



\widetilde{L}_r –FINITE TYPE NULL HYPERSURFACES IN GENERALIZED ROBERTSON-WALKER SPACETIMES

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Abstract. This paper explores the \widetilde{L}_r -finite type null hypersurfaces within generalized Robertson-Walker spacetimes, where \widetilde{L}_r stands for the linearized operator of the first variation of the $(r + 1)$ -th mean curvature arising from normal variations of the hypersurface equipped with its rigged Riemannian structure. We provide necessary and/or sufficient conditions characterizing \widetilde{L}_r - p -type and \widetilde{L}_r -null- p -type null hypersurfaces ($p = 1, 2$), followed by various examples.

Keywords: \widetilde{L}_r -finite type null hypersurfaces, isoparametric, Generalized Robertson-Walker spacetime

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1. INTRODUCTION

In the late 1970s, while interested in finding the best possible estimate of the total mean curvature of a compact submanifold in Euclidean space and to define a notion of degree for these submanifolds, Chen, B. Y. introduced the concept of finite type submanifolds ([6]). Since then, this field in differential geometry associated with position vector fields has evolved very fastly as highlighted in the comprehensive survey by Chen, B.-Y [7, Chapter 6].

A submanifold $\psi : M^n \rightarrow \mathbb{R}^{n+k}$ is said to be of finite type if the position vector field ψ has the following spectral decomposition

$$\psi = \psi_0 + \psi_1 + \cdots + \psi_p \quad \text{with } \Delta\psi_i = \lambda_i\psi_i, \tag{1.1}$$

for some positive integer $p \in \mathbb{N}$, such that $\psi_0 \in \mathbb{R}^{n+k}$ is a constant vector, ψ_i ($i = 1, \dots, p$) are non-constant \mathbb{R}^{n+k} -valued maps on M^n and $\lambda_i \in \mathbb{R}$ for each i , where Δ is the Laplacian operator on M . If all λ_i 's ($i = 1, \dots, p$) are mutually distinct, the submanifold M^n is termed p -type, and if any of them is zero, M is called null- p -type.

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It is well-known that the Laplacian operator for an isometrically immersed hypersurface $M^n \subset \mathbb{R}^{n+1}$ is an intrinsic second-order linear differential operator. It naturally arises as the linearized operator of the first variation of the mean curvature for normal variations of the hypersurface. Thus, the Laplacian operator Δ can be considered as the first in a series of n operators $L_0 = \Delta, L_1, \dots, L_{n-1}$, where L_r represents the linearized operator of the first variation of the $(r + 1)$ -th order mean curvature due to normal variations. These operators are defined by

$$L_r(f) = \text{tr}(T_r \circ \nabla^2 f),$$

for any smooth function f on M , where T_r denotes the r -th Newton transformation associated with the shape operator of the hypersurface, and $\nabla^2 f$ is metrically dual operator to the Hessian of f . Although generally not elliptic, the operators L_r share properties with the Laplacian operator of M , as discussed in [1] and the concepts of finite p -type and null- p -type extend naturally in $L_k - p$ -type and $L_k - \text{null} - p$ -type, respectively for pseudo-Riemannian submanifolds M_t^n in the pseudo-Euclidean spaces \mathbb{R}_s^m . In [11], the authors study L_k -finite type hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1} \subset \mathbb{R}_1^{n+2}$ for $k \geq 1$, examining properties specific to these spaces, such as the principal curvatures and operators associated with hyperbolic geometry. They also address results related to specific cases of hypersurfaces in larger hyperbolic spaces, exploring geometric consequences, particularly for totally umbilical and k -minimal hypersurfaces. A key result stands as follows : k -minimal H_k -hypersurfaces in \mathbb{H}^4 and open portions of a non-flat totally umbilical hypersurface in \mathbb{H}^4 are the only $L_k - 1$ - type hypersurfaces in \mathbb{H}^4 . They also show that if $\psi : M^3 \rightarrow \mathbb{H}^4 \subset \mathbb{R}_1^5$ is an orientable H_k -hypersurface, then, if M is of $L_k - 2$ - type, then H_{k+1} is a nonzero constant. In n -dimensional case, they show that orientable H_k -hypersurface M^n that has at most two distinct principal curvatures is of $L_k - 2$ - type if and only if M^n is an open portion of

$$M_m^n(r) = \left\{ (x_1, \dots, x_{n+2}) \mid -x_1^2 + \sum_{i=2}^{m+1} x_i^2 = -1 - r^2, \sum_{j=m+2}^{n+2} x_j^2 = r^2 \right\},$$

for some positive integer m , $1 \leq m \leq n - 1$, and for some positive number r . It is worth mentioning that, as far as we know, all the hypersurfaces involved in known results are nondegenerate. To contribute filling the gap, the present paper aims to explore the null case in Generalized Robertson Walker spacetimes.

In upcoming Section 2, we fix basic normalization equations, isoparametric properties and background informations on Newton transformations and \widetilde{L}_r -operators as presented in [2] and [12]. Our main results are located in Section 3 and Section 4. In Section 3, we show that \widetilde{L}_r -1-type condition is equivalent to r -maximality for null hypersurfaces in de Sitter and anti de Sitter spaces (Theorem 3.1) and similar necessary and sufficient condition for the flat case $c = 0$ (Theorem 3.2). In Section 4, we focused on $\widetilde{L}_r - 2$ -type condition. Theorem 4.1 asserts that for non zero constant $\overset{\star}{S}_2$, $c = \pm 1$ and $r = 0, 1, 2$, such null hypersurfaces are necessarily isoparametric. For more general case, Theorem 4.2 asserts that for non-zero constant $\overset{\star}{S}_2$ and $\overset{\star}{S}_r$, $\widetilde{L}_r - \text{Null} - 2$ -type condition is equivalent to isoparametric when $c = \pm 1$. Finally, we show that in the flat Minkowski space ($c = 0$), the $\widetilde{L}_r - 2$ -type condition reduces also to isoparametric one (Theorem 4.3). Different results are supported by various examples.

2. NULL HYPERSURFACES IN GRW SPACETIMES

2.1. Normalization and rigged structure. A hypersurface M of a Lorentzian manifold $(\overline{M}, \overline{g})$ is null if the metric tensor is degenerate on it, i.e the induced structure from the Lorentzian ambient manifold is degenerate.

A rigging for a null hypersurface M is a vector field ζ defined in some open neighbourhood of M such that $\zeta_p \notin T_p M$ for all $p \in M$. If ζ is defined only over M , then we call it a restricted rigging. If a rigging exists, then we can take the unique null vector field $\xi \in \mathfrak{X}(M)$ such that $\overline{g}(\zeta, \xi) = 1$ (called rigged vector field) and the (screen) distribution given by $\mathcal{S}_p = \zeta_p^\perp \cap T_p M$ for all $p \in M$. A rigging with integrable screen distribution \mathcal{S} induces a screen foliation \mathcal{F} and will be termed \mathcal{F} -rigging. We can also define the rigged metric as the Riemannian metric on M given by $\tilde{g} = g + \omega \otimes \omega$, where $\omega = i^* \alpha$, $g = i^* \overline{g}$ and α the \overline{g} -metrically equivalent one-form to ζ with $i : M \rightarrow \overline{M}$ the canonical inclusion map. The rigged vector field ξ is unitary and orthogonal to \mathcal{S} with respect to \tilde{g} . Moreover, ω is \tilde{g} -metrically equivalent to ξ , and is called the rigged one-form. The vector field $N = \zeta - \frac{1}{2} \overline{g}(\zeta, \zeta) \xi$ is the unique null vector field defined on M , orthogonal to the screen distribution \mathcal{S} and such that $\overline{g}(N, \xi) = 1$.

Moreover, we have the following decompositions :

$$T_p \overline{M} = T_p M \oplus \text{span}(N_p), \quad T_p M = \text{span}\{\xi_p\} \oplus \mathcal{S}_p \quad (2.2)$$

for all $p \in M$.

The rigging technique presents two main advantages. The first one is that all the geometric objects defined above from the rigging are tuned together in a way that allows linking properties of the null hypersurface with properties of the ambient space. The second one is the presence of the Riemannian rigged metric \tilde{g} , which geometry is reasonably well coupled with the ambient geometry in most cases and it allows us to use Riemannian tools for the study of the null hypersurface [10].

We get from decompositions (2.2)

$$\overline{\nabla}_U V = \nabla_U V + B(U, V)N, \quad \overline{\nabla}_U N = -A(U) + \tau(U)N \quad (2.3)$$

where $\overline{\nabla}$, ∇ are the Levi-Civita connection of \overline{M} and the induced (projected) connection on M , respectively. The induced connection ∇ is torsion free but, in general, is not metric, which makes it less useful in the theory. The second fundamental form B , the one-form τ (also called rotation one form) and the screen second fundamental form C are given by

$$\begin{aligned} B(U, V) &= -g(\nabla_U \xi, V), & \tau(U) &= -g(\nabla_U \xi, \zeta), \\ C(U, V) &= -g(\overline{\nabla}_U N, P(V)) = -g(\overline{\nabla}_U \zeta, P(V)), \end{aligned}$$

for all $U, V \in \mathfrak{X}(M)$, where $P : TM \rightarrow \mathcal{S}$ is the canonical projection associated to the second decomposition in (2.2). The vector field $\overline{\nabla}_U \xi = \nabla_U \xi$ is tangent to the null hypersurface M and can be decomposed as

$$\overline{\nabla}_U \xi = -\tau(U)\xi - \overset{\star}{A}(U),$$

where $\overset{\star}{A}(U) \in \mathcal{S}$. The endomorphism $\overset{\star}{A}$ is the shape operator of \mathcal{S} and satisfies

$$B(U, V) = g(\overset{\star}{A}(U), V) = g(U, \overset{\star}{A}(V)), \quad B(\xi, U) = 0.$$

Some useful identities in the theory are the following:

$$-2C(U, X) = d\omega(U, X) + (L_\zeta g)(U, X) + g(\zeta, \zeta)B(U, X), \quad (2.4)$$

the Gauss-Codazzi equation

$$g(R_{UV}W, \xi) = g\left(\left(\nabla_U \overset{\star}{A}\right)(V), W\right) - g\left(\left(\nabla_V \overset{\star}{A}\right)(U), W\right) + \tau(U)g(\overset{\star}{A}(V), W) - \tau(V)g(\overset{\star}{A}(U), W), \quad (2.5)$$

$$(L_\xi \widetilde{g})(X, Y) = -2B(X, Y) \quad (2.6)$$

for all $U, V, W \in \mathfrak{X}(M), X, Y \in \mathcal{S}$, and the Raychaudhuri equation [3, Remark 3] :

$$\overline{Ric}(\xi, \xi) = \xi(H) + \tau(\xi)H - \|\overset{\star}{A}\|^2,$$

where H denotes the (non-normalized) null mean curvature of the null hypersurface given by

$$H_p = \sum_{i=1}^n B(e_i, e_i),$$

with $\{e_1, \dots, e_n\}$ an orthonormal basis in \mathcal{S}_p . In particular, $H = -\widetilde{\text{div}}\xi$.

If $B = 0$, then it is said that M is totally geodesic and if $B = \rho g$ for certain $\rho \in C^\infty(M)$, then M is totally umbilical. Observe that these definitions do not depend on the chosen rigging, although the tensors B, τ and C do depend. Throughout, the Levi-Civita connection on the normalized rigged structure (M, \widetilde{g}) will be denoted $\widetilde{\nabla}$ and we have for all $X, Y, Z \in \mathcal{S}$

$$C(\xi, X) = -\tau(X) - \widetilde{g}(\widetilde{\nabla}_\xi \xi, X), \quad \widetilde{\nabla}_X Y = \overset{\star}{\nabla}_X Y - \widetilde{g}(\widetilde{\nabla}_X \xi, Y)\xi,$$

being $\overset{\star}{\nabla}$ the connection on the screen bundle \mathcal{S} . In particular

$$\widetilde{g}(\widetilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z) = \overline{g}(\overline{\nabla}_X Y, Z) \quad \forall X, Y, Z \in \mathcal{S}.$$

From now on, we assume \overline{M} to be a generalized Robertson-Walker (GRW) spacetime of constant sectional curvature $c \in \{-1, 0, 1\}$, which will be denoted $\overline{M}_1^{n+2}(c)$ throughout. We have

$$\overline{M}_1^{n+2}(c) = (I \times_f F, \overline{g}), \quad \overline{g} = -dt^2 + f^2(t)g_F$$

where f (the warping function) is a smooth positive function on I . It is known that such spacetime admits timelike closed and conformal vector field, say $\zeta = f\partial_t$. So, the target space $\overline{M}_1^{n+2}(c)$ of immersion is locally isometric to one of the modele spaces : de Sitter spacetime \mathbb{S}_1^{n+2} of curvature $c = 1$, the Lorentz-Minkowski spacetime \mathbb{R}_1^{n+2} when $c = 0$ or the anti de Sitter spacetime \mathbb{H}_1^{n+2} (actually the universal covering of this pseudohyperbolic space \mathbb{H}_1^{n+2}) of curvature $c = -1$. Hence, we consider the following orientable isometric immersion

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}_{1+t}^m$$

of the null hypersurface in $\overline{M}_1^{n+2}(c)$ where $m = n+2+c^2$ and $t = c(c-1)/2$ with $c = 1, 0, -1$.

Due to the causal character (spacelike or null) of tangent vectors to a null hypersurface in Lorentzian space, the induced singular metric on the null hypersurface has signature $(0, n)$. So the timelike concircular vector field ζ can act as rigging vector field for M . The closed and conformal vector field ζ has the outstanding property that there exists a smooth function $\sigma \in C^\infty(\overline{M})$ (the conformal factor) such that $\overline{\nabla}_U \zeta = \sigma U$ for all $U \in \mathfrak{X}(\overline{M})$. In particular

$L_\zeta \bar{g} = 2\sigma \bar{g}$. For a closed and conformal rigging, the rotation 1-form vanishes identically ($\tau = 0$) and ξ is \bar{g} -geodesic. Moreover, due to the closedness of ζ , $\tilde{\nabla}_U \xi = -\overset{\star}{A}(U)$ and

$$\tilde{\nabla}_U V = \nabla_U V + [B(U, V) - C(U, PV)] \xi, \quad (2.7)$$

for all $U, V \in \mathfrak{X}(M)$. Also, using (2.4) we derive the following useful relation linking the shape operators A and $\overset{\star}{A}$.

$$A = -\frac{1}{2} \lambda \overset{\star}{A} - \sigma P, \quad (2.8)$$

where $\lambda = \bar{g}(\zeta, \zeta)$ denotes the length function of ζ .

For the closed rigging ζ , the screen distribution $p \mapsto \mathcal{S}_p = \zeta_p^\perp \cap T_p M$ is integrable and gives rise to a foliation \mathcal{F} on the null hypersurface. Moreover, we have shown in [4, Lemma 7] that the conformal factor σ and the length function λ are constant through the (screen) leaves \mathcal{F}_p , $p \in M$. In other words,

$$X \cdot \sigma = 0 \quad \text{and} \quad X \cdot \lambda = 0$$

for all $X \in \mathcal{S}$.

In [12, 13], null screen isoparametric hypersurfaces are defined to be normalized null hypersurfaces with screen foliation on which the screen principal curvatures are leafwise constant. It is not difficult to see that, if for a given rigging ζ on M , the screen principal curvatures (say) $\overset{\star}{\kappa}_i$ are constant, then for any rigging ζ' for M with associated screen principal curvatures $\overset{\star}{\kappa}'_i$, there exists a smooth non vanishing function ϕ on M such that $\phi \overset{\star}{\kappa}'_i$ are also constant. This motivates the following definition of isoparametric null hypersurfaces, without prior reference to a fixed normalization.

Definition 2.1. *A null hypersurface M is said to be isoparametric if it admits a \mathcal{F} -rigging ζ with associated screen principal curvatures $\overset{\star}{\kappa}_i$ and a smooth non vanishing function $\phi \in C^\infty(M)$ such that $\phi \overset{\star}{\kappa}_i$ is leafwise constant for each i .*

Screen foliated totally umbilical null hypersurfaces are isoparametric. In particular (future) null cone $\Lambda_0^n \subset \mathbb{R}_1^{n+1}$ are isoparametric null hypersurfaces.

2.2. Rigged linearized operators \tilde{L}_r . The shape operator $\overset{\star}{A}$ is self-adjoint and satisfies $\overset{\star}{A} \xi = 0$. Its $n+1$ real valued eigenfunctions $\overset{\star}{k}_0 = 0, \overset{\star}{k}_1, \dots, \overset{\star}{k}_n$ are the screen principal curvatures and we let $(X_0 = \xi, X_1, \dots, X_n)$ denote a \tilde{g} -orthonormal basis of eigenvector fields of $\overset{\star}{A}$, with $\text{span}(X_1, \dots, X_n) = \mathcal{S}$. For $0 \leq r \leq n$, the r -th null mean curvature $\overset{\star}{H}_r$ of the null hypersurface with respect to the shape operator $\overset{\star}{A}$ is given by

$$\binom{n+1}{r} \overset{\star}{H}_r = \sum_{0 \leq i_1 < \dots < i_r \leq n} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r} \quad \text{and} \quad \overset{\star}{H}_0 = 1,$$

and the null hypersurface is said to be r -maximal if $\overset{\star}{H}_r = 0$ identically on M . The following notations will be in use :

$$\overset{\star}{S}_r = \sum_{0 \leq i_1 < \dots < i_r \leq n} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r}, \quad \overset{\star}{S}_r^\alpha = \sum_{\substack{0 \leq i_1 < \dots < i_r \leq n \\ i_1, \dots, i_r \neq \alpha}} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r}.$$

In particular $\overset{\star}{S}_0 = 1$ and $\overset{\star}{S}_1 = H$ (the null mean curvature).

For $0 \leq r \leq n+1$, the r -th Newton transformation $\overset{\star}{T}_r$ with respect to the shape operator $\overset{\star}{A}$ is the $\text{End}(\Gamma(TM))$ element given by

$$\overset{\star}{T}_r = \sum_{a=0}^r (-1)^a \overset{\star}{S}_a \overset{\star}{A}^{r-a}.$$

Inductively,

$$\overset{\star}{T}_0 = I \text{ and } \overset{\star}{T}_r = (-1)^r \overset{\star}{S}_r I + \overset{\star}{A} \circ \overset{\star}{T}_{r-1},$$

where I denotes the identity of $\Gamma(TM)$ and $\overset{\star}{T}_{n+1} = 0$ (follows Cayley-Hamilton's theorem). By algebraic computations, one shows the following.

Proposition 2.1 ([3]). .

- (1) $\overset{\star}{T}_r$ is self-adjoint and commute with $\overset{\star}{A}$ for any r ;
- (2) $\overset{\star}{T}_r X_\alpha = (-1)^r \overset{\star}{S}_r X_\alpha$ (for a fixed α);
- (3) $\text{tr}(\overset{\star}{T}_r) = (-1)^r (n+1-r) \overset{\star}{S}_r$;
- (4) $\text{tr}(\overset{\star}{A} \circ \overset{\star}{T}_{r-1}) = (-1)^{r-1} r \overset{\star}{S}_r$;
- (5) $\text{tr}(\overset{\star}{A}^2 \circ \overset{\star}{T}_{r-1}) = (-1)^{r-1} (\overset{\star}{S}_1 \overset{\star}{S}_r - (r+1) \overset{\star}{S}_{r+1})$;
- (6) $\text{tr}(\overset{\star}{T}_{r-1} \circ \nabla_X \overset{\star}{A}) = (-1)^{r-1} X \cdot \overset{\star}{S}_r$.

For each Newton transformation $\overset{\star}{T}_r$, we can consider the second-order linear differential operator $\widetilde{L}_r : C^\infty(M) \rightarrow C^\infty(M)$ given by

$$\widetilde{L}_r(f) = \text{tr}(\overset{\star}{T}_r \circ \widetilde{\nabla}^2 f) \tag{2.9}$$

where $\widetilde{\nabla}^2 f := \widetilde{\nabla} \widetilde{\nabla} f$ stands for the \widetilde{g} -dual of the Hessian $\widetilde{Hess} f$ of f with respect to \widetilde{g} on M . Observe that when $r = 0$, $\widetilde{L}_0 = \widetilde{\Delta}$ is nothing but the Laplacian operator on the Riemannian rigged structure (M, \widetilde{g}) . Also, the second-order linear differential operator \widetilde{L}_r defined here in (2.9) is different from $L_r(f) = \text{tr}(\overset{\star}{T}_r \circ \nabla(\widetilde{\nabla} f))$ as defined in [5] where a hybrid use of the (projected) induced connection ∇ and the rigged Levi-Civita connection $\widetilde{\nabla}$ on (M, \widetilde{g}) is made. But these two connections do not coincide in general. These operators satisfy for all $f, h \in C^\infty(M)$,

$$\widetilde{L}_r(fh) = f \widetilde{L}_r(h) + h \widetilde{L}_r(f) + 2\widetilde{g}(\widetilde{\nabla} f, \overset{\star}{T}_r \widetilde{\nabla} h). \tag{2.10}$$

For the following orientable isometric immersion

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}_{1+t}^m$$

of the null hypersurface in $\overline{M}_1^{n+2}(c)$ where $m = n+2+c^2$ and $t = c(c-1)/2$ with $c = 1, 0, -1$, one calculates \widetilde{L}_r acting on the coordinate components of the immersion ψ , i.e a function given by $\langle \psi, a \rangle$ where $a \in \mathbb{R}_{1+t}^m$ is an arbitrary fixed vector and then extend \widetilde{L}_r to the \mathbb{R}_t^m -valued function ψ by setting

$$\widetilde{L}_r \psi = (\widetilde{L}_r \psi_1, \dots, \widetilde{L}_r \psi_m) = \sum_{i=1}^m \varepsilon_i \widetilde{L}_r \langle \psi, e_i \rangle e_i,$$

where $\psi_i = \varepsilon_i \langle \psi, e_i \rangle$ and (e_1, \dots, e_m) stands for an orthonormal basis of \mathbb{R}_{1+t}^m and $\varepsilon_i = \langle e_i, e_i \rangle = \pm 1$. We have [2]

$$\begin{aligned} \tilde{L}_r \psi &= (-1)^{r+1} \left[(n-r)\sigma \overset{\star}{S}_r + (r+1)(\lambda+1) \overset{\star}{S}_{r+1} \right] \xi \\ &\quad + (-1)^r (r+1) \overset{\star}{S}_{r+1} \zeta + (-1)^{r+1} (n-r)c \overset{\star}{S}_r \psi, \end{aligned} \quad (2.11)$$

Also, the following holds [2, Proposition 3.3]

Proposition 2.2. *Let*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}_{1+t}^m$$

be a isometric immersion of a null hypersurface in the Robertson-Walker space $\overline{M}_1^{n+2}(c)$ where $m = n + 2 + c^2$, $t = c(c-1)/2$ with $c = 1, 0, -1$, furnished with a timelike closed and conformal rigging vector field ζ . If $\lambda = \langle \zeta, \zeta \rangle$ denotes the squared length function of ζ and σ its conformal factor, Then,

$$\begin{aligned} \tilde{L}_r^2 \psi &= \left[(r+1)(\lambda+1) \overset{\star}{S}_{r+1} + (n-r)\sigma \overset{\star}{S}_r \right] \tilde{\nabla} \overset{\star}{S}_{r+1} \\ &\quad + 2(-1)^r (r+1)(\lambda+1) (\overset{\star}{T}_r \circ \overset{\star}{A}) \tilde{\nabla} \overset{\star}{S}_{r+1} \\ &\quad + 2(-1)^r (n-r)\sigma (\overset{\star}{T}_r \circ \overset{\star}{A}) \tilde{\nabla} \overset{\star}{S}_r \\ &\quad + 2(-1)^r (r+1)\sigma \overset{\star}{T}_r \tilde{\nabla} \overset{\star}{S}_{r+1} \\ &\quad + 2(-1)^{r+1} (n-r)c \overset{\star}{T}_r \tilde{\nabla} \overset{\star}{S}_r \\ &\quad + \Lambda_r^\xi \xi + \Lambda_r^\zeta \zeta + \Lambda_r^\psi \psi; \end{aligned} \quad (2.12)$$

with Λ_r^ξ , Λ_r^ζ and Λ_r^ψ as follows :

$$\begin{aligned} \Lambda_r^\xi &= (-1)^{r+1} (r+1)(\lambda+1) \tilde{L}_r \overset{\star}{S}_{r+1} + (-1)^{r+1} \sigma (n-r) \tilde{L}_r \overset{\star}{S}_r \\ &\quad + (r+1)\lambda \left(\frac{1}{2}(\lambda+1) \overset{\star}{S}_1 - 2(r+1)\sigma \right) \overset{\star}{S}_{r+1}^2 + c(n-r)^2 \sigma \overset{\star}{S}_r^2 \\ &\quad + (n-r) \left(\frac{1}{2}\lambda\sigma \overset{\star}{S}_1 + (r+1)(c\lambda - 2\sigma^2) \right) \overset{\star}{S}_r \overset{\star}{S}_{r+1} \\ &\quad + \frac{1}{2}(r+1)(r+2)\lambda(\lambda+1) \overset{\star}{S}_{r+1} \overset{\star}{S}_{r+2} \\ &\quad + \frac{1}{2}(r+2)(n-r)\lambda\sigma \overset{\star}{S}_r \overset{\star}{S}_{r+2} + 2(n-r)c \overset{\star}{S}_r (\xi \cdot \overset{\star}{S}_r) \\ &\quad - \left[(r+1)(\lambda+1) \overset{\star}{S}_{r+1} + \sigma(n+3r+4) \overset{\star}{S}_r \right] (\xi \cdot \overset{\star}{S}_{r+1}); \end{aligned} \quad (2.13)$$

$$\Lambda_r^\zeta = (r+1) \left[(-1)^r \tilde{L}_r \overset{\star}{S}_{r+1} + (r+1)\sigma \overset{\star}{S}_{r+1}^2 - (n-r)c \overset{\star}{S}_r \overset{\star}{S}_{r+1} \right] \quad (2.14)$$

and

$$\begin{aligned} \Lambda_r^\psi &= c \left[(-1)^{r+1} (n-r) \tilde{L}_r \overset{\star}{S}_r + (n-r)^2 c \overset{\star}{S}_r^2 - (r+1)^2 \lambda \overset{\star}{S}_{r+1}^2 \right. \\ &\quad \left. - 2(r+1)(n-r)\sigma \overset{\star}{S}_r \overset{\star}{S}_{r+1} - 2(r+1) \overset{\star}{S}_r (\xi \cdot \overset{\star}{S}_{r+1}) \right]. \end{aligned} \quad (2.15)$$

Remark 2.1. *Observe that*

$$\tilde{\nabla} \overset{\star}{S}_r = P \tilde{\nabla} \overset{\star}{S}_r + (\xi \cdot \overset{\star}{S}_r) \xi, \quad \overset{\star}{T}_r \tilde{\nabla} \overset{\star}{S}_r = P \left[\overset{\star}{T}_r \tilde{\nabla} \overset{\star}{S}_r \right] + (-1)^r \overset{\star}{S}_r (\xi \cdot \overset{\star}{S}_r) \xi$$

and similar formulas for $\widetilde{\nabla}^* S_{r+1}$ and $\overset{*}{T}_r \widetilde{\nabla}^* S_{r+1}$. So we get the following useful equivalent formula for (2.12)

$$\begin{aligned} \widetilde{L}_r^2 \psi &= \left[(r+1)(\lambda+1) \overset{*}{S}_{r+1} + (n-r)\sigma \overset{*}{S}_r \right] P \widetilde{\nabla}^* \overset{*}{S}_{r+1} \\ &\quad + 2(-1)^r (r+1)(\lambda+1) (\overset{*}{T}_r \circ \overset{*}{A}) \widetilde{\nabla}^* \overset{*}{S}_{r+1} \\ &\quad + 2(-1)^r (n-r)\sigma (\overset{*}{T}_r \circ \overset{*}{A}) \widetilde{\nabla}^* \overset{*}{S}_r \\ &\quad + 2(-1)^r (r+1)\sigma P \overset{*}{T}_r \widetilde{\nabla}^* \overset{*}{S}_{r+1} \\ &\quad + 2(-1)^{r+1} (n-r)cP \overset{*}{T}_r \widetilde{\nabla}^* \overset{*}{S}_r \\ &\quad + \Lambda_r^\xi \xi + \Lambda_r^\zeta \zeta + \Lambda_r^\psi \psi; \end{aligned} \tag{2.16}$$

with

$$\begin{aligned} \Lambda_r^\xi &= (-1)^{r+1} (r+1)(\lambda+1) \widetilde{L}_r \overset{*}{S}_{r+1} + (-1)^{r+1} \sigma (n-r) \widetilde{L}_r \overset{*}{S}_r \\ &\quad + (r+1)\lambda \left(\frac{1}{2}(\lambda+1) \overset{*}{S}_1 - 2(r+1)\sigma \right) \overset{*}{S}_{r+1}^2 + c(n-r)^2 \sigma \overset{*}{S}_r^2 \\ &\quad + (n-r) \left(\frac{1}{2}\lambda\sigma \overset{*}{S}_1 + (r+1)(c\lambda - 2\sigma^2) \right) \overset{*}{S}_r \overset{*}{S}_{r+1} \\ &\quad + \frac{1}{2}(r+1)(r+2)\lambda(\lambda+1) \overset{*}{S}_{r+1} \overset{*}{S}_{r+2} \\ &\quad + \frac{1}{2}(r+2)(n-r)\lambda\sigma \overset{*}{S}_r \overset{*}{S}_{r+2} - 2(r+1)\sigma \overset{*}{S}_r (\xi \cdot \overset{*}{S}_{r+1}). \end{aligned} \tag{2.17}$$

For further use we recall the following.

Theorem 2.1. [2, Theorem 3.1]. *Let (M^{n+1}, ζ) be a normalized null hypersurface of a Lorentzian space form $(\overline{M}_1^{n+2}(c), \bar{g})$ with rigged vector field ξ and $\tau = 0$. Then,*

$$\xi \cdot \overset{*}{S}_r = (-1)^{r-1} \text{tr} \left(\overset{*}{A} \circ \overset{*}{T}_{r-1} \right) \stackrel{\text{Prop. 2.1 (5)}}{=} \left(\overset{*}{S}_1 \overset{*}{S}_r - (r+1) \overset{*}{S}_{r+1} \right). \tag{2.18}$$

Consequently, if $\overset{*}{S}_r = 0$ for some $r = 1, \dots, n$, then $\overset{*}{S}_k = 0$ for all $k \geq r$.

3. $\widetilde{L}_r - 1$ - TYPE NULL HYPERSURFACES IN GRW SPACETIMES

Assume $\psi : M^{n+1} \rightarrow \overline{M}_1^{n+2}(c) \subset \mathbb{R}_{1+t}^m$ is \widetilde{L}_r -1-type ($0 \leq r \leq n-1$). Then $\psi - \psi_0 = \psi_1$ with $\widetilde{L}_r \psi_1 = \alpha \psi_1$, ($\alpha \in \mathbb{R}$). Hence, we have $\widetilde{L}_r \psi = \widetilde{L}_r \psi_1 = \alpha \psi_1 = \alpha(\psi - \psi_0) = \alpha\psi - \alpha\psi_0$. If we put $b = -\alpha\psi_0$, we have:

$$\widetilde{L}_r \psi = \alpha\psi + b. \tag{3.19}$$

Let $U = PU + \omega(U)\xi \in \mathfrak{X}(M)$. Taking derivative of (3.19) with respect to U leads to

$$\begin{aligned} &\left[\sigma(r+1) \overset{*}{S}_{r+1} \omega(U) + (\lambda+1)(r+1)U \cdot \overset{*}{S}_{r+1} + \sigma(n-r)U \cdot \overset{*}{S}_r + (-1)^r \alpha \omega(U) \right] \xi \\ &- \left[(r+1)U \cdot \overset{*}{S}_{r+1} \right] \zeta + c \left[(n-r)U \cdot \overset{*}{S}_r + (r+1) \overset{*}{S}_{r+1} \omega(U) \right] \psi \\ &- \left[(\lambda+1)(r+1) \overset{*}{S}_{r+1} + \sigma(n-r) \overset{*}{S}_r \right] \overset{*}{A}(U) + \left[c(n-r) \overset{*}{S}_r - \sigma(r+1) \overset{*}{S}_{r+1} + (-1)^r \alpha \right] PU = 0 \end{aligned} \tag{3.20}$$

which is equivalent to

$$(r+1)U \cdot \overset{*}{S}_{r+1} = 0 \tag{3.21}$$

$$c \left[(n-r)U \cdot \overset{\star}{S}_r + (r+1)\overset{\star}{S}_{r+1}\omega(U) \right] = 0 \quad (3.22)$$

$$\sigma \left[(r+1)\overset{\star}{S}_{r+1}\omega(U) + (n-r)U \cdot \overset{\star}{S}_r \right] + (r+1)(\lambda+1)U \cdot \overset{\star}{S}_{r+1} + (-1)^r \alpha \omega(U) = 0 \quad (3.23)$$

$$- \left[(\lambda+1)(r+1)\overset{\star}{S}_{r+1} + \sigma(n-r)\overset{\star}{S}_r \right] \overset{\star}{A}(U) + \left[c(n-r)\overset{\star}{S}_r - \sigma(r+1)\overset{\star}{S}_{r+1} + (-1)^r \alpha \right] PU = 0 \quad (3.24)$$

Furthermore, combining (3.19), (2.11) and the following decomposition

$$b = Pb^\top + \langle b, \zeta - \lambda\xi \rangle \xi + \langle b, \xi \rangle \zeta + c\langle b, \psi \rangle \psi,$$

lead to

$$(-1)^{r+1} \left[(\lambda+1)(r+1)\overset{\star}{S}_{r+1} + \sigma(n-r)\overset{\star}{S}_r \right] = \langle b, \zeta - \lambda\xi \rangle \quad (3.25)$$

$$(-1)^r (r+1)\overset{\star}{S}_{r+1} = \langle b, \xi \rangle \quad (3.26)$$

$$(-1)^{r+1} c(n-r)\overset{\star}{S}_r = \alpha + c\langle b, \psi \rangle \quad (3.27)$$

$$Pb^\top = 0 \quad (3.28)$$

We discuss separately the cases when $c \in \{-1, +1\}$ and $c = 0$.

Case: $c \in \{-1, +1\}$

Equations (3.22) and (3.23) imply $\alpha = 0$ and $b = -\alpha\psi_0 = 0$. Substituting $\alpha = 0$ and $b = 0$ into Eqs. (3.26) and (3.27) yields for $0 \leq r < n$ that $\overset{\star}{S}_r = 0$ and $\overset{\star}{S}_{r+1} = 0$. Making use of Theorem 2.1 we can thus state the following.

Theorem 3.1. *Let $\psi : M^{n+1} \rightarrow \overline{M}_1^{n+2}(c) \subset \mathbb{R}_{1+t}^m$ be a connected isometric immersion of a null hypersurface in a GRW spacetime $\overline{M}_1^{n+2}(\pm 1)$, endowed with a timelike closed and conformal rigging vector field ζ . Then, M is \tilde{L}_r -1-type ($0 \leq r < n$) if and only if $\overset{\star}{S}_r = 0$.*

Example 3.1. *Let $n \geq m \geq 2$ be integers. Consider*

$$M = \{x \in \mathbb{L}^{n+3} \mid -x_0^2 + x_1^2 + \cdots + x_{m+1}^2 = 0, \quad x_{m+2}^2 + \cdots + x_{n+2}^2 = 1\} \cap \{x_0 > 0\}.$$

It is easy to see that $M = \Lambda_0^{m+1} \times \mathbb{S}^{n-m}$ is a null hypersurface of the De Sitter spacetime \mathbb{S}_1^{n+2} given by the product of the lightcone Λ_0^{m+1} of dimension $m+1$ with the $n-m$ standard sphere \mathbb{S}^{n-m} (a null cone torus). A timelike closed and conformal rigging for M is given by

$$\zeta = \partial_0 + x_0 x,$$

with (null) rigged vector field

$$\xi = -\frac{1}{x_0} \cdot (x_0, x_1, \dots, x_{m+1}, 0, \dots, 0).$$

Then the shape operator is

$$\overset{\star}{A} \simeq \left[\begin{array}{ccc|c} 0 & \cdots & \cdots & 0 \\ \vdots & \frac{1}{x_0} I_m & & 0 \\ \vdots & 0 & & 0_{n-m} \\ 0 & & & \end{array} \right],$$

and we get that

$$\overset{\star}{H}_r = \begin{cases} \binom{n+1}{r}^{-1} \binom{m}{r} \cdot \frac{1}{(x_0)^r} & \text{if } 0 \leq r \leq m \\ 0 & \text{if } m+1 \leq r \leq n+1 \end{cases} \tag{3.29}$$

Hence, the null cone torus $M = \Lambda_0^{m+1} \times \mathbb{S}^{n-m}$ is $\tilde{L}_r - 1$ -type for $m+1 \leq r \leq n+1$.

Case: $c = 0$

When $c = 0$, equation (3.27) implies $\alpha = 0$ and $b = 0$. Thus, equation (3.26) gives $\overset{\star}{S}_{r+1} = 0$ and, for $0 \leq r < n$, equation (3.25) implies $\sigma = 0$ or $\overset{\star}{S}_r$. It follows that $(\sigma|_M = 0$ and $\overset{\star}{S}_{r+1} = 0)$ or $\overset{\star}{S}_r = 0$. Since by Theorem 2.1, $\overset{\star}{S}_r = 0$ implies $\overset{\star}{S}_{r+1} = 0$, above condition is equivalent to $\overset{\star}{S}_{r+1} = 0$ and $(\overset{\star}{S}_r = 0$ or $\sigma|_M = 0)$. Thus, we can state the following:

Theorem 3.2. *Let $\psi : M^{n+1} \rightarrow \overline{M}_1^{n+2}(0) \subset \mathbb{R}_{1+t}^m$ be a connected isometric immersion of a null hypersurface in GRW spacetime $\overline{M}_1^{n+2}(0)$, furnished with a timelike closed and conformal rigging vector field ζ with conformal factor σ . Then, M is \tilde{L}_r -1-type ($0 \leq r \leq n-1$) if and only if $\overset{\star}{S}_{r+1} = 0$ and $(\overset{\star}{S}_r = 0$ or $\sigma|_M = 0)$.*

Example 3.2. *Let $m \geq 1$ be an integer and consider the in the Minkowski spacetime \mathbb{R}_1^{2m+2} with coordinates $(t, x_1, \dots, x_{2m+1})$ the null hypersurface*

$$\Sigma_m = \left\{ (t, x_1, \dots, x_{2m+1}) \in \mathbb{R}_1^{2m+2} \mid t = -\sqrt{x_{m+1}^2 + \dots + x_{2m+1}^2} < 0 \right\}.$$

It is easy to see that Σ_m is a null hypersurface, as the gradient of the defining function $u = t + \sqrt{x_{m+1}^2 + \dots + x_{2m+1}^2}$ is a null vector field. A rigging for Σ_m is the timelike closed and conformal vector field $\zeta = \partial_t$ for which σ vanishes identically. The associated rigged vector field (null generator normalized by $g(\xi, \zeta) = 1$) is

$$\xi = -\partial_t + \sum_{k=m+1}^{2m+1} \frac{x_k}{R} \partial_{x_k}, \quad \text{where } R = \sqrt{x_{m+1}^2 + \dots + x_{2m+1}^2}.$$

Then, with respect to a screen distribution $S = \text{span}\{\partial_{x_1}, \dots, \partial_{x_m}\} \oplus TS^{m-1}$, the shape operator takes the form

$$\overset{\star}{A} \simeq \left[\begin{array}{c|c} 0_m & 0 \\ \hline 0 & -\frac{1}{R} I_{m+1} \end{array} \right],$$

and we obtain that the r -th mean curvature is

$$\overset{\star}{H}_r = \begin{cases} \binom{m}{r}^{-1} \left(-\frac{1}{R}\right)^r & \text{if } 0 \leq r \leq m+1, \\ 0 & \text{if } m+1 < r \leq 2m+1. \end{cases}$$

Hence, Σ_m is $\tilde{L}_r - 1$ -type for $m+1 < r \leq 2m+1$ since $\sigma \equiv 0$ and $\overset{\star}{S}_r = 0$ for $m+1 < r \leq 2m+1$.

Corollary 3.1. *Let $\psi : M^{n+1} \rightarrow \overline{M}_1^{n+2}(0) \subset \mathbb{R}_{1+t}^m$ denote a connected isometric immersion of a null hypersurface in a Generalized Robertson-Walker (GRW) spacetime $\overline{M}_1^{n+2}(0)$, endowed with a timelike closed and conformal rigging vector field ζ . If M is of \widetilde{L}_r -1-type ($0 \leq r \leq n-1$), then at least one of the following conditions holds:*

- (1) $\overset{\star}{S}_r = 0$.
- (2) $\sigma = \overset{\star}{S}_{r+1} = 0$.

4. $\widetilde{L}_r - 2-$ TYPE NULL HYPERSURFACES IN GRW SPACETIMES

Let $\psi : M^{n+1} \rightarrow \overline{M}_1^{n+2}(c) \subset \mathbb{R}_{1+t}^m$ be \widetilde{L}_r -2-type ($0 \leq r \leq n-1$). Then $\psi - \psi_0 = \psi_1 + \psi_2$ with $\widetilde{L}_r \psi_1 = \lambda_1 \psi_1$ and $\widetilde{L}_r \psi_2 = \lambda_2 \psi_2$. We have

$$\widetilde{L}_r^2 \psi = (\lambda_1 + \lambda_2) \widetilde{L}_r \psi - \lambda_1 \lambda_2 (\psi - \psi_0). \quad (4.30)$$

Therefore, (2.11) and Proposition 2.2 in (4.30) provide

$$(-1)^{r+1} (\lambda_1 + \lambda_2) \left[(\lambda + 1) (r + 1) \overset{\star}{S}_{r+1} + \sigma (n - r) \overset{\star}{S}_r \right] + \lambda_1 \lambda_2 \langle \psi_0, \zeta - \lambda \xi \rangle = \overset{\star}{\Lambda}_r^\xi \quad (4.31)$$

$$(-1)^r (\lambda_1 + \lambda_2) (r + 1) \overset{\star}{S}_{r+1} + \lambda_1 \lambda_2 \langle \psi_0, \xi \rangle = \Lambda_r^\zeta \quad (4.32)$$

$$(-1)^r (\lambda_1 + \lambda_2) c (n - r) \overset{\star}{S}_r - \lambda_1 \lambda_2 + c \lambda_1 \lambda_2 \langle \psi_0, \psi \rangle = \Lambda_r^\psi \quad (4.33)$$

$$\lambda_1 \lambda_2 P \psi_0^\top = P \widetilde{L}_r^2 \psi \quad (4.34)$$

Consider $\overset{\star}{S}_r$ to be a non-zero constant and assume first that $c \neq 0$. It follows from Theorem 2.1 that $\xi \cdot \overset{\star}{S}_r = 0 = \overset{\star}{S}_1 \overset{\star}{S}_r - (r + 1) \overset{\star}{S}_{r+1}$, which implies $\overset{\star}{S}_{r+1} = \frac{\overset{\star}{S}_r}{r + 1} \overset{\star}{S}_1$. Thus,

$$\widetilde{\nabla} \overset{\star}{S}_{r+1} = \frac{\overset{\star}{S}_r}{r + 1} \widetilde{\nabla} \overset{\star}{S}_1. \text{ Moreover, } \xi \cdot \overset{\star}{S}_{r+1} = \frac{\overset{\star}{S}_r}{(r + 1)} \left| \overset{\star}{A} \right|^2 \text{ and we get } \overset{\star}{S}_{r+2} = \frac{2}{(r + 1)(r + 2)} \overset{\star}{S}_2 \overset{\star}{S}_r.$$

Therefore, $\widetilde{\nabla} \overset{\star}{S}_{r+2} = \frac{2 \overset{\star}{S}_r}{(r + 1)(r + 2)} \widetilde{\nabla} \overset{\star}{S}_2$. It can also be shown that

$$\widetilde{\nabla} \left(\xi \cdot \overset{\star}{S}_{r+1} \right) = \frac{2 \overset{\star}{S}_r}{(r + 1)} \left(\overset{\star}{S}_1 \widetilde{\nabla} \overset{\star}{S}_1 - \widetilde{\nabla} \overset{\star}{S}_2 \right).$$

Consequently,

$$P \widetilde{\nabla} \overset{\star}{S}_{r+1} = \frac{\overset{\star}{S}_r}{r + 1} P \widetilde{\nabla} \overset{\star}{S}_1 \quad (4.35)$$

and

$$P \widetilde{\nabla} \left(\xi \cdot \overset{\star}{S}_{r+1} \right) = \frac{2 \overset{\star}{S}_r}{(r + 1)} \left(\overset{\star}{S}_1 P \widetilde{\nabla} \overset{\star}{S}_1 - P \widetilde{\nabla} \overset{\star}{S}_2 \right) \quad (4.36)$$

Besides, combining equations (4.31), (4.32), and (4.33), we have:

$$(\lambda + 1) \left[\lambda_1 \lambda_2 \langle \psi_0, \xi \rangle - \Lambda_r^\zeta \right] + \sigma \left[c \Lambda_r^\psi + c \lambda_1 \lambda_2 - \lambda_1 \lambda_2 \langle \psi_0, \psi \rangle \right] + \lambda_1 \lambda_2 \langle \psi_0, \zeta - \lambda \xi \rangle = \overset{\star}{\Lambda}_r^\xi, \quad (4.37)$$

and this, after reduction provides

$$\begin{aligned} & \lambda_1 \lambda_2 \langle \psi_0, \xi \rangle - \sigma \overset{\star}{S}_1^2 \overset{\star}{S}_r^2 + c(n-r) \overset{\star}{S}_r^2 \overset{\star}{S}_1 + c\sigma \lambda_1 \lambda_2 - \sigma \lambda_1 \lambda_2 \langle \psi_0, \psi \rangle + \lambda_1 \lambda_2 \langle \psi_0, \zeta \rangle \\ &= \frac{3}{2(r+1)} \overset{\star}{S}_r^2 \left(\overset{\star}{S}_1^3 + \overset{\star}{S}_1 \overset{\star}{S}_2 \right) + \frac{3(n-r)}{2(r+1)} \overset{\star}{S}_r^2 \left(\lambda \sigma \overset{\star}{S}_1^2 + \lambda \sigma \overset{\star}{S}_2 \right). \end{aligned}$$

Now, we prove the following:

Theorem 4.1. *Let $\psi : M^{n+1} \rightarrow \overline{M}_1^{n+2}(c) \subset \mathbb{R}_{1+t}^m$ be an isometric immersion of a null hypersurface into a generalized Robertson-Walker spacetime $\overline{M}_1^{n+2}(c)$, where $m = n + 2 + c^2$, $t = \frac{1}{2}c(c - 1)$ with $c = \pm 1$, endowed with a closed and conformal rigging field ζ . Suppose $\overset{\star}{S}_2$ non zero constant. For $r \in \{0, 1, 2\}$, the $\widetilde{L}_r - 2-$ type property implies that the null hypersurface is isoparametric.*

Proof. Consider the relation (4.38) with $r = 2$ in which $\overset{\star}{S}_2$ is a non-zero constant. Then,

$$\begin{aligned} & \lambda_1 \lambda_2 \langle \psi_0, \xi \rangle - \sigma \overset{\star}{S}_1^2 \overset{\star}{S}_2^2 + c(n-2) \overset{\star}{S}_2^2 \overset{\star}{S}_1 + c\sigma \lambda_1 \lambda_2 - \sigma \lambda_1 \lambda_2 \langle \psi_0, \psi \rangle + \lambda_1 \lambda_2 \langle \psi_0, \zeta \rangle \\ &= \frac{1}{2} \overset{\star}{S}_2^2 \left(\overset{\star}{S}_1^3 + \overset{\star}{S}_1 \overset{\star}{S}_2 \right) + \frac{(n-2)}{2} \overset{\star}{S}_2^2 \lambda \sigma \left(\overset{\star}{S}_1^2 + \overset{\star}{S}_2 \right). \end{aligned} \tag{4.38}$$

By differentiating the relation (4.38) three times with respect to ξ noting that

$$\xi \cdot \left| \overset{\star}{A} \right|^2 = 2 \overset{\star}{S}_1 \left| \overset{\star}{A} \right|^2 \quad \text{and} \quad \xi \cdot \overset{\star}{S}_1 = \left| \overset{\star}{A} \right|^2,$$

and then repeatedly differentiating with respect to $X \in \mathcal{S}^\zeta$, we obtain:

$$X \cdot \overset{\star}{S}_1 = 0 \quad \text{or} \quad \overset{\star}{S}_2 = 0 \quad \text{or} \quad \lambda = 0. \tag{4.39}$$

But, $\overset{\star}{S}_2 \neq 0$ and $\lambda < 0$ since ζ is timelike. It follows that $\overset{\star}{S}_1$ is (screen) leafwise constant. Observe that under the condition $\overset{\star}{S}_2$ is a non zero constant, for each $k \in \mathbb{N}^*$ the following holds.

$$\left\{ \begin{aligned} \overset{\star}{S}_{2k} &= \frac{\left(2 \overset{\star}{S}_2 \right)^k}{(2k)!}, \\ \widetilde{\nabla} \overset{\star}{S}_{2k+1} &= \frac{1}{2k+1} \overset{\star}{S}_{2k} \widetilde{\nabla} \overset{\star}{S}_1 \end{aligned} \right. \tag{4.40}$$

which implies

$$P \widetilde{\nabla} \overset{\star}{S}_{2k} = 0 \quad \text{and} \quad P \widetilde{\nabla} \overset{\star}{S}_{2k+1} = \frac{1}{2k+1} \overset{\star}{S}_{2k} P \widetilde{\nabla} \overset{\star}{S}_1 = 0.$$

It follows that for all k , $\overset{\star}{S}_k$ is (screen) leafwise constant and this is equivalent to saying that the principal curvatures are also (screen) leafwise constant. We conclude that M is isoparametric.

