|_M$, as characterized in equation (2.1), is multiplicative measurable. □

Definition 2.4. *Let g be a multiplicative measurable function. If*

$$\int_{\mathbb{R}}^{M} \left(|g(x)|_M\right)^{d\mu(x)} < \infty \tag{2.2}$$

holds, then the function g is said to be multiplicative integrable function. Therefore, multiplicative integral of the function g is given as

$$\int_{\mathbb{R}}^{M} g^{d\mu} = \frac{\int_{\mathbb{R}}^{M} \left(g^{+}\right)^{d\mu}}{\int_{\mathbb{R}}^{M} \left(g^{-}\right)^{d\mu}}.$$

Remark 2.1. *Given that both Riemann and Lebesgue integral calculus produce identical outcomes in $\mathbb{R}$ (excluding cases where functions are not Riemann-integrable but are Lebesgue-integrable), the following integration method presented by* [2] *is applicable for computing the Lebesgue integral. Let g be multiplicative measurable function and (2.2) holds, then*

$$\int_{\mathbb{R}}^{M} g(x)^{d\mu(x)} = \exp\left(\int_{\mathbb{R}} \ln g(x)\, d\mu(x)\right). \tag{2.3}$$

Now, we will provide a lemma that demonstrates that the inequality in (2.2) also guarantees that the integral in (2.3) has a finite value.

Lemma 2.2. *Let g be multiplicative measurable function. Then we get*

$$\left|\int_{\mathbb{R}}^{M} g(x)^{d\mu(x)}\right|_M \leq \int_{\mathbb{R}}^{M} \left(|g(x)|_M\right)^{d\mu(x)}.$$

Proof. Let g be a multiplicative measurable function. By Proposition 2.2 the function $|g|_M$ is also a multiplicative measurable function. Hence, we have

$$\begin{aligned}\left|\int_{\mathbb{R}}^{M} g(x)^{d\mu(x)}\right|_M &= \left|\exp\left(\int_{\mathbb{R}} \ln g(x)\, d\mu(x)\right)\right|_M \\ &= \exp\left|\ln\left(\exp\left(\int_{\mathbb{R}} \ln g(x)\, d\mu(x)\right)\right)\right| \\ &= \exp\left|\int_{\mathbb{R}} \ln g(x)\, d\mu(x)\right| \\ &\leq \exp\int_{\mathbb{R}} |\ln g(x)|\, d\mu(x) \\ &= \exp\left(\int_{\mathbb{R}} \ln(\exp(|\ln g(x)|))\, d\mu(x)\right) \\ &= \int_{\mathbb{R}}^{M} (\exp(|\ln g(x)|))^{d\mu(x)} = \int_{\mathbb{R}}^{M} (|g(x)|_M)^{d\mu(x)}. \end{aligned} \tag{2.4}$$

This is the desired result. Also, combining (2.2) and (2.4) we obtain

$$\left|\int_{\mathbb{R}}^{M} g(x)^{d\mu(x)}\right|_M < \infty.$$

In conclusion, the integral in (2.3) has a finite value if the inequality (2.2) is valid. □

The set of multiplicative integrable functions from $\mathbb{R}$ to $\mathbb{R}(G)$ is denoted as $\mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$.

Theorem 2.1. *The following statements are valid.*

i. Let g be a function from $\mathbb{R}$ to $\mathbb{R}(G)$. If $g(x) = 1$ almost everywhere, then $g \in \mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$ and $\int_{\mathbb{R}}^{M} g(x)^{d\mu(x)} = 1$.

ii. Let $c \in \mathbb{R}(G)$ and $g, h \in \mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$. Then the functions gh and $c^{\ln g} = g^{\ln c}$ belong to $\mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$ and

$$\int_{\mathbb{R}}^{M} g(x)\, h(x)^{d\mu(x)} = \int_{\mathbb{R}}^{M} g(x)^{d\mu(x)} \int_{\mathbb{R}}^{M} h(x)^{d\mu(x)}, \tag{2.5}$$

$$\int_{\mathbb{R}}^{M} \left(c^{\ln g(x)}\right)^{d\mu(x)} = c^{\ln\left(\int_{\mathbb{R}}^{M} g(x)^{d\mu(x)}\right)}. \tag{2.6}$$

Additionally, through the utilization of these two operators gh and $c^{\ln g} = g^{\ln c}$, the set $\mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$ is a vector space in the multiplicative sense.

Proof.

i. Let g be a function from $\mathbb{R}$ to $\mathbb{R}(G)$, where $g(x) = 1$ almost everywhere. The multiplicative measurability of the constant function g is similar to the known measurability in $\mathbb{R}$. Also, we have

$$\int_{\mathbb{R}}^{M} \left(|g(x)|_M\right)^{d\mu(x)} = \exp \int_{\mathbb{R}} |\ln g(x)|\, d\mu(x) = \exp \int_{\mathbb{R}} 0 d\mu(x) = 1 < \infty$$

and

$$\int_{\mathbb{R}}^{M} \left(|g(x)|_M\right)^{d\mu(x)} = \exp \int_{\mathbb{R}} |\ln g(x)|\, d\mu(x) = \exp \int_{\mathbb{R}} 0 d\mu(x) = 1.$$

This means that $g \in \mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$.

ii. Let $c \in \mathbb{R}(G)$ and $g, h \in \mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$. Initially, we indicate that gh constitutes a multiplicative measurable function, as established by Proposition 2.1. Also, we write

$$\begin{aligned}
\int_{\mathbb{R}}^{M} (g(x)\, h(x))^{d\mu(x)} &= \exp \int_{\mathbb{R}} \ln (g(x)\, h(x))\, d\mu(x) \\
&= \exp \int_{\mathbb{R}} (\ln g(x) + \ln h(x))\, d\mu(x) \\
&= \exp \left(\int_{\mathbb{R}} \ln g(x)\, d\mu(x) + \int_{\mathbb{R}} \ln h(x)\, d\mu(x) \right) \\
&= \exp \left(\int_{\mathbb{R}} \ln g(x)\, d\mu(x) \right) \exp \left(\int_{\mathbb{R}} \ln h(x)\, d\mu(x) \right) \\
&= \int_{\mathbb{R}}^{M} g(x)^{d\mu(x)} \int_{\mathbb{R}}^{M} h(x)^{d\mu(x)}.
\end{aligned}$$

This indicates that $gh \in \mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$. Now, we will demonstrate that $c^{\ln g} = g^{\ln c}$ belongs to $\mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$. Considering Proposition 2.1, we confirm that the function

$c^{\ln g} = g^{\ln c}$ is a multiplicative measurable function. Furthermore, we write

$$\begin{aligned}\int_{\mathbb{R}}^{M} \left(c^{\ln g(x)}\right)^{d\mu(x)} &= \exp \int_{\mathbb{R}} \ln \left(c^{\ln g(x)}\right) d\mu(x) \\ &= \exp \int_{\mathbb{R}} \ln g(x) \ln c \; d\mu(x) \\ &= \exp \left(\ln c \int_{\mathbb{R}} \ln g(x)\, d\mu(x) \right) \\ &= \left(\exp \int_{\mathbb{R}} \ln g(x)\, d\mu(x) \right)^{\ln c} \\ &= \left(\int_{\mathbb{R}}^{M} g(x)^{d\mu(x)} \right)^{\ln c} = c^{\ln\left(\int_{\mathbb{R}}^{M} g(x)^{d\mu(x)}\right)}.\end{aligned}$$

It is straightforward to demonstrate that the set $\mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$ is a vector space with operators gh and $c^{\ln g} = g^{\ln c}$.

□

Let $g, h \in \mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$. Therefore, let us define a function $d_{\mathcal{L}_{\mathbb{R}(G)}}(g,h)$ as

$$d_{\mathcal{L}_{\mathbb{R}(G)}}(g,h) = \exp \int_{\mathbb{R}} \left| \ln \frac{g(x)}{h(x)} \right| d\mu(x).$$

This function fulfills conditions ii and iii of a multiplicative metric but raises an issue regarding the one component of condition i. Let us now examine this matter. Consider $d_{\mathcal{L}_{\mathbb{R}(G)}}(g,h) = 1$. Thus, we write

$$\int_{\mathbb{R}} \left| \ln \frac{g(x)}{h(x)} \right| d\mu(x) = 0.$$

This means that $\ln \frac{g(x)}{h(x)} = 0$ almost everywhere. Thus, we get $g(x) = h(x)$ almost everywhere. Consequently, the space $L_{\mathbb{R}(G)}(\mathbb{R})$, characterized as a collection of equivalence classes of functions that are almost everywhere equal and belong to $\mathcal{L}_{\mathbb{R}(G)}(\mathbb{R})$, will solve the aforementioned issue present in the axioms of a multiplicative metric space.

Definition 2.5. *Let us define multiplicative Lebesgue spaces as follows. Consider*

$$\mathcal{L}^{p}_{\mathbb{R}(G)}(\mathbb{R}) = \left\{ h \middle| \; h \text{ is multiplicative measurable and } \exp \int_{\mathbb{R}} |\ln h(x)|^{p} d\mu(x) < \infty \right\}$$

for $1 \le p < \infty$. *One can describe the spaces* $L^{p}_{\mathbb{R}(G)}(\mathbb{R})$, *in which their elements consist of equivalence classes of functions that are equal almost everywhere and belong to* $\mathcal{L}^{p}_{\mathbb{R}(G)}(\mathbb{R})$.

Definition 2.6. *Let* g *be an* $\mathbb{R}(G)$*-valued multiplicative measurable function on* $\mathbb{R}$. *Then the function* g *is an essentially bounded on* $\mathbb{R}$, *if there exists a constant* $B \in [1, \infty)$ *such that*

$$\mu\{x \in \mathbb{R} | \; |g(x)|_{M} > B\} = 0.$$

In other words, g is the essentially bounded on $\mathbb{R}$, if $|g(x)|_M \leq B$ almost everywhere on $\mathbb{R}$. We define the essential supremum of g as the multiplicative infimum of the essential bounds of g on $\mathbb{R}$, where the multiplicative infimum of a set $A \subset \mathbb{R}(G)$ is

$$\inf_M A := \exp\left(\inf_{a\in A} (\ln a)\right).$$

The following definition is given for the essential supremum of g:

$$ess\sup_M g = \inf_M \{ B \in [1,\infty) | \ \ |g(x)|_M \leq B \ almost\ everywhere\}.$$

As a result, the space $\mathcal{L}^{\infty}_{\mathbb{R}(G)}$ is described as following.

$$\mathcal{L}^{\infty}_{\mathbb{R}(G)}(\mathbb{R}) = \left\{ g | \ \ g \text{ is multiplicative measurable and } ess\sup_M g < \infty \right\}$$

One can describe the space $L^{\infty}_{\mathbb{R}(G)}(\mathbb{R})$, in which its elements consist of equivalence classes of functions that are equal almost everywhere and belong to $\mathcal{L}^{\infty}_{\mathbb{R}(G)}$.

Lemma 2.3. *Let $1 \leq p < \infty$. Then $g \in L^p_{\mathbb{R}(G)}(\mathbb{R})$ if and only if $\ln g \in L^p_{\mathbb{R}}(\mathbb{R})$.*

Proof. Initially, it is evident that the statements $f = h$ almost everywhere and $\ln f = \ln h$ almost everywhere are identical. Consider $1 \leq p < \infty$. Let us take $g \in L^p_{\mathbb{R}(G)}(\mathbb{R})$. Then, the function g is a multiplicative measurable function. By Lemma 2.1, the function $\ln g$ is a measurable function. Based on the definition of set $L^p_{\mathbb{R}(G)}(\mathbb{R})$, it follows that

$$\exp\left(\int_{\mathbb{R}} |\ln g(x)|^p \, d\mu(x)\right) < \infty \tag{2.7}$$

and so

$$\int_{\mathbb{R}} |\ln g(x)|^p \, d\mu(x) < \infty. \tag{2.8}$$

This means that $\ln g \in L^p_{\mathbb{R}}(\mathbb{R})$. Let $\ln g \in L^p_{\mathbb{R}}(\mathbb{R})$. Then, the function $\ln g$ is a measurable function. By Lemma 2.1, the function g is a multiplicative measurable function. According to the definition of the set $L^p_{\mathbb{R}}(\mathbb{R})$ and the inequality (2.8), the inequality (2.7) holds. This indicates that $g \in L^p_{\mathbb{R}(G)}(\mathbb{R})$. □

Theorem 2.2. *Consider $1 \leq p < \infty$. Let $c \in \mathbb{R}(G)$ and $g, h \in L^p_{\mathbb{R}(G)}(\mathbb{R})$. Then the set $L^p_{\mathbb{R}(G)}(\mathbb{R})$ is a vector space in the multiplicative sense with two operators given as gh and $c^{\ln g} = g^{\ln c}$.*

Proof. Consider $1 \leq p < \infty$. Let $c \in \mathbb{R}(G)$ and $g, h \in L^p_{\mathbb{R}(G)}(\mathbb{R})$. By Lemma 2.3, we have $\ln g, \ln h \in L^p_{\mathbb{R}}(\mathbb{R})$. Given that the space $L^p_{\mathbb{R}}(\mathbb{R})$ is a vector space,

$$\ln g + \ln h = \ln(gh)$$

is an element of the space $L^p_{\mathbb{R}}(\mathbb{R})$. Therefore, Lemma 2.3 implies that $gh \in L^p_{\mathbb{R}(G)}(\mathbb{R})$. Considering the scalar $\ln c$, due to the vector space structure of $L^p_{\mathbb{R}}(\mathbb{R})$, it follows that

$$\ln c \ln g = \ln g^{\ln c} \in L^p_{\mathbb{R}}(\mathbb{R}).$$

Consequently, Lemma 2.3 establishes that $g^{\ln c}$ belongs to $L^p_{\mathbb{R}(G)}(\mathbb{R})$. The remaining axioms defining a vector space in the multiplicative sense are readily apparent. □

Proposition 2.3. *Consider $1 \leq p < \infty$. Let us take $g \in L^p_{\mathbb{R}(G)}(\mathbb{R})$. Then one can give a function $\|.\|_{L^p_{\mathbb{R}(G)}}$ that is defined as*

$$\|g\|_{L^p_{\mathbb{R}(G)}} = \exp\left(\int\limits_{\mathbb{R}} |\ln g(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}.$$

Therefore, the space $\left(L^p_{\mathbb{R}(G)}(\mathbb{R}), \|.\|_{L^p_{\mathbb{R}(G)}}\right)$ is a multiplicative normed space.

Proof. Consider $1 \leq p < \infty$. Let us take $g \in L^p_{\mathbb{R}(G)}(\mathbb{R})$. By Lemma 2.3, we have $\ln g \in L^p_{\mathbb{R}}(\mathbb{R})$. It is established that the space $L^p_{\mathbb{R}}(\mathbb{R})$ constitutes a normed space endowed with the norm that is given as

$$\|h\|_{L^p_{\mathbb{R}}} = \left(\int\limits_{\mathbb{R}} |h(x)|^p \, d\mu(x)\right)^{\frac{1}{p}}$$

for all $h \in L^p_{\mathbb{R}}(\mathbb{R})$. Therefore, we get

$$\|g\|_{L^p_{\mathbb{R}(G)}} = \exp\left(\|\ln g\|_{L^p_{\mathbb{R}}}\right), \tag{2.9}$$

for all $g \in L^p_{\mathbb{R}(G)}(\mathbb{R})$. By employing this final equality (2.9) along with the basic properties of the norm $\|.\|_{L^p_{\mathbb{R}}}$, we show that the function $\|.\|_{L^p_{\mathbb{R}(G)}}$ indeed constitutes a multiplicative norm. Let $g, h \in L^p_{\mathbb{R}(G)}(\mathbb{R})$ and $c \in \mathbb{R}(G)$.

i) If

$$\|g\|_{L^p_{\mathbb{R}(G)}} = \exp\left(\|\ln g\|_{L^p_{\mathbb{R}}}\right) = 1,$$

then

$$\|\ln g\|_{L^p_{\mathbb{R}}} = 0.$$

By the norm property of $\|.\|_{L^p_{\mathbb{R}}}$, we obtain $\ln g = 0$ almost everywhere and so $g = 1$ almost everywhere. If $g = 1$ almost everywhere, then

$$\|1\|_{L^p_{\mathbb{R}(G)}} = \exp\left(\|\ln 1\|_{L^p_{\mathbb{R}}}\right) = \exp\left(\|0\|_{L^p_{\mathbb{R}}}\right) = \exp(0) = 1.$$

ii)

$$\begin{aligned}\left\|g^{\ln c}\right\|_{L^p_{\mathbb{R}(G)}} &= \exp\left(\left\|\ln\left(g^{\ln c}\right)\right\|_{L^p_{\mathbb{R}}}\right) = \exp\left(\|\ln c \ln g\|_{L^p_{\mathbb{R}}}\right) = \exp\left(|\ln c| \, \|\ln g\|_{L^p_{\mathbb{R}}}\right) \\ &= \left(\exp\left(\|\ln g\|_{L^p_{\mathbb{R}}}\right)\right)^{|\ln c|} = \left(\|g\|_{L^p_{\mathbb{R}(G)}}\right)^{\ln(\exp(|\ln c|))} = \left(\|g\|_{L^p_{\mathbb{R}(G)}}\right)^{\ln\left(|c|_M\right)}.\end{aligned}$$

iii)

$$\begin{aligned}\|gh\|_{L^p_{\mathbb{R}(G)}} &= \exp\left(\|\ln(gh)\|_{L^p_{\mathbb{R}}}\right) \\ &= \exp\left(\|\ln g + \ln h\|_{L^p_{\mathbb{R}}}\right) \\ &\leq \exp\left(\|\ln g\|_{L^p_{\mathbb{R}}} + \|\ln h\|_{L^p_{\mathbb{R}}}\right) \\ &= \exp\left(\|\ln g\|_{L^p_{\mathbb{R}}}\right)\exp\left(\|\ln h\|_{L^p_{\mathbb{R}}}\right) \\ &= \|g\|_{L^p_{\mathbb{R}(G)}} \|h\|_{L^p_{\mathbb{R}(G)}}.\end{aligned} \tag{2.10}$$

This final equation (2.10) is referred to as the multiplicative form of Minkowski's inequality. □

Theorem 2.3 (Multiplicative Hölder's inequality)**.** *Consider* $1 \leq p, q < \infty$*. Let us take* $g \in L^p_{\mathbb{R}(G)}(\mathbb{R})$ *and* $h \in L^q_{\mathbb{R}(G)}(\mathbb{R})$*, where* $\frac{1}{p} + \frac{1}{q} = 1$*. Then* $g^{\ln h} = h^{\ln g} \in L^1_{\mathbb{R}(G)}(\mathbb{R})$ *and*

$$\left\| g^{\ln h} \right\|_{L^1_{\mathbb{R}(G)}} \leq \left(\|g\|_{L^p_{\mathbb{R}(G)}} \right)^{\ln\left(\|h\|_{L^q_{\mathbb{R}(G)}} \right)}.$$

Proof. Consider $1 \leq p, q < \infty$. Let us take $g \in L^p_{\mathbb{R}(G)}(\mathbb{R})$ and $h \in L^q_{\mathbb{R}(G)}(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} = 1$. By using the equality (2.9) and Hölder's inequality, we determine

$$\begin{aligned}
\left\| g^{\ln h} \right\|_{L^1_{\mathbb{R}(G)}} &= \exp\left(\left\| \ln\left(g^{\ln h} \right) \right\|_{L^1_{\mathbb{R}}} \right) \\
&= \exp\left(\|\ln h \ln g\|_{L^1_{\mathbb{R}}} \right) \\
&\leq \exp\left(\|\ln g\|_{L^p_{\mathbb{R}}} \|\ln h\|_{L^q_{\mathbb{R}}} \right) \\
&= \left(\exp\left(\|\ln g\|_{L^p_{\mathbb{R}}} \right) \right)^{\|\ln h\|_{L^q_{\mathbb{R}}}} \\
&= \left(\exp\left(\|\ln g\|_{L^p_{\mathbb{R}}} \right) \right)^{\ln\left(\exp\left(\|\ln h\|_{L^q_{\mathbb{R}}} \right) \right)} \\
&= \left(\|g\|_{L^p_{\mathbb{R}(G)}} \right)^{\ln\left(\|h\|_{L^q_{\mathbb{R}(G)}} \right)}.
\end{aligned}$$

Since $g \in L^p_{\mathbb{R}(G)}(\mathbb{R})$ and $h \in L^q_{\mathbb{R}(G)}(\mathbb{R})$, the functions g and h are multiplicative measurable functions. According to Proposition 2.1, the function gh is a multiplicative measurable function. From the previous inequality, given that

$$\left\| g^{\ln h} \right\|_{L^1_{\mathbb{R}(G)}} \leq \left(\|g\|_{L^p_{\mathbb{R}(G)}} \right)^{\ln\left(\|h\|_{L^q_{\mathbb{R}(G)}} \right)} < \infty,$$

it follows that $g^{\ln h} = h^{\ln g} \in L^1_{\mathbb{R}(G)}(\mathbb{R})$. □

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