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ON THE STRUCTURE OF ALMOST RICCI-BOURGUIGNON SOLITONS IN LORENTZIAN PARA-KENMOTSU GEOMETRY

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Abstract. This paper investigates almost Ricci-Bourguignon solitons on Lorentzian para-Kenmotsu (LP-Kenmotsu) manifolds. We show that when the soliton vector field coincides with the timelike vector field, the manifold admits an Einstein structure with constant scalar curvature, and the soliton is classified as shrinking or expanding depending on the soliton parameter. In the gradient case, the structure reduces either to an Einstein manifold or a gradient $u^{\#}$-Yamabe soliton. For manifolds of constant scalar curvature, we establish that the geometry is locally isometric to a Lorentzian hyperbolic space, while the Ricci-Bourguignon condition further yields a gradient conformal structure. These results provide a classification of almost Ricci-Bourguignon solitons in the LP-Kenmotsu setting and open avenues for exploring their role in Lorentzian paracontact geometry and spacetime models inspired by general relativity.

Keywords: LP-Kenmotsu manifolds, Ricci-Bourguignon solitons, Einstein manifolds.

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1. Introduction

The concept of geometric flows has become an important tool in modern differential geometry and mathematical physics. Among them, the Ricci flow, introduced by Hamilton [11], plays a central role, as it smooths out irregularities of the metric and leads to self-similar solutions known as Ricci solitons, which generalize Einstein metrics and provide insights into singularity formation in geometry and cosmology. Since then, Ricci solitons have been widely examined in both Riemannian and Lorentzian settings due to their applications in general relativity and cosmology [18].

To capture more general behaviors of evolving metrics, several extensions of Ricci solitons have been introduced. Pigola et al. [23] studied Ricci almost solitons, where the soliton constant is allowed to be a smooth function instead of a fixed constant. Later, Ricci–Yamabe solitons were proposed as a unifying framework involving both Ricci and Yamabe flows [13], and they were also investigated on LP-Kenmotsu manifolds in [12]. In addition, Bourguignon [4]

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introduced a modified Ricci flow, known as the Ricci-Bourguignon flow, obtained by adding a scalar curvature term to the Ricci tensor. In this direction, Haseeb and Prasad [14] investigated curvature properties and metric connections specific to LP-Kenmotsu manifolds, providing essential structural insights that underpin soliton behavior. Complementing this, Ahmad et al. [1] studied ρ-Einstein solitons within the framework of LP-Kenmotsu manifolds, focusing on their geometric properties and the conditions under which these solitons exist. Their work contributes to the understanding of Ricci-type solitons in Lorentzian geometry, particularly in the context of para-Kenmotsu structures. Later, Siddesha and Sangeetha [31] extended this work to Riemannian manifolds with concurrent-recurrent vector fields (Riemannian CR manifolds) and further examined the properties of conharmonic and conformal curvature tensors on such manifolds admitting ρ-Einstein solitons. Dwivedi [9] introduced the notion of Ricci-Bourguignon almost solitons and established several results pertaining to it by deriving integral formulas for compact gradient cases and showing that, under suitable curvature or symmetry conditions, such solitons are isometric to Euclidean spheres. Building on this, Dey and Suh [8] studied almost $\star$-$u^{\#}$-Ricci-Bourguignon solitons in the framework of almost contact metric manifolds, particularly Kenmotsu manifolds. They showed that if such a manifold admits an almost $\star$-$u^{\#}$-Ricci-Bourguignon solitons with the potential vector field collinear to the timelike vector field, then the manifold is $u^{\#}$-Einstein. Recently, Naveen Kumar et al. [21] investigated generalized Ricci-type solitons on LP-Kenmotsu manifolds and demonstrated that such a manifold attains Einstein status when its metric conforms to a generalized Ricci-type soliton. For further insights, we suggest [3,5–7,15,16,22–30,32,35]. These generalizations illustrate the richness of soliton theory and motivate the following definition of almost Ricci-Bourguignon solitons:

Definition 1.1. *Let (M, g) be a smooth pseudo-Riemannian manifold. The metric g is said to define an almost Ricci-Bourguignon soliton if there exists a smooth function λ such that the Ricci tensor S of M satisfies*

$$\pounds_V g + 2S + 2(\lambda - \rho r)g = 0, \tag{1.1}$$

where $\pounds_V g$ denotes the Lie derivative of g with respect to a vector field V.

These solitons are classified as shrinking, steady, or expanding according to the sign of λ. If λ is constant, the soliton is referred to as a Ricci-Bourguignon soliton. Furthermore, when V is a Killing vector field, an almost Ricci-Bourguignon soliton reduces to a Ricci-Bourguignon soliton. If the vector field V is the gradient of a smooth function f, i.e., $V = \nabla f$, then the soliton is called a gradient almost Ricci-Bourguignon soliton, and the governing equation becomes

$$\nabla^2 f + S + (\lambda - \rho r)g = 0, \tag{1.2}$$

where S is the Ricci tensor, r is the scalar curvature, ρ a real parameter, and λ a constant.

Having introduced the framework of almost Ricci-Bourguignon solitons, we now place them in the Lorentzian setting, which provides a natural foundation for spacetime models in general relativity. Several recent works have explored soliton structures on Lorentzian manifolds. For instance, the interplay between Cotton and Bach tensors has been studied, leading to characterizations of quasi-Einstein Lorentzian manifolds and their conformal properties [19].

Likewise, the role of semi-symmetric metric connections in defining and classifying Ricci almost solitons has been analyzed [18]. These results illustrate how Lorentzian manifolds extend classical soliton theory to physically relevant settings.

Within this framework, LP-Kenmotsu manifolds appear as a natural generalization of contact-type structures in Lorentzian geometry. Their curvature properties and compatibility with soliton structures have been investigated in several recent works, especially in relation to Ricci–Yamabe solitons [12]. LP-Kenmotsu manifolds are always odd-dimensional and display rich geometric behavior driven by the timelike vector field.

Motivated by these developments, the present work is devoted to the study of almost Ricci-Bourguignon solitons on LP-Kenmotsu manifolds. We examine conditions under which such solitons yield Einstein structures, gradient solitons, or Lorentzian hyperbolic spaces. Our results contribute to the classification of soliton structures in Lorentzian geometry and suggest potential connections with spacetime models in general relativity.

The structure of the paper is as follows: Section 2 provides a review of the basic facts on LP-Kenmotsu manifolds. Section 3 develops the framework of almost Ricci-Bourguignon solitons and presents the main classification results under different geometric conditions. Section 4 is devoted to the gradient case and its consequences. Finally, Section 5 gives the summary of the results obtained and outlines possible directions for future work.

2. LP-Kenmotsu Manifolds

Consider a differentiable manifold M of dimension $(2n+1)$. A Lorentzian almost paracontact metric structure on M is given by a tensor field ϕ of $(1,1)$-type, a vector field u, a 1-form $u^{\#}$, and a Lorentzian metric g, which satisfy the following conditions [2, 20]:

$$\phi^2 = I + u^{\#} \otimes u, \quad u^{\#}(u) = -1, \quad u^{\#} \circ \phi = 0, \tag{2.3}$$

$$g(\phi\zeta_1, \phi\zeta_2) = g(\zeta_1, \zeta_2) + u^{\#}(\zeta_1)u^{\#}(\zeta_2), \quad g(\zeta_1, u) = u^{\#}(\zeta_1), \tag{2.4}$$

$\forall$ vector fields $\zeta_1, \zeta_2 \in \chi(M)$, where I denotes the identity map on the tangent bundle of M. The quadruple $(\phi, u, u^{\#}, g)$ is then called a Lorentzian almost paracontact metric structure.

Such a manifold M is said to be an *LP-Kenmotsu manifold* if its structure tensors satisfy

$$(\nabla_{\zeta_1}\phi)\zeta_2 = -g(\phi\zeta_1, \zeta_2)u - u^{\#}(\zeta_2)\phi\zeta_1, \tag{2.5}$$

for all $\zeta_1, \zeta_2 \in \chi(M)$, here ∇ denotes the Levi-Civita connection associated with g. From relation (2.5), it follows that

$$\nabla_{\zeta_1}u = -\zeta_1 - u^{\#}(\zeta_1)u, \tag{2.6}$$

$$(\nabla_{\zeta_1}u^{\#})\zeta_2 = -g(\zeta_1, \zeta_2) - u^{\#}(\zeta_1)u^{\#}(\zeta_2). \tag{2.7}$$

Furthermore, in any LP-Kenmotsu manifold the following curvature relations hold [14]:

$$u^{\#}(R(\zeta_1, \zeta_2)\zeta_3) = g(\zeta_2, \zeta_3)u^{\#}(\zeta_1) - g(\zeta_1, \zeta_3)u^{\#}(\zeta_2), \tag{2.8}$$

$$R(\zeta_1, \zeta_2)u = u^{\#}(\zeta_2)\zeta_1 - u^{\#}(\zeta_1)\zeta_2, \tag{2.9}$$

$$R(u, \zeta_1)\zeta_2 = g(\zeta_1, \zeta_2)u - u^{\#}(\zeta_2)\zeta_1, \tag{2.10}$$

$$R(u, \zeta_1)u = \zeta_1 + u^{\#}(\zeta_1)u, \tag{2.11}$$

$$S(\zeta_1, u) = 2nu^{\#}(\zeta_1), \tag{2.12}$$

$$Qu = 2nu, \tag{2.13}$$

where R is the Riemannian curvature tensor, S and Q are the Ricci tensor and Ricci operator, respectively.

A $(2n+1)$-dimensional LP-Kenmotsu manifold is called *Einstein* if its Ricci tensor is a scalar multiple of the metric tensor, i.e., $S = fg$ for some smooth function f. According to Li et al. [17], any $(2n+1)$-dimensional LP-Kenmotsu manifold also satisfies the identities

$$(\nabla_{\zeta_1} Q)u = Q\zeta_1 - 2n\zeta_1, \tag{2.14}$$

$$(\nabla_u Q)\zeta_1 = 2Q\zeta_1 - 4n\zeta_1. \tag{2.15}$$

3. Almost Ricci-Bourguignon Solitons on LP-Kenmotsu Manifolds

In this section, we study almost Ricci-Bourguignon solitons on LP-Kenmotsu manifolds. Ahmad et al. [1] recently investigated ρ-Einstein solitons in the Lorentzian para-Kenmotsu setting and showed that such manifolds are necessarily $u^{\#}$-Einstein. For constant scalar curvature, they obtained the soliton constant as

$$\lambda = \rho(n^2 - 1),$$

and the Ricci tensor

$$S(\zeta_1, \zeta_2) = (\lambda - \rho r)g(\zeta_1, \zeta_2) + (\rho r)\, u^{\#}(\zeta_1)u^{\#}(\zeta_2),$$

revealing a quasi-Einstein structure.

Motivated by this, we establish that if the potential vector field V is parallel to the timelike vector field u, then the soliton is trivial and the manifold becomes Einstein with scalar curvature $r = 2n(2n+1)$. We also identify a condition under which the almost Ricci-Bourguignon soliton reduces to a Ricci-Bourguignon soliton, thus aligning with the conclusions of Ahmad et al. [1].

Theorem 3.1. *Let M be a $(2n+1)$-dimensional (> 3) LP-Kenmotsu manifold. If its metric g admits an almost Ricci-Bourguignon soliton with potential vector field $V = \sigma u$, where u is the unit timelike vector field and σ is a smooth function, then:*

- *M is an Einstein manifold,*
- *the scalar curvature $r = 2n(2n+1)$,*
- *the gradient $\nabla\sigma$ is collinear with u.*

Proof. Let the metric g on M represent an almost Ricci-Bourguignon soliton. In this case, M satisfies equation (1.1). Since the potential vector field V is parallel to the unit timelike vector field u, it can be expressed as $V = \sigma u$, where σ is a non-zero smooth function defined on M. Therefore, we obtain

$$\nabla_{\zeta_1} V = \zeta_1(\sigma)u + \sigma(-\zeta_1 - u^{\#}(\zeta_1)u),$$

where the covariant derivative of V along $\zeta_1 \in \Gamma(TM)$ has been computed using equation (2.6). Utilizing the foregoing equation in (1.1), we find

$$2S(\zeta_1, \zeta_2) + \zeta_1(\sigma)u^{\#}(\zeta_2) + \zeta_2(\sigma)u^{\#}(\zeta_1) = 2(\sigma - \lambda + \rho r)g(\zeta_1, \zeta_2) + 2\sigma u^{\#}(\zeta_1)u^{\#}(\zeta_2), \tag{3.16}$$

$\forall\ \zeta_1, \zeta_2 \in \chi(M)$. Now, substituting u in place of ζ_2 in the preceding equation and applying relation (2.12), we obtain

$$\zeta_1(\sigma) = [u(\sigma) + 2(2n + \lambda - \rho r)]u^{\#}(\zeta_1),\ \ \zeta_1 \in \chi(M). \tag{3.17}$$

Assigning $\zeta_1 = u$ in equation (3.17) yields $\lambda - \rho r = -[u(\sigma) + 2n]$. Plugging this result back into equation (3.17), we get

$$\zeta_1(\sigma) = -u(\sigma)u^{\#}(\zeta_1),\ \ \zeta_1 \in \chi(M), \tag{3.18}$$

where

$$u(\sigma) = -\lambda + \rho r - 2n. \tag{3.19}$$

As a result of equation (3.18) and the relation $\lambda - \rho r = -[u(\sigma) + 2n]$, equation (3.16) takes the form

$$S(\zeta_1, \zeta_2) = [\sigma + u(\sigma) + 2n]g(\zeta_1, \zeta_2) + [u(\sigma) + \sigma]u^{\#}(\zeta_1)u^{\#}(\zeta_2),\ \ \zeta_1, \zeta_2 \in \chi(M). \tag{3.20}$$

Let $\{e_i\}_{i=1}^{n}$ be a local orthonormal basis on M. By substituting $\zeta_1 = \zeta_2 = e_i$ into the above equation and taking sum over i, we obtain

$$\sigma + u(\sigma) = \frac{r}{2n} - (2n + 1). \tag{3.21}$$

Substituting equation (3.21) into equation (3.20) yields

$$S(\zeta_1, \zeta_2) = \left(\frac{r}{2n} - 1\right) g(\zeta_1, \zeta_2) + \left(\frac{r}{2n} - (2n + 1)\right) u^{\#}(\zeta_1)u^{\#}(\zeta_2). \tag{3.22}$$

Therefore, M is a quasi-Einstein manifold. Using the quasi-Einstein condition, it has been demonstrated that the scalar curvature r of M with dimension greater than 3 satisfies the following equation (see Lemma (2.5) of [13]):

$$\zeta_1(r) = -u(r)u^{\#}(\zeta_1),\ \ \zeta_1 \in \chi(M). \tag{3.23}$$

Furthermore, by differentiating equation (3.22) covariantly with respect to ζ_3 and applying relation (2.6), we obtain

$$\begin{aligned}(\nabla_{\zeta_3} S)(\zeta_1, \zeta_2) \ = \ & -\left(\frac{r}{2n} - (2n + 1)\right) [g(\zeta_1, \zeta_3)u^{\#}(\zeta_2) + g(\zeta_2, \zeta_3)u^{\#}(\zeta_1) \\ & +2u^{\#}(\zeta_1)u^{\#}(\zeta_2)u^{\#}(\zeta_3)] + \frac{\zeta_3(r)}{2n}[g(\zeta_1, \zeta_2) \\ & +u^{\#}(\zeta_1)u^{\#}(\zeta_2)],\ \ \zeta_1, \zeta_2 \in \chi(M).\end{aligned} \tag{3.24}$$

By contracting equation (3.24) over ζ_2 and ζ_3, and then applying equation (3.23), we get

$$u(r) = 2[r - (2n + 1)(2n)]. \tag{3.25}$$

Assuming that the unit timelike vector field u leaves the scalar curvature r invariant which means $ur = 0$, from relation (3.25) it follows that $r = (2n + 1)(2n)$, which indicates that the scalar curvature is constant. Consequently, from equation (3.22), we deduce

$$S(\zeta_1, \zeta_2) = (2n)g(\zeta_1, \zeta_2),\ \zeta_1, \zeta_2 \in \chi(M).$$

Thus, M is an Einstein manifold. Given $r = (2n+1)(2n)$, equation (3.21) implies $u(\sigma) = -\sigma$, thereby concluding the proof of the theorem. □

Now, if σ is assumed to be a non-zero constant rather than a function, equation (3.19) simplifies to $\lambda = \rho r - 2n$, which is a constant. Substituting the value of r into this expression gives

$$\lambda = 2n[\rho(2n+1) - 1].$$

This shows that λ is positive if $\rho > 0$ and negative if $\rho < 0$. Accordingly, we present the following corollary:

Corollary 3.1. *If the metric g of a $(2n+1)$-dimensional $((2n+1) > 3)$ LP-Kenmotsu manifold M represents an almost Ricci-Bourguignon soliton where the non-zero potential vector field V is a constant multiple of the unit timelike vector field u, then the soliton is classified as shrinking if $\rho > 0$ or expanding if $\rho < 0$.*

4. Gradient Almost Ricci-Bourguignon Solitons on LP-Kenmotsu Manifolds

In this section, we analyze the nature of gradient almost Ricci-Bourguignon solitons on LP-Kenmotsu manifolds. Prior studies, such as [34], have shown that in three-dimensional Kenmotsu geometry, the existence of gradient almost Ricci solitons leads either to constant negative curvature or forces the potential field to align with the timelike vector field. Similar conclusions were drawn in [10], where the soliton was found to be expanding and the manifold was shown to be $u^{\#}$-Einstein. Since LP-Kenmotsu manifolds provide a Lorentzian analogue to Kenmotsu structures, extending these results to the LP-setting offers deeper geometric insights. We establish the following theorem.

Theorem 4.1. *Let M be an LP-Kenmotsu manifold whose metric g admits a gradient almost Ricci-Bourguignon soliton. Then M is either Einstein or the metric defines a gradient $u^{\#}$-Yamabe soliton.*

Proof. Assume that the Lorentzian metric g on the manifold M admits a gradient almost Ricci-Bourguignon soliton structure. Then, by applying equation (1.2), we obtain

$$\nabla_{\zeta_1}\nabla f + Q\zeta_1 = (-\lambda + \rho r)\zeta_1, \quad \zeta_1 \in \chi(M), \tag{4.26}$$

where f is the potential function. Taking the curvature operator into account and differentiating appropriately yields

$$R(\zeta_1,\zeta_2)\nabla f = (\nabla_{\zeta_2}Q)\zeta_1 - (\nabla_{\zeta_1}Q)\zeta_2 + \zeta_1(-\lambda+\rho r)\zeta_2 - \zeta_2(-\lambda+\rho r)\zeta_1. \tag{4.27}$$

Replacing the vector field ζ_1 with the unit timelike vector field u in the above expression and utilizing identity (2.10), we arrive at

$$(\zeta_2 f)u - (uf)\zeta_2 = (\nabla_{\zeta_2}Q)u - (\nabla_u Q)\zeta_2 + u(-\lambda+\rho r)\zeta_2 - \zeta_2(-\lambda+\rho r)u. \tag{4.28}$$

Furthermore, utilizing equations (2.14) and (2.15), we obtain the following identities involving the Ricci operator Q

$$g(\zeta_2, \nabla(f - \lambda + \rho r))u - u(f - \lambda + \rho r)\zeta_2 = 2n\zeta_2 - Q\zeta_2. \tag{4.29}$$

By contracting with u and simplifying, one finds:

$$d(f - \lambda + \rho r) = -u(f - \lambda + \rho r)u^{\#}, \tag{4.30}$$

where d is the exterior derivative. This shows that the function $f-\lambda+\rho r$ remains constant along the distribution $\mathcal{D}$ orthogonal to the unit timelike vector field u. Substituting equation (4.30) into (4.29), we deduce that

$$-u(f-\lambda+\rho r)\{u^{\#}(\zeta_2)u+\zeta_2\}=2n\zeta_2-Q\zeta_2. \tag{4.31}$$

Now, taking the trace of the above equation yields

$$u(f-\lambda+\rho r)=\left((2n+1)-\frac{r}{2n}\right). \tag{4.32}$$

Inserting (4.32) into (4.31), one immediately obtains an $u^{\#}$-Einstein condition (3.22). Further computations provide:

$$S(\zeta_2,\nabla f)=\left(\frac{r}{2n}-1\right)\zeta_2 f+\left(\frac{r}{2n}-(2n+1)\right)(uf)u^{\#}(\zeta_2). \tag{4.33}$$

Subsequently, by tracing equation (4.27) along ζ_1 yields

$$S(\zeta_2,\nabla f)=\frac{1}{2}\zeta_2(r)-2n\zeta_2(-\lambda+\rho r).$$

Equating the preceding expressions, we observe that

$$\zeta_2(r)=4n\zeta_2(-\lambda+\rho r)+2\left(\frac{r}{2n}-1\right)(\zeta_2 f)+2\left(\frac{r}{2n}-(2n+1)\right)(uf)u^{\#}(\zeta_2). \tag{4.34}$$

From this, one infers that $dr\wedge u^{\#}=0$, and hence

$$\zeta_2(r)=2[2n(2n+1)-r]u^{\#}(\zeta_2). \tag{4.35}$$

Now, assume that the vector field ζ_2 in equation (4.34) belongs to the distribution $\mathcal{D}$, orthogonal to u. Since $f-\lambda+\rho r$ is constant along $\mathcal{D}$, and using identities (4.30) and (4.35), we arrive at $\{2n(2n+1)-r\}(\zeta_2 f)=0$, for every $\zeta_2\in\mathcal{D}$. As a result, it follows that

$$(2n(2n+1)-r)(\nabla f+(uf)u)=0. \tag{4.36}$$

If $r=2n(2n+1)$, then by equation (3.22), the Ricci tensor reduces to $S(\zeta_1,\zeta_2)=2n\,g(\zeta_1,\zeta_2)$, and thus the manifold M is Einstein. On the other hand, if $r\neq 2n(2n+1)$ on some open subset $\mathcal{O}\subset M$, then equation (4.36) implies that $\nabla f=-(uf)u$. This alignment of the gradient with the unit timelike vector field leads to the relation $df=-(uf)u^{\#}$. Taking the exterior derivative of this expression yields $d^2f=-[d(uf)\wedge u^{\#}+(uf)du^{\#}]=0$, from which it follows that uf is constant, as $d^2f=0$ and $du^{\#}=0$ on an LP-Kenmotsu manifold. Differentiating the relation $\nabla f=-(uf)u$ covariantly along a vector field ζ_2 yields $\nabla_{\zeta_2}\nabla f=-\zeta_2(uf)u-(uf)(-\zeta_2-u^{\#}(\zeta_2)u)$, so that $\mathrm{Hess}(f)=(uf)(g+u^{\#}\otimes u^{\#})$. This confirms that the structure corresponds to a gradient $u^{\#}$-Yamabe soliton, which completes the proof. □

Moreover, when the scalar curvature satisfies $r=2n(2n+1)$, the manifold M becomes Einstein, and this scalar curvature remains constant. Substituting this value into equation (4.32), we find that $u(f)=u(\lambda-\rho r)$, which implies that $\nabla f=\nabla(\lambda-\rho r)$. As a result, the equation (4.26) reduces to

$$\nabla_{\zeta_2}\nabla(\lambda-\rho r)=(-\lambda+\rho r-2n)\zeta_2,\quad \text{for all } \zeta_2\in\chi(M).$$

Applying Theorem 2 of Tashiro [33], we conclude that if the manifold M is complete, then it is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}$.

Corollary 4.1. *Let M be a complete $(2n+1)$-dimensional LP-Kenmotsu manifold with constant scalar curvature admitting a gradient almost Ricci-Bourguignon soliton. Then M is locally isometric to a Lorentzian hyperbolic space, provided $\nabla f \neq -(uf)u$.*

If the soliton is a Ricci-Bourguignon soliton (i.e., λ is constant), then the soliton equation becomes:

$$\nabla^2 f = \mu g, \quad \text{with} \quad \mu = -\lambda + \rho r - 2n,$$

implying the soliton is gradient conformal.

Corollary 4.2. *If an LP-Kenmotsu manifold with constant scalar curvature admits a gradient Ricci-Bourguignon soliton, then the manifold is gradient conformal.*

5. Conclusion

In this work, we investigated almost Ricci-Bourguignon solitons on LP-Kenmotsu manifolds. We showed that when the soliton vector field is aligned with the timelike vector field, the manifold becomes Einstein with constant scalar curvature, and the soliton is classified as shrinking or expanding depending on the parameter. In the gradient case, the geometry reduces either to an Einstein structure or to a gradient $u^{\#}$-Yamabe soliton. For constant scalar curvature, the manifold is locally isometric to a Lorentzian hyperbolic space, while in the case of a Ricci-Bourguignon soliton, the metric additionally admits a gradient conformal structure. Overall, these results provide a classification of Ricci-Bourguignon solitons in the LP-Kenmotsu setting and motivate further extensions to other paracontact geometries. Future work may also explore their physical interpretations in spacetime models inspired by general relativity.

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