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ALMOST POTENT MANIFOLDS

BAYRAM ŞAHIN *

Abstract. In this paper we introduce a new manifold, namely potent manifolds. We give examples and investigate the integrability conditions. We also check the curvature relations of Kaehler potent manifolds and show that such manifolds are flat when it has constant sectional curvature. Then we introduce potent sectional curvature and obtain a spacial form of the curvature tensor field when its potent sectional curvature is a constant.
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1. Introduction

Manifolds on which extra structures are defined present a geometrically rich structure. In this sense, the first structures are complex structures [29] and contact structures [2]. In the process, many new structures have been defined on manifolds and have received new names accordingly. These manifolds can be listed as product manifolds [29], quaternion manifolds [13], biproduct complex manifolds [6], para-contact manifolds [25], para-quaternionic manifolds [9], [11], [26], hyper Kaehler manifolds [3], metric mixed 3-structures [14], [27], almost tangent manifolds, [20], [28] and the like. Recently, with the definition of Golden manifolds [5] in the literature, new manifold classes such as metallic manifolds [10], poly-Norden manifolds [23], meta-Golden manifolds [21], meta-Metallic manifolds [7], bi-tangent quaternion manifolds [16], [18] have been introduced and their geometric properties have been examined.

In this paper, inspired by the concept of idempotent transformation, we introduce almost potent manifolds and investigate the geometric properties of such manifolds. We also introduce Hermitian potent manifolds, Kaehler potent manifolds and potent space forms.

The paper is organized as follows. First, the definition of an almost potent manifold is presented and examples are given. In addition, the Nijenhuis tensor field of an almost potent manifold is calculated and the integrability condition is given accordingly. In this

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* Corresponding author
Bayram Şahin ⋄ bayram.sahin@ege.edu.tr ⋄ https://orcid.org/0000-0002-9372-1151.

case, the concept of a Kaehler potent manifold is given. In the third section, the constant sectional curvature of a Kaehler potent manifold is considered and it is shown that it is flat if it has constant sectional curvature. For this reason, holomorphic-like sectional curvature and holomorphic bisectional-like curvatures are investigated, but it turns out that these are also zero. Therefore, the notion of potent sectional curvature of a Kaehler potent manifold, is given. In the case that the potent sectional curvature is constant, a special case of the curvature tensor field of a Kaehler potent manifold is obtained.

2. Almost Potent Manifolds

In this section we introduce almost potent manifolds and almost Hermitian potent manifolds. We then investigate the integrability of almost potent manifolds and define Kaehler potent manifolds.

Definition 2.1. *Let M be a differentiable manifold and F an endomorphism on M. If F is idempotent, i.e.*

$$F^2X = FX \tag{2.1}$$

for all $X \in \chi(M)$, then F is called an almost potent structure on M. In this case (M, F) is called an almost potent manifold.

Example 2.1. *Consider $\mathbb{R}^2$ with the map $A(x, y) = (\frac{1}{2}(x + y), \frac{1}{2}(x + y))$. Then $(\mathbb{R}^2, A)$ is an almost potent manifold.*

Example 2.2. *Consider the right circular cylinder M given by*

$$X(u, v) = (\cos u, \sin u, v).$$

Then the matrix of the shape operator of M is

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.2}$$

Then it is easy to see that $S^2 = S$. Thus (M, S) is an almost potent manifold.

Example 2.3. *Consider the Clifford algebra $Cl(2, 0)$ with the basis e_1, e_2 such that $e_1^2 = e_2^2 = 1$. $Cl(2, 0)$ can be represented by the algebra $M(2, \mathbb{R})$ of all real matrices by taking*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now consider an idempotent matrix B and define F as

$$F = Be_i$$

Then F is also an idempotent matrix and $Cl(2, 0), F)$ is an almost potent manifold.