The proof for $r = 1$ or $r = 0$ is exactly the same. □

Theorem 4.2. *Let $\psi : M^{n+1} \rightarrow \overline{M}_1^{n+2}(c) \subset \mathbb{R}_{1+t}^m$ be a connected isometric immersion of a null hypersurface in GRW spacetime $\overline{M}_1^{n+2}(c)$ ($c = \pm 1$), furnished with a timelike closed and conformal rigging vector field ζ . Assume that $\overset{\star}{S}_r$ and $\overset{\star}{S}_2$ are non-zero constant*

Then M is \tilde{L}_r -null-2-type ($0 \leq r \leq n-1$) if and only if M is isoparametric.

Proof. Since $\lambda_1 = 0$ or $\lambda_2 = 0$, equation (4.38) reduces to:

$$-\sigma \overset{\star}{S}_1^2 \overset{\star}{S}_r^2 + c(n-r) \overset{\star}{S}_r^2 \overset{\star}{S}_1 = \frac{3}{2(r+1)} \overset{\star}{S}_r^2 \left(\overset{\star}{S}_1^3 + \overset{\star}{S}_1 \overset{\star}{S}_2 \right) + \frac{3(n-r)}{2(r+1)} \overset{\star}{S}_r^2 \lambda \sigma \left(\overset{\star}{S}_1^2 + \overset{\star}{S}_2 \right) \quad (4.41)$$

Repeatedly differentiating (4.41) with respect to a generic $X \in \mathcal{S}^\zeta$ leads to $X \cdot \overset{\star}{S}_1 = 0$ and we see that $\overset{\star}{S}_1$ is (screen) leafwise constant. Use ((4.40), to conclude that $\overset{\star}{S}_k$ is (screen) leafwise constant for all k and M is isoparametric. \square

Example 4.1. Let $\bar{M} = \mathbb{R} \times_{e^t} \mathbb{R}^{n+1}$ with metric $\bar{g} = -dt^2 + e^{2t} \left(\sum_{i=1}^{n+1} (dx^i)^2 \right)$. This is a GRW space with constant curvature $c = 1$. Consider

$$L = \{x^1 = e^{-t}\}.$$

Define the function $\varphi = x^1 - e^{-t}$, where. We see that $|\nabla\varphi|^2 = 0$ and L is a null hypersurface. Then take the rigging vector field to be the timelike closed and conformal vector field $\zeta = \partial_t$. The associated rigged vector field is given by $\xi = -\partial_t + e^{-t}\partial_{x^1}$. Also, for each $p \in M$ the tangent space is $T_p L = \text{Span}\{\xi = \partial_t + e^{-t}\partial_{x^1}, \partial_{x^2}, \dots, \partial_{x^{n+1}}\}$ and we have for all $i = 2, \dots, n+1$,

$$\bar{\nabla}_{\partial_{x^i}} \xi = \nabla_{\partial_{x^i}} \xi = -\partial_{x^i} = -\overset{\star}{A}(\partial_{x^i}).$$

Since $\mathcal{S}^\zeta = \text{Span}\{\partial_{x^2}, \dots, \partial_{x^{n+1}}\}$, this implies that for all $U \in \mathcal{S}^\zeta$, $\overset{\star}{A}(U) = U$ from which the conditions of Theorem 4.2 are fulfilled and L is \tilde{L}_r -null-2-type ($0 \leq r \leq n-1$).

Let us now deal with the case where $c = 0$ which by Eqs. (4.33) and (2.15) imply $\lambda_1 \lambda_2 = 0$. Without loss of generality, assume $\lambda_1 = 0$. Then, combining equations (4.31), (4.32) and (4.33) yields:

$$\begin{aligned} (-1)^{r+1} \lambda_2 \sigma (n-r) \overset{\star}{S}_r &= \left[-\lambda \sigma + \frac{1}{2(r+1)} \lambda \sigma (n-r) - 2\sigma^2 (n-r) - \sigma \right] \overset{\star}{S}_r \overset{\star}{S}_1^2 \\ &+ \frac{1}{2(r+1)} \lambda (\lambda + 1) \overset{\star}{S}_r^2 \overset{\star}{S}_1^3 + \frac{1}{r+1} \lambda (\lambda + 1) \overset{\star}{S}_1 \overset{\star}{S}_r^2 \overset{\star}{S}_2 \\ &+ \frac{\lambda \sigma}{r+1} (n-r) \overset{\star}{S}_r^2 \overset{\star}{S}_2 + 4\sigma \overset{\star}{S}_r^2 \overset{\star}{S}_2 \end{aligned} \quad (4.42)$$

Now, consider $\overset{\star}{S}_2$ to be non zero constant. By differentiating (4.42) (repeatedly) with respect to $X \in \mathcal{S}^\zeta$, it follows that $\lambda = 0$, $\lambda = -1$ or $X \cdot \overset{\star}{S}_1 = 0$. But again, $\lambda < 0$ as ζ is timelike. In fact, both cases $\lambda = -1$ or $X \cdot \overset{\star}{S}_1 = 0$ leads to M is isoparametric. Indeed, for $\lambda = -1$, we obtain

$$\begin{aligned} (-1)^{r+1} \lambda_2 \sigma (n-r) \overset{\star}{S}_r &= \left[-\frac{1}{2(r+1)} \sigma (n-r) - 2\sigma^2 (n-r) \right] \overset{\star}{S}_r \overset{\star}{S}_1^2 \\ &- \frac{\sigma}{r+1} (n-r) \overset{\star}{S}_r^2 \overset{\star}{S}_2 + 4\sigma \overset{\star}{S}_r^2 \overset{\star}{S}_2. \end{aligned} \quad (4.43)$$

We conclude that $\overset{\star}{S}_1$ is screen lifewise constant as it is solution of a second order algebraic equation with (screen) leafwise constant coefficients. Now, using again (4.40) under the condition that $\overset{\star}{S}_2$ is leafwise constant leads to isoparametry as all $\overset{\star}{S}_r$ are leafwise constant.

Therefore, we can state the following.

Theorem 4.3. *Let $\psi : M^{n+1} \rightarrow \mathbb{R}_1^{n+2}$ be a connected isometric immersion of a null hypersurface in the Minkowski space \mathbb{R}_1^{n+2} , furnished with a timelike closed and conformal rigging vector field ζ . Assume that $\overset{\star}{S}_r$ and $\overset{\star}{S}_2$ are non-zero constant.*

Then M is \widetilde{L}_r -2-type ($0 \leq r \leq n - 1$) if and only if M is isoparametric.

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