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A STUDY ON THE TOPOLOGY OF GRAPH COMPLEXES

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Abstract. In this paper, we consider the homotopy types of independence complexes of some graphs. Moreover, we study the homotopy types of graphs which are expanded from a given graph via certain operations. For any graph whose independence complex is contractible, we calculate the homotopy type of clique complex of its central graph. In addition to these, we build a complex from a bipartite graph to calculate homotopy types of some complexes.

Keywords: Independence complex, Clique complex, Homotopy type, Central graph.
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1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A set $S \subseteq V(G)$ is called *independent set* if any two vertices x and y in S are non-adjacent in G. The *independence complex* of a simple undirected graph G is the simplicial complex whose simplices are the independent sets of G. It is denoted by $\mathrm{Ind}(G)$. If any two edges in $M \subseteq E(E)$ are non-adjacent, then M is called a *matching*. The *matching complex* $\mathrm{M}(G)$ of a graph G is a simplicial complex with vertices are the edges of G and faces are the matchings of G. The *clique complex* $\Delta(G)$ of a graph G is the simplicial complex whose vertex set is the vertices of G and faces are the cliques of G. The independence complex of a graph G is the clique complex of its complement. Also, the independence complex of the line graph of G is the matching complex of G. These arguments justify the study of independence complexes of graphs in relation to their clique and matching complexes. Numerous papers have been written on the topic of independence complexes of graphs from an algebraic perspective [8, 9, 16], and topological perspective [1, 5, 6, 7, 10, 11]. Main studies about the complexes arising from the graphs are to determine the its homotopy types.

In [13], D. Kozlov calculated homotopy types of independence complexes of cycle and path graphs. Engström studied the homotopy types of claw-free graphs [6]. In [11], Kawamura investigated homotopy types of independence complexes of chordal graphs. Ehrenborg and Hetyei studied the independence complexes of forests [5]. In [4], Csorba investigated the

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homotopy types of independence complexes of graphs whose edges are subdivided. Jonsson proved that the independence complexes of bipartite graphs have the same homotopy type as those of the suspensions of simplicial complexes [10]. Also, in [1] Barmak introduced a notion called *star cluster* and provided a novel tool to study the topology of independence complexes. Many of our proofs are based on this notion. In addition to these, many authors studied the face enumeration of independence complexes [9, 12].

In this paper, we deal with graph complexes such as independence, clique and matching complexes. The paper is organized as follows: Section 2 focuses on fundamental definitions and previously established results that will serve as the foundation for the remainder of the paper. In Section 3, we study the topology of Lozin transformation of a graph. Also, we computed the homotopy types of the clique complex of central graphs for any contractible graph. In the last section, we introduced a complex arising from a bipartite graph and calculated homotopy types of some complexes arising from bipartite graphs.

2. Preliminaries

Given a graph $G = (V, E)$, the set $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ is called the *open neighborhood* of u in G and $N_G[u] = N_G(u) \cup \{u\}$ is called the *closed neighborhood* of u. The induced subgraph of G on $S \subseteq V(G)$ is the graph consists of vertex set $V(S)$ and two vertices in S are adjacent if and only if they are adjacent in G, this subgraph is denoted by $G[S]$. The subgraph $G\backslash U$ is obtained by deleting the vertices of U and remove all the edges connecting the vertices of U. A *complete graph* K_n on n vertices is a graph in which for every vertices u and v, there is an edge uv in K_n.

A finite *(abstract) simplicial complex* Δ on the vertex set $V(\Delta)$ is a collection of subsets of $V(\Delta)$ which satisfies; $\sigma \in \Delta$ and $\gamma \subseteq \sigma$, then $\gamma \in \Delta$. The elements of Δ are called *faces* or *simplices*. The maximal faces of Δ with respect to inclusion are called *facets*. The dimension of a face σ of Δ is defined by $\dim(\sigma) = \mid \sigma \mid -1$.

The k-skeleton of a simplcial complex Δ is a simplicial complex which consisting of i-simplices of Δ with $i \leq k$.

A subcomplex Δ' of a complex Δ is called an *induced subcomplex* of Δ; if $\sigma \in \Delta$ and $\sigma \subseteq V(\Delta')$, then $\sigma \in \Delta'$. The induced subcomplex of Δ on $U \subseteq V(\Delta)$ is denoted by $\Delta[U]$. Let Δ and Γ are simplicial complexes with disjoint vertex sets. Then the *join* of Δ and Γ is the simplicial complex whose faces consists of $\sigma \cup \tau$ such that $\sigma \in \Delta$ and $\tau \in \Gamma$. The join of Δ and Γ is denoted by $\Delta * \Gamma$.

Next we give definitions of star, link and deletion of a face of a simplicial complex.

Definition 2.1. *If σ is a face of Δ, then the link, deletion and star of σ are defined as follows:*

$$\mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cup \tau \in \Delta, \sigma \cap \tau = \emptyset\} \text{ is called link of } \sigma,$$
$$\mathrm{del}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cap \tau = \emptyset\} \text{ is called deletion of } \sigma,$$
$$\mathrm{st}_\Delta(\sigma) = \{\tau \in \Delta : \sigma \cup \tau \in \Delta\} \text{ is called star of } \sigma.$$

The definitions of link and deletion of a vertex x of a graph G can be translated for independence complexes as follows:

$$\mathrm{lk}_{\mathrm{Ind}(G)}(x) = \mathrm{Ind}(G\backslash N_G[x]) \text{ and } \mathrm{del}_{\mathrm{Ind}(G)}(x) = \mathrm{Ind}(G\backslash x).$$

In the following, the definitions of cone and suspension of a simplicial complex are given.

Definition 2.2. *The cone* $\mathrm{C}(\Delta)$ *of a simplicial complex* Δ *with apex* $v \in V(\Delta)$ *is* $v * \Delta$*.*

Definition 2.3. *The suspension* $\Sigma(\Delta)$ *of a simplicial complex* Δ *is* $\{u, v\} * \Delta$*.*

In the following, the definitions of independence complex, clique complex and matching complex will be given.

Definition 2.4. *Let* $G = (V, E)$ *be a graph. The independence complex of* G *is the simplicial complex on* V *whose faces are the independent sets of* G *and denoted by* $\mathrm{Ind}(G)$*.*

Definition 2.5. *Let* $G = (V, E)$ *be a graph. The clique complex of* G *is the simplicial complex on* V *whose faces are the cliques of* G *and denoted by* $\Delta(G)$*.*

Definition 2.6. *Let* $G = (V, E)$ *be a graph. The matching complex of* G *is the simplicial complex on* V *whose vertices are the edges and faces are the matchings of* G*. This complex is denoted by* $\mathrm{M}(G)$*.*

Definition 2.7. *Let* Δ *be a simplicial complex.* Δ *is said to be a flag complex if and only if every missing face of* Δ *is of size two.*

From the above definitions, the following proposition can be given.

Proposition 2.1. *Let* G *be a simple undirected graph. Then the complexes* $\mathrm{Ind}(G)$*,* $\Delta(G)$ *and* $\mathrm{M}(G)$ *are flag complexes.*

Star cluster of a face is defined by Barmak in [1], we recall its definition in the following:

Definition 2.8. [1] *Let* σ *be a simplex of a simplicial complex* Δ*. The star cluster of* σ *in* Δ *as the subcomplex* $\mathrm{SC}_\Delta(\sigma) = \bigcup_{v\in\sigma} \mathrm{st}_\Delta(v)$*.*

The following lemma is very important for the rest of the paper and proved in [1].

Lemma 2.1. *(Theorem 3.6,* [1]*) Let* G *be a graph and let* $v \in G$ *be a non-isolated vertex which is contained in no triangle i.e. no two vertices of* $N_G(v)$ *are adjacent. Then* $N_G(v)$ *is a simplex of* $\mathrm{Ind}(G)$ *and* $\mathrm{Ind}(G) \simeq \Sigma(\mathrm{st}_{\mathrm{Ind}(G)}(v) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(v)))$*.*

The next theorem is about the homotopy types of triangle-free graphs.

Theorem 2.1. *(Theorem 3.5,*[1]*) Let* G *be a graph such that there exits a vertex* $v \in G$ *which is contained in no triangle. Then the independence complex of* G *has the homotopy type of a suspension.*

Remark 2.1. *Let* x *be a vertex of a graph* G *which is contained in no triangle. Then the faces of the simplicial complex* $\mathrm{st}_{\mathrm{Ind}(G)}(x) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(x))$ *consist of the independent sets* σ *such that* $\sigma \cup \{x\}$ *and* $\sigma \cup \{y\}$ *are independent for* $y \in N_G(x)$*. If the independent set* σ *consists of only vertices in* $N_G(y) \backslash \{x\}$ *for* $y \in N_G(x)$*, then* σ *is the boundary of* $(\mid \sigma - 1 \mid)$*-simplex in* $\mathrm{st}_{\mathrm{Ind}(G)}(x) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(x))$*.*

Definition 2.9. *A space* X *is called contractible, if* X *is homotopy equivalent to a point.*

The next lemma is about homotopy types of the suspension of a contractible space.

Lemma 2.2. [5] *If Δ is a contractible complex, then $\Sigma(\Delta)$ is a contractible complex.*

Proposition 2.2. [5, 15] *The homotopy types of the suspension of wedge sum of spheres are as follows;*

$$\Sigma\left(\mathbb{S}^{k_1} \vee \mathbb{S}^{k_2} \vee \ldots \vee \mathbb{S}^{k_i}\right) \simeq \mathbb{S}^{k_1+1} \vee \mathbb{S}^{k_2+1} \vee \ldots \vee \mathbb{S}^{k_i+1}.$$

3. Graph operations and its topology

In this section we study the topology of complexes of graphs expand from a graph via a particular operation. The Lozin transformation of a graph is an operation when applied it increases graphs induced matching number and introduced in [14]. The homotopy type of independence complex of Lozin transformed graph studied by the authors in [2]. We study this by the notion star cluster.

Definition 3.1. *Let G be a graph and x be a vertex of G. The Lozin's transformation $\mathcal{L}_x(G)$ of G with respect to x is defined as follows:*

(i) Partition the neighborhood $N_G(x)$ of the vertex x into two subsets Y and Z in arbitrary way,

(ii) add a $P_4 = (\{y, a, b, z\}, \{ya, ab, bz\})$ to the rest of the graph,

(iii) connect vertex y of the P_4 to each vertex in Y, and connect z to each vertex in Z.

The following figure shows Lozin transformation of G with respect to vertex x. The partition was done with respect to vertex x, the edge uv forms the partition Y and vertices p and w form the partition Z.

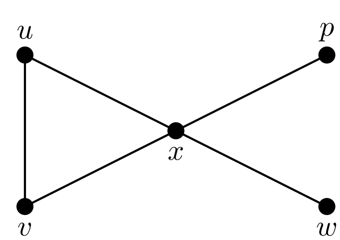

Figure 1. A graph G

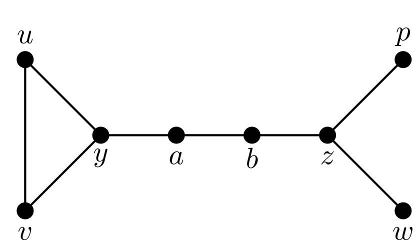

Figure 2. $\mathcal{L}_x(G)$

The Lozin transtormation expands a graph from one vertex. In the following theorem, we give the homotopy type of the Lozin transformed graph by means of the original graph.

Theorem 3.1. *([2]) Let G be a graph and $\mathcal{L}_x(G)$ be its Lozin transformation with respect to x. Then $\mathrm{Ind}(\mathcal{L}_x(G))$ is homotopy equivalent to $\Sigma(\mathrm{Ind}(G))$.*

Proof. Let $\mathcal{L}_x(G)$ be the Lozin transformation of G with respect to the vertex x. If we partition $N_G(x)$ into two subsets namely Y and Z i.e. $N_G(x) = Y \cup Z$. Then the vertices a and b are contained in no triangle, since $y - a - b - z$ is a path. We prove this for a and by symmetry it is similar for b. The complex $\mathrm{Ind}_{\mathcal{L}_x(G)}(N(a))$ is the 1-simplex yb. It is enough to show that $\mathrm{st}_{\mathrm{Ind}(\mathcal{L}_x(G))}(a) \cap \mathrm{SC}_{\mathrm{Ind}(\mathcal{L}_x(G))}(N_{\mathcal{L}_x(G)}(a)) = \mathrm{Ind}(G)$. Let $\sigma \in \mathrm{st}_{\mathrm{Ind}(\mathcal{L}_x(G))}(a) \cap \mathrm{SC}_{\mathrm{Ind}_{\mathcal{L}_x(G)}}(N_{\mathcal{L}_x(G)}(a))$ be a maximal face. Then σ is a maximal independent set of $\mathcal{L}_x(G)$ which can be extended to a and also can be extended to y or b. If $\sigma \cup \{a\}$ and $\sigma \cup \{y\}$ are independent sets and $\sigma \cup \{b\}$ is not independent. Then $\sigma \cup \{z\}$ is an independent set. Thus $\sigma \cap Y = \emptyset$ and $\sigma \cap Z = \emptyset$. If $\sigma \cup \{a\}$ and $\sigma \cup \{b\}$ are independent sets and $\sigma \cup \{y\}$ and $\sigma \cup \{z\}$ are not independent. Then $\sigma \cap Y \neq \emptyset$ or $\sigma \cap Z \neq \emptyset$ or both. Thus one can conclude that σ is a maximum independent set of $\mathrm{Ind}(G)$. Conversely, let $\sigma \in \mathrm{Ind}(G)$ be a maximum independent set. If $x \in \sigma$ then $\sigma \cap N_G(x) = \emptyset$. So σ can be extended to a and y or σ can be extended to a and b in $\mathcal{L}_x(G)$. Thus $\sigma \in \mathrm{st}_{\mathrm{Ind}(\mathcal{L}_x(G))}(a) \cap \mathrm{SC}_{\mathrm{Ind}(\mathcal{L}_x(G))}(N_{\mathcal{L}_x(G)}(a))$. □

If a graph contains an induced path P_4, then we can contract endpoints of P_4 into one vertex. In other words, we can reverse the Lozin operation of a graph if the graph has P_4.

Corollary 3.1. *Let G be a graph. If G has a subgraph P_4 whose internal vertices are of degree two and end vertices are not adjacent. Then $\mathrm{Ind}(H)$ is homotopy equivalent to $\Sigma(\mathrm{Ind}(G))$ where H is formed by contracting the end vertices of P_4 into one vertex.*

Proof. The contraction of a P_4 from end vertices into one vertex in a graph is clearly reversing the Lozin operation. □

When applying the Lozin transformation to a graph G with respect to the vertex x, we partition the vertex set of $N_G(x)$ into two disjoint sets Y and Z. However, the next theorem states that if there exists a vertex $t \in Y \cap Z$, then the Corrollary 3.1 still applies.

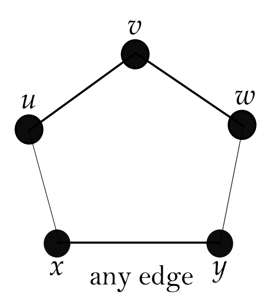

FIGURE 3. Adding P_3 to any edge of a graph

Theorem 3.2. *Let G be a graph and G' is obtained by attaching a P_3 by adding edges from endpoints of P_3 to endpoints of any edge from end vertices to make a C_5. Then $\mathrm{Ind}(G') \simeq \Sigma(\mathrm{Ind}(G))$.*

Proof. Let $u - v - w$ be a path attached to the edge xy of a graph G. Then $x - u - v - w - y$ is a C_5. So v is in no-triangle and $N_G(v)$ is a 1-simplex uw. So Theorem 3.1 implies that $\mathrm{Ind}(G') \simeq \Sigma(\mathrm{st}_{\mathrm{Ind}(G)}(v) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(v)))$. Now we show that $\mathrm{st}_{\mathrm{Ind}(G)}(v) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(v)) = \mathrm{Ind}(G)$. Let $\sigma \in \mathrm{st}_{\mathrm{Ind}(G)}(v) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(v))$ be a face. Then σ is an independent set which can be extended to independent set $\sigma \cup \{v\}$. Also σ can be extended to $\sigma \cup \{u\}$ or $\sigma \cup \{w\}$ or both. Thus $\sigma \in \mathrm{Ind}(G)$.

Now assume that $\sigma \in \mathrm{Ind}(G)$ is an independent set. If $x \in \sigma$ and $y \notin \sigma$ then $\sigma \cup \{v\}$ and $\sigma \cup \{w\}$ is independent. If $x \notin \sigma$ and $y \in \sigma$ then $\sigma \cup \{v\}$ and $\sigma \cup \{u\}$ is independent. Suppose that both x and y are not in σ. Then $\sigma \cup \{v\}$ and $\sigma \cup \{u, w\}$ are independent sets of G'. Thus $\sigma \in \mathrm{st}_{\mathrm{Ind}(G)}(v) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(v))$. Therefore $\mathrm{st}_{\mathrm{Ind}(G)}(v) \cap \mathrm{SC}_{\mathrm{Ind}(G)}(N_G(v)) = \mathrm{Ind}(G)$, this completes proof. □

Example 3.1. *Let G_n be the graph constructed by gluing cycles C_5 as described in Figure 4. Then the homotopy type of* $\mathrm{Ind}(G_n)$ *can be calculated by Theorem 3.2. Thus* $\mathrm{Ind}(G_n) \simeq \Sigma^n(\mathbb{S}^0) \simeq \mathbb{S}^n$.

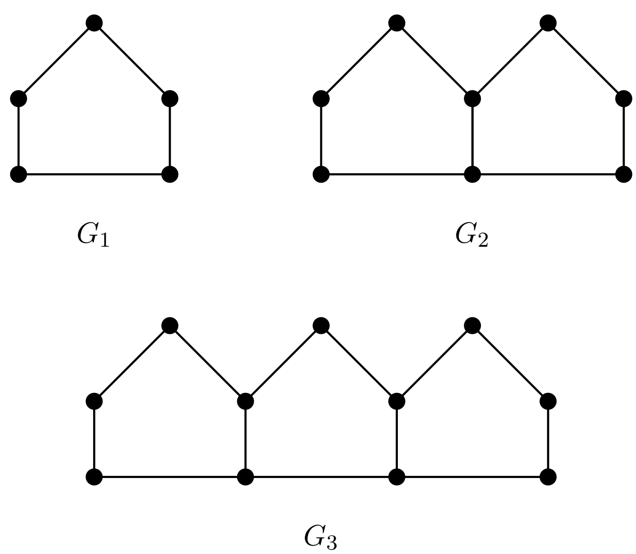

FIGURE 4. $\mathrm{Ind}(G_n)$

In the following we give the definition of central graph of an undirected simple graph. This operation increases the number of vertices of the original graph.

Definition 3.2. *Let G be a simple and undirected graph and let $V(G)$ and $E(G)$ are vertex and edge sets of G, respectively. The central graph of G, denoted by $C(G)$, is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in $C(G)$.*

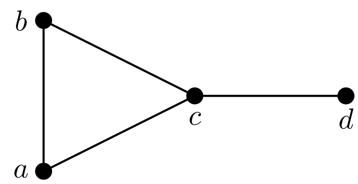

FIGURE 5. A graph G

Example 3.2. *The graph depicted in Figure 6 is the central graph of G which is depicted in Figure 5.*

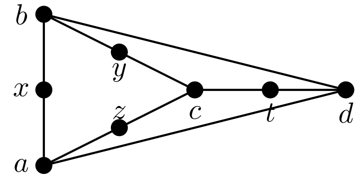

FIGURE 6. Central graph of G

Theorem 3.3. [3] *Let Δ be a simplicial complex with vertex V. If there exits a subset $A \subseteq V$ such that* $\dim(\Delta[A]) = 0$ *and* $\Delta[V \backslash A]$ *is contractible, then* $\Delta \simeq \bigvee_{x\in A} \Sigma(\mathrm{lk}_\Delta(x))$.

Proposition 3.1. *Let $G = (V, E)$ be a graph and $C(G)$ be its central graph. If* $\mathrm{Ind}(G)$ *is contractible, then* $\Delta(C(G)) \simeq \bigvee_{|E(G)|} \mathbb{S}^1$.

Proof. Assume that S is the set of vertices added to $V(G)$ when subdividing the edges. It is clear that the complex $\Delta(C(G)[V \backslash S])$ is $\mathrm{Ind}(G)$. So it is contractible by assumption. Since $\Delta(C(G)[S])$ is a discrete set of vertices, this implies that $\dim \Delta(C(G)[S]) = 0$. Then, by Theorem 3.3, one can decude that $\Delta(C(G)) \simeq \bigvee_{x\in S} \Sigma(\mathrm{lk}_{\Delta(C(G))}(x))$. In $\Delta(C(G))$, for every vertex $x \in S$, the complex $\mathrm{lk}_{\Delta C(G)}(x)$ consists of two disjoint vertices, since they are adjacent to vertices which form an edge in G. Thus $\mathrm{lk}_{\Delta(C(G))}(x) \simeq \mathbb{S}^0$ for every $x \in S$. Therefore we have $\Delta_{C(G)} \simeq \bigvee_{|E(G)|} \mathbb{S}^1$ by Proposition 2.2. □

In [11], the author stated that for each wedge $\bigvee \mathbb{S}^{k_t}$ of finitely many spheres, there exists a chordal graph G such that $\mathrm{Ind}(G)$ is homotopy equivalent to $\bigvee \mathbb{S}^{k_t}$ for $k_t \geq \gamma(G) - 1$, where $\gamma(G)$ is the domination number of G. In the following theorem, spheres are 1-dimensional and graph is arbitrary.

Theorem 3.4. *For each $\bigvee \mathbb{S}^1$ of finitely many 1-spheres, there exists a graph G such that* $\mathrm{Ind}(G)$ *is homotopy equivalent to* $\bigvee \mathbb{S}^1$.

Proof. Our proof is based on the construction of a complex which has homotopy type $\bigvee \mathbb{S}^1$ and homotopy equivalent to an independence complex of a graph. Assume that the number of 1-spheres in the product $\mathbb{S}^1 \vee \mathbb{S}^1 \cdots \vee \mathbb{S}^1$ equals to m. Let H be a graph consists of two connected components; a vertex x and a path P_{m+1}. Since $\mathrm{Ind}(H)$ is a contractible graph with m edges, then the clique complex of its central graph $\Delta(C(H))$ is homotopy equivalent to $\bigvee_m \mathbb{S}^1$ by Proposition 3.1. Therefore $\mathrm{Ind}(G) = \Delta(C(H)) \cong \bigvee_m \mathbb{S}^1$ with $G = \overline{C(H)}$. □

Example 3.3. *Given a graph P_4 which is contractible, its central graph and complement shown in Figure 7 and 8. So* $\mathrm{Ind}(\overline{C(P_4)}) = \Delta(C(P_4))$ *is homotopy equivalent to* $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1$.

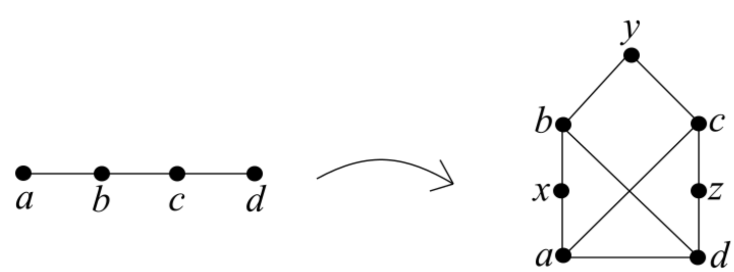

FIGURE 7. Graph P_4 and its central graph $C(P_4)$

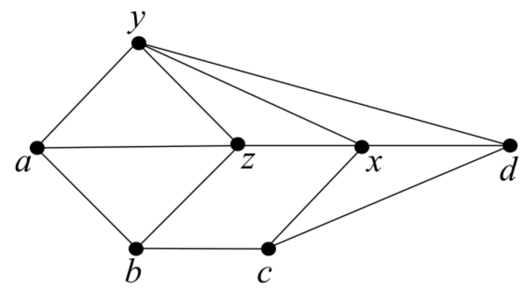

FIGURE 8. Complement of $C(P_4)$.

4. Building a complex from a bipartite graph

A complex is homeomorphic to its barycentric subdivision and this subdivision is an independence complex of a graph. Thus it is a well-known fact that every simplicial complex is homotopy equivalent to an independence complex $\text{Ind}(G)$ of a graph G.

In [10], Jonsson defined a complex $\Gamma_{G,V} \subseteq 2^V$ as follows:
Let $G = V \cup W$ be a bipartite graph. A set $\sigma \subseteq V$ belongs to $\Gamma_{G,V}$ if and only if there is a vertex $w \in W$ such that $\sigma \cup \{w\}$ is an independent set in G.

The following theorem states that the suspension of this complex is homotopy equivalent to independence complex of bipartite graph G.

Theorem 4.1. [10] *Let G be a bipartite graph with nonempty parts V and W. Then* $\text{Ind}(G) \simeq \Sigma(\Gamma_{G,V})$.

From above arguments, we can define a simplicial complex on a bipartite graph B as follows:

Definition 4.1. *Let B be a bipartite graph with bipartition U and W. Then the simplicial complex associated to this graph B is a complex whose vertex set is U and faces are the subsets $\sigma \subseteq U$ such that $\sigma \cup \{v\}$ is an independent set of B for some $u \in W$. This complex is denoted by Δ_B.*

Remark 4.1. *The complex* $\text{Ind}(B)$ *is homotopy equivalent to* $\Sigma(\Delta_B)$.

Proposition 4.1. *If $B = U \cup W$ is a complete bipartite graph $K_{m,n}$, then Δ_B is an empty complex. Moreover,* $\text{Ind}(B) \simeq \Sigma(\Delta_B) \simeq S^0$.

Proof. Since for any $u \in U$ there exits no $\sigma \subseteq W$ such that $\sigma \cup \{u\}$ is an independent set of B, this implies that Δ_B is an empty complex. From Proposition 2.17, one can conclude that $\text{Ind}(B) \simeq S^0$. □

Proposition 4.2. *If the graph B consists of n disjoint edges, then Δ_B is the boundary complex of a $(n-1)$-simplex and* $\text{Ind}(G) \simeq \mathbb{S}^{n-1}$.

Proof. Since, for any x_i, the set $\{x_1, x_2, ..., \hat{x}_i, ..., x_n\}$, obtained by omitting, forms a facet of Δ_B, it follows that Δ_B is the boundary complex of a $(n-1)$-simplex. Then by Proposition 2.17, one can therefore conclude that $\text{Ind}(G) \simeq \Sigma(\Delta_B) \simeq \mathbb{S}^{n-1}$. □

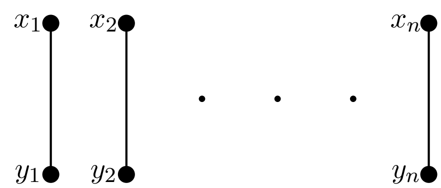

FIGURE 9. n disjoint edges

Definition 4.2. *Let Δ be a simplicial complex on vertex set $V(\Delta)$. The Alexander dual of Δ is the simplicial complex $\Delta^* = \{A \subset V(\Delta) : V(\Delta)\backslash A \notin \Delta\}$.*

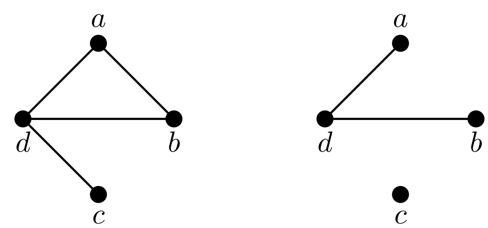

FIGURE 10. A simplicial complex and its Alexander dual

In [4], Csorba showed that the independence complex of a graph whose edges are subdivided exactly once is homotopy equivalent to the suspension of Alexander dual of the independence complex of that graph. In the next theorem, we will build a complex and give its homotopy type. It provides a different method.

Theorem 4.2. *([4], Theorem 5) Let G_2 be the edge subdivision of a graph G. Then the independence complex of G_2 has the same homotopy type of $\Sigma(\text{Ind}(G)^*)$.*

Proof. Let $V(G) = X = \{x_1, ..., x_m\}$. If the set $\{y_1, y_2, ..., y_m\}$ consists of the vertices added to G when subdividing the edges, then we have $V(G_2) = V(G) \cup \{y_1, y_2, ..., y_m\}$. If we set $Y = \{y_1, y_2, ..., y_m\}$, then G_2 is a bipartite graph with bipartition $G_2 = X \cup Y$. Thus $\Delta_{G_2} = \{F_i \subset X : F_i \cup \{y_i\}$ is independent set for some $y_i \in Y\}$. Since $V\backslash F_i$ is an edge of G and not a face of $\text{Ind}(G)$. Therefore we have $\text{Ind}(G_2) \simeq \Sigma(\text{Ind}(G)^*)$. □

5. Conclusion

We use the star cluster notion to determine the homotopy types of complexes arising from triangle-free graphs. We construct a complex from a bipartite graph which is a triangle-free graph. Further studies may be a concern to construct new methodologies for other graph complexes such as matching and clique complexes.

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