Example 2.4. *Let M be a Riemannian manifold and $\{e_1, e_2, e_3\}$ an orthonormal frame of M. Assume that there is an endomorphism F on M such that $Fe_1 = e_1$, $Fe_2 = e_2$ and $Fe_3 = 0$. Then (M, F) is an almost potent manifold.*

We now define Hermitian potent manifold.

Definition 2.2. *Let (M, F) be an almost potent manifold. If there is a Riemannian metric g on M such that*

$$g(FX, Y) = g(X, FY) \tag{2.3}$$

for $X, Y \in \chi(M)$, then (M, F, g) is called almost Hermitian potent manifold. In this case we have

$$g(FX, FY) = g(FX, Y) = g(X, FY). \tag{2.4}$$

Example 2.5. *Right circular cylinder given in Example 2.2 with the shape operator is an example of an almost Hermitian potent manifold.*

We now investigate the integrability of almost potent structure F. We first note that the Nijenhuis tensor field of an endomorphism or $(1, 1)$ tensor field is given by

$$N_F(X, Y) = [FX, FY] + F^2[X, Y] - F[FX, Y] - F[X, FY] \tag{2.5}$$

for $X, Y \in \chi(M)$. Using (2.1) and (2.5) we obtain the following

$$N_F(X, Y) = [FX, FY] + F[X, Y] - F[FX, Y] - F[X, FY]. \tag{2.6}$$

From (2.6), by direct computation, we have the following lemma.

Lemma 2.1. *Let (M, F, g) be almost Hermitian potent manifold. Then we have*

$$N_F(X, Y) = (\nabla_{FX}F)Y - F(\nabla_X F)Y - (\nabla_{FY}F)X + F(\nabla_Y F)X \tag{2.7}$$

for $X, Y \in \chi(M)$.

Thus from (2.7), we have the following result.

Proposition 2.1. *Let (M, F, g) be almost Hermitian potent manifold. Then an almost potent structure is integrable if it is parallel.*

Thus we give the following definition.

Definition 2.3. *Let (M, F, g) be almost Hermitian potent manifold. If F is parallel, i.e.*

$$(\nabla_X F)Y = 0 \tag{2.8}$$

for $X, Y \in \chi(M)$, then (M, F, g) is called Kaehler potent manifold.

Example 2.6. *Right circular cylinder given in Example 2.2 with the shape operator S is an example of a Kaehler potent manifold.*

We note the following properties from linear algebra [19] of inner product spaces with an idempotent map. For any idempotent operator F, its only possible eigenvalues are 0 and 1. An idempotent operator on almost Hermitian potent manifold is typically a projection operator. It projects vector fields onto a subspace of the space. If F is an idempotent operator, then the image $Im(F)$ is a subspace of $\chi(M)$. Thus we have

$$\chi(M) = Im(F) \oplus Ker(F) \tag{2.9}$$

where $KerF$ is the kernel of F. Moreover if $F = F^*$ (where F^* is the adjoint of F, $Im(F)$ and $Ker(F)$ are orthogonal complement to each other. From now on, we will denote the vector fields belonging to the image space of the endomorphism F on almost Hermitian potent manifold M by $\chi(ImF(M))$.

3. Curvature relations of a Kaehler potent manifold

In this section, we are going to investigate the curvature tensor fields of a Kaehler potent manifold and show that it is a flat when it is a real space form. We also check holomorphic-like and holomorphic bi-sectional like curvature. We first give the following algebraic properties of curvature tensor fields.

Lemma 3.1. *Let (M, F, g) a Kaehler potent manifold and R the curvature tensor field of M. Then we have*

$$R(X,Y)FZ = FR(X,Y)Z \tag{3.10}$$

$$g(R(X,Y)FZ, FW) = g(R(X,Y)Z, FW) \tag{3.11}$$

$$R(FX, FY) = R(X, FY) = R(FX, Y) \tag{3.12}$$

for $X, Y, Z, W \in \chi(ImF(M))$.

Proof. (3.10) is clear from (2.8). For (3.11), using (3.10) we have

$$g(R(X,Y)FZ, FW) = g(FR(X,Y)Z, FW).$$

Then from (2.4) we get (3.11).For (3.12) we first have

$$g(R(FX, FY)Z, W) = g(R(Z,W)FX, FY).$$

Using (3.11) we obtain

$$g(R(FX, FY)Z, W) = g(R(Z,W)X, FY) = g(R(X, FY)Z, W)$$

which gives the first part of (3.11). Second part follows from (2.4). □

If the Kaehler potent manifold has constant sectional curvature, the following situation occurs.

Proposition 3.1. *Let M be an $n-$ dimensional Kaehler potent manifold. If M has constant sectional curvature c at every point $p \in M$, then M is flat provided $n \geq 2$.*

Proof. Since M has constant sectional curvature c, the curvature tensor field of the manifold is

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}$$

for $X, Y, Z \in \chi(M)$. Using (3.10) we get

$$c\{g(Y, FZ)X - g(X, FZ)Y\} = c\{g(Y,Z)FX - g(X,Z)FY\}$$

for $X, Y, Z \in \chi(ImF(M))$. Taking $Y = FX$ and using (2.3) we arrive at

$$c\{g(FX, Z)X - g(X, FZ)FX\} = c\{g(FX, Z)FX - g(X,Z)FX\}.$$

Using (2.1) we have

$$c\{g(FX, Z)FX - g(X, FZ)FX\} = c\{g(FX, Z)FX - g(X,Z)FX\}.$$

Hence using (2.3) we derive

$$0 = c\{g(FX, Z)FX - g(X,Z)FX\}.$$

Thus taking $Z \perp X$ such that FX and Z are not ortgonal to each other, we get

$$0 = cg(FX, Z)FX.$$

This implies that $c = 0$. □

Remark 3.1. *The choice of vector fields used in the proof is very crucial. It is easy to see that this is valid in* $(\mathbb{E}^2 = (\mathbb{R}^2, <, >)$*. For example, if* $X = (1, 1)$*,* $Z = (-1, 1)$ *are chosen and the idempotent matrix is chosen as* $F = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$*, it is seen that this situation is valid.*

As in complex geometry, for a Kaehler potent manifold, the holomorphic-like sectional curvature and the holomorphic-like bi-sectional curvature can be defined as

$$H(X) = g(R(X, FX)FX, X)$$

and

$$H(X, Y) = g(R(X, FX)FY, Y)$$

for $X, Y \in \chi(ImF(M))$. However, the following propositions will show us that these notions do not work for Kaehler potent manifolds.

Proposition 3.2. *Every Kaehler potent manifold has zero holomorphic-like sectional curvature.*

Proof. From (3.10) we have

$$g(R(X, FX)FX, X) = g(FR(X, FX)X, X).$$

Then (2.3) gives

$$g(R(X, FX)FX, X) = g(R(X, FX)X, FX).$$

Hence $g(R(X, FX)FX, X) = -g(R(X, FX)FX, X)$ which completes proof. □

Proposition 3.3. *Every Kaehler potent manifold has zero holomorphic-like bi-sectional curvature.*

Proof. From (3.10) and (2.3) we have

$$\begin{aligned} H(X, Y) &= g(R(X, FX)FY, Y) \\ &= g(F(R(X, FX)Y, Y) \\ &= g(R(X, FX)Y, FY) \end{aligned}$$

for $X, Y \in \chi(ImF(M))$. Hence we derive

$$\begin{aligned} H(X, Y) &= g(R(X, FX)FY, Y) \\ &= -g(R(X, FX)FY, Y). \end{aligned}$$

Thus $H(X, Y) = 0$. □

4. Potent sectional curvature and potent space forms

From the previous section, it was seen that the Kaehler potent manifold with constant sectional curvature is flat and the concepts of holomorphic-like sectional curvature and holomorphic bi-sectional curvature do not work. Therefore, in this section, the notion of potent sectional curