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MOTION OF GALILEAN PARTICLES WITH CURVATURE AND TORSION

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Abstract. This paper examines the motion of particles governed by an action that depends on the curvature and torsion of their trajectories in the Galilean 3-space G_3. We derive the Euler-Lagrange equation corresponding to the action $H(\gamma) = \int_\gamma f(\kappa, \tau) ds$ in G_3. We present examples to clarify the solutions of the system, clearly explaining their properties and relevance. With examples specifically focusing on the natural Hamiltonian problem derived from the Frenet frame of the curve and a generalization of these natural Hamiltonians, we aim to illustrate their key features and underlying principles.

Keywords: Curvature, Euler-Lagrange equations, Galilean geometry, Torsion, Variational calculus

2020 Mathematics Subject Classification: 53A04, 53A35, 37K25

1. Introduction

The study of particle motion governed by geometric properties such as curvature and torsion provides an understanding of underlying physical and mathematical principles. The interplay between curvature and torsion in defining particle trajectories has long been a subject of interest in both classical mechanics and modern theoretical studies. These geometric properties not only characterize the shape and behavior of curves in space but also play a critical role in variational principles, where the goal is often to identify extremal trajectories that satisfy specific physical constraints. Such analyses have applications across disciplines, including physics, where they model the dynamics of systems, and mathematics, where they enrich the theory of differential geometry and the calculus of variations (see, [1, 4, 5, 6, 9, 10, 11, 12, 15, 17], etc.).

Building on these foundations, variational problems emerge as a central framework for analyzing particle motion and other systems. Deeply rooted in the calculus of variations, they hold an essential role in mathematical analysis and find extensive applications across disciplines such as physics and engineering. These problems aim to identify extrema (minimum or maximum values) of functionals, which are mappings from a space of functions to real

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numbers. The objective of a typical variational problem is to find a function that optimizes a specific quantity, often expressed as an integral. For instance, the Hamiltonian of a space curve, defined as $H = \int f(\kappa, \tau, \kappa', \tau', ...)ds$, where f is a real arbitrary function, depends on the curvature (κ) and torsion (τ) of the curve. A simplified form of this Hamiltonian, expressed as $\int f(\kappa)ds$, depends solely on curvature, offering a more tractable framework for analyzing specific classes of variational problems. This reduction highlights the significance of curvature in determining the geometry and behavior of particle trajectories, particularly in cases where torsion does not contribute to the dynamics. Such models are widely used in applications ranging from the study of elastic rods and thin filaments to the analysis of geodesics and other naturally occurring curves. By focusing exclusively on curvature, these simplified Hamiltonians allow for deeper insights into the fundamental principles governing the shape and stability of particle paths. Extending this concept, a more general form of the Hamiltonian, $\int f(\kappa, \tau)ds$, incorporates both curvature and torsion as central variables. This generalization captures a broader range of geometric and physical phenomena, enabling the study of more complex particle trajectories. The inclusion of torsion reflects the three-dimensional twisting of the trajectory, adding a critical layer of complexity that is essential for understanding systems where both bending and twisting motions play a role. This formulation is particularly useful in problems involving helical structures, dynamical systems, and energy-minimizing configurations in elastic and physical systems. By considering both curvature and torsion, $\int f(\kappa, \tau)ds$ provides a comprehensive framework for exploring the interplay between these geometric properties in shaping particle motion. Capovilla et al. [4] examined the equilibrium conditions of space curves under local energy penalties associated with their curvature and torsion. They derived the Euler–Lagrange equations using the Frenet–Serret frame and exploited Noether's theorem to identify conservation laws tied to Euclidean invariance. The study highlighted specific integrable cases of the Hamiltonian $H = \int f(\kappa, \tau)ds$ connecting the results to physical applications like polymer stiffness and elastic properties of DNA. Following this, Ferrández et al. [9] explored the motion of relativistic particles in 3D pseudo-Riemannian spaces, governed by a Lagrangian as a general function of curvature and torsion. They derived Euler-Lagrange equations, identified dynamical invariants using Killing vector fields, and provided moduli spaces of solutions through integrable critical curves, extending the study of geometrically constrained motions. On the other hand, Tükel [19] contributed to this field by adopting a variational approach to determine critical points of the total squared torsion functional for curves in Euclidean and Minkowski 3-space, further enriching the understanding of these intricate geometric structures.

Shifting focus to Galilean geometry, we enter a framework where space is treated as a rigid, three-dimensional entity and time flows uniformly for all observers, independent of motion. Unlike the relativistic interplay of space and time, the Galilean model offers a simpler, classical foundation for understanding motion and forces, making it a natural setting to explore the mechanics of particles influenced by curvature and torsion. Bilir et al. [7] study investigates the classical variational problem of elastic curves in the Galilean plane, deriving Euler-Lagrange equations, determining the curvature of arc-length parameterized curves, and providing explicit examples. Tükel and Turhan [20] examined elastic curves in Galilean $3-$space G_3, deriving the Euler-Lagrange equations for the bending energy functional under boundary conditions. They solved the resulting differential equations and provided explicit examples to characterize elastic curves within this geometric framework. They examined natural Hamiltonians derived from the derivatives of the principal normal and binormal vectors of Frenet curves in Galilean and pseudo Galilean 3-space, solving the variational problem for the total squared torsion functional and identifying critical points characterized by constant torsion or curvature [16, 21]. Turhan, in his latest work [18], investigated hyperelastic curves in G_3, focusing on their characterization as extremals of a curvature energy functional, which is also a specific case of a natural Hamiltonian functional. By deriving Euler-Lagrange equations, he provided insights into the geometric behavior of these curves under the Galilean metric structure and illustrated their applications through detailed examples.

This paper focuses on the variational problem defined by the functional $\int f(\kappa, \tau)ds$ representing the curvature and torsion of curves in G_3. By employing the principles of the calculus of variations, we derive the associated Euler-Lagrange equations to characterize critical points of this functional under specific boundary conditions. We provide examples to highlight their key properties and potential applications. We center our analysis on natural Hamiltonian systems derived from the Frenet frame of curves and extend it to a generalization of these Hamiltonians.

2. Preliminaries

Let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ vectors in G_3. So, the Galilean scalar product of vectors is given as

$$< x, y >_{G_3} = \begin{cases} x_1 y_1, \text{ if } x_1 \neq 0 \text{ or } y_1 \neq 0, \\ x_2 y_2 + x_3 y_3, \text{ if } x_1 = 0 \text{ and } y_1 = 0. \end{cases}$$

The vectors x and y are said to be perpendicular in the Galilean sense if $< x, y >= 0$. The vector $x = (x_1, x_2, x_3)$ is known as isotropic (non-isotropic) if $x_1 = 0$ $(x_1 \neq 0)$. Any unit non-isotropic vector has the form $x = (1, x_2, x_3)$. For the vector x, the Galilean norm is written as

$$\|x\|_{G_3} = \begin{cases} |x_1|, \text{ if } x_1 \neq 0, \\ \sqrt{x_2^2 + x_3^2}, \text{ if } x_1 = 0, \end{cases}$$

[22].

A curve $\alpha : I \subset R \to G_3$ is called as admissible if it has no inflection points and no isotropic tangents (see, [3, 8, 23]). For a unit speed admissible curve $\alpha(s)$ parametrized by

$$\alpha(s) = (s, \alpha_2(s), \alpha_3(s)),$$

where s is the arclength parameter of α, we can give the curvature $\kappa(s)$ and the torsion $\tau(s)$ as follows

$$\kappa(s) = \sqrt{\alpha_2''(s) + \alpha_3''(s)}$$

and

$$\tau(s) = \frac{\det(\alpha'(s), \alpha''(s), \alpha'''(s))}{\kappa^2(s)}.$$

On the other hand, the Frenet frame of the curve $\alpha(s)$ in G_3 is given by

$$\begin{aligned} T(s) &= \alpha'(s), \\ N(s) &= \frac{1}{\kappa(s)}\alpha''(s), \\ B(s) &= \frac{1}{\kappa(s)}(0, -\alpha_3''(s), \alpha_2''(s)), \end{aligned}$$

where T, N, and B are respectively known as the tangent vector, principal normal vector and binormal vectors of $\alpha(s)$. So, the Frenet equations of $\alpha(s)$ are written in matrix form as

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ 0 & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{2.1}$$

[8, 20].

3. Particles with Galilean Curvature and Torsion

Let G_3 be the Galilean 3-space and γ be an admissible curve with speed $\nu = \|\gamma'(t)\|$, curvature κ, torsion τ and Frenet frame $\{T, N, B\}$. This section concerns the model, whose action is given by the functional $H(\gamma) = \int_\gamma f(\kappa, \tau)ds$ in G_3. Let $\Gamma = \Gamma(t, r)$ be a variation of $\gamma : [0, \ell] \to G_3$ with $\Gamma(t, 0) = \gamma(t)$. Associated with Γ, we consider the variation vector field W along $\gamma(t)$. The vector fields $V(t, r)$, $W(t, r)$ can be defined, where $V(0, t) = \gamma'(t)$ and $W(t) = W(0, t)$ is a variational vector field along $\gamma(t)$ (see, [2, 9, 18]). If s denotes the arclength parameter, then $\gamma(s, r)$, $\kappa^2(s, r)$, $V(s, r)$, etc. can be written for the corresponding reparametrizations, where $s \in [0, \ell]$ and ℓ is arc length of γ.

We arrive at the following Lemma from the Frenet equations in (2.1).

Lemma 3.1 ([21])**.** *Let $\gamma(t, r)$ be a variation of curve $\gamma \in G_3$. Then the following formulas are satified:*

i) $W(\nu) = <W', T> \nu$,
ii) $W(\kappa) = <W'', N> -2\kappa <W', T>$,
iii) $W(\tau) = \left(\frac{1}{\kappa}(<W'', B>)\right)' - <W', \tau T>$.

Now we assume that γ is a stationary point of the functional $H(\gamma)$. Then, we have

$$\partial H(W) = \left.\frac{\partial H(\gamma + \varepsilon W)}{\partial \varepsilon}\right|_{\varepsilon=0} = 0.$$

Thus, we obtain

$$\left.\frac{\partial H}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_0^\ell \left(f_\kappa W(\kappa) + f_\tau W(\tau) + <W', T>\right) ds.$$

Taking into consideration Lemma 3.1, we find

$$\left.\frac{\partial H}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_0^\ell \begin{pmatrix} <W'', f_\kappa N> + <W', -2\kappa f_\kappa T> + f_\tau <W''', \frac{1}{\kappa}B> \\ + <W'', f_\tau(\frac{1}{\kappa}B)'> + <W', -f_\tau \tau T> + <W', fT> \end{pmatrix} ds.$$

Then, by using standard arguments involving the above formulas and integration by parts, the first variation of $H(\gamma)$ along γ in the direction of W is given by

$$\left.\frac{\partial H}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_0^\ell < E, W > ds + B[W,\gamma]|_0^\ell,$$

where

$$E = \left((-f + \tau f_\tau + 2\kappa f_\kappa)\, T + (f_\kappa N)' - \left(\frac{f_\tau'}{\kappa} B\right)'\right)'$$

and

$$\begin{aligned} B[W,\gamma]|_0^\ell &= \left. < W'', \frac{f_\tau}{\kappa} B > \right|_0^\ell + \left. < W', f_\kappa N - \frac{f_\tau'}{\kappa} B > \right|_0^\ell \\ &\quad + \left. < W, (f - \tau f_\tau - 2\kappa f_\kappa)\, T - (f_\kappa N)' + \left(\frac{f_\tau'}{\kappa} B\right)' > \right|_0^\ell . \end{aligned}$$

Point out that, we used f_κ and f_τ to denote the partial derivatives of f with respect to κ and τ, respectively. Also, we restrict ourselves to variations with fixed endpoints having the same Frenet frames on them. Then, the boundary term $B[W,\gamma]|_0^l = 0$, so that the critical curves are characterized by the vanishing of the Euler–Lagrange operator $E' = 0$. If the necessary calculations are made and the Frenet equations are used, we have

$$(-f + \tau f_\tau + 2\kappa f_\kappa)' = 0, \tag{3.2}$$

$$-f\kappa + \tau\kappa f_\tau + 2\kappa^2 f_\kappa + f_\kappa'' - \tau^2 f_\kappa + (\frac{f_\tau'}{\kappa})'\tau + (\frac{f_\tau'\tau}{\kappa})' = 0 \tag{3.3}$$

and

$$f_\kappa'\tau + (f_\kappa\tau)' - (\frac{f_\tau'}{\kappa})'' + \frac{f_\tau'}{\kappa}\tau^2 = 0. \tag{3.4}$$

From (3.2), we obtain

$$f = \tau f_\tau + 2\kappa f_\kappa + A, \tag{3.5}$$

where A constant. Substituting (3.5) into (3.3), we obtain

$$-\kappa A + f_\kappa'' - \tau^2 f_\kappa + (\frac{f_\tau'}{\kappa})'\tau + (\frac{f_\tau'\tau}{\kappa})' = 0. \tag{3.6}$$

Theorem 3.1. *The motion equations of the action, defined by the functional* $H(\gamma) = \int_\gamma f(\kappa,\tau)ds$ *in* G_3 *are characterized by the Euler-Lagrange equations (3.4) and (3.6).*

To further illustrate this theory, we now provide examples that demonstrate the application of the Euler-Lagrange equations and highlight the geometric and physical implications of the solutions.

In the Galilean 3-space G_3, curves can be classified based on their curvature and torsion values, as outlined in [14]. This classification includes special cases such as straight lines, plane curves, circular helices, generalized helices, Salkowski curves, and anti-Salkowski curves, each defined by specific geometric properties. In the following examples, we will refer to this classification as a basis for analyzing critical curves

Example 3.1. *Let* γ *is an admissible curve in* G_3*. We examine whether* γ *is critical for* H *based on its curvature and torsion values, as outlined in the cases below to illustrate its geometric properties:*

i) Straight line. A straight line γ in G_3 is a critical point of the functional H.

ii) Planar curve. Let γ be a planar curve in G_3. If γ is is a critical point of the functional H, then, γ satisfies the equation $-\kappa A + f''_\kappa = 0$.

iii) Circular helices (W-curves). Any circular helix or $W-$curve in G_3 is a solution of the functional H when the function $f(\kappa, \tau) = 0$.

iv) Salkowski and anti Salkowski curves. A Salkowski curve in G_3 is critical for the functional H if it satisfies the following Euler-Lagrange equation

$$-\kappa^2 A + f''_\tau \tau + (\frac{f'''_\tau}{\tau})' = 0.$$

Moreover, it is evident from the Euler-Lagrange equations (3.4) and (3.6), that there is no anti-Salkowski curve in G_3 that is critical for H.

Example 3.2. *Let $\gamma(s) = (s, \dfrac{(s - \sin s \cos s)}{4}, \dfrac{(\sin s^2 - s^2)}{4})$ be a regular curve in G_3 with $\kappa(s) = \sin s$ and $\tau(s) = 1$* [14]. *If the values of curvature and torsion are incorporated into the Euler-Lagrange equations, it can be observed that the curve in question serves as an example for the derived results for $s = \frac{\pi}{2} + k\pi$, $k \in Z$.*

We know that a simple model for the Hamiltonian is in the form $\int f(\kappa)ds$ which depends on the curvature. Especially, a natural Hamiltonian $\int \kappa^2 ds$ generated by $< T', T' >$ is known as a bending energy functional and critical points of this functional under suitable condition are called as elastic curves [13]. Elastic curves and its generalization under given first order boundary data have been worked and developed by many authors up to now. In his study, Turhan characterized hyperelastic curves in G_3 by addressing a generalization of the functional formed by the inner product of first derivative of the tangent vector of the curve, which is known as a natural Hamiltonian [18]. Upon examining the derived Euler-Lagrange equation, it is an undeniable fact that the results serve as an example for the problem addressed in this study. Beyond this, another obvious question arises: how can the critical points of the natural Hamiltonian functional formed by other frame elements (produced by $< N', N' >$ and $< B', B' >$) of the Frenet frame be determined? In another example, the critical points of the natural Hamiltonian constructed using the binormal vector field of the curve are examined.

Example 3.3. *We consider an admissible curve γ with Galilean Frenet frame $\{T, N, B\}$ and curvature κ and torsion τ in G_3. A generalization of the natural Hamiltonian is considered as the functional $\int < B', B' >^{n/2} ds$. This functional is a generalized torsion energy action given by $\int \tau^n ds$. The critical points of this functional are characterized by the following Euler-Lagrange equations*

$$n(n-1)\tau^{n-1}\tau' = 0, \tag{3.7}$$

$$(n-1)\tau^n \kappa + \left(\frac{n(n-1)\tau^{n-2}\tau'}{\kappa}\right)' \tau + \left(\frac{n(n-1)\tau^{n-1}\tau'}{\kappa}\right)' = 0 \tag{3.8}$$

and

$$\frac{n(n-1)\tau^n\tau'}{\kappa} - \left(\frac{n(n-1)\tau^{n-2}\tau'}{\kappa}\right)'' = 0. \tag{3.9}$$

From (3.7), we get τ is a constant value. If τ is zero, then Eqs. (3.8) and (3.9) are satisfied for any value of κ. If $\tau \neq 0$, then τ is constant and from (3.8), we obtain κ is zero.

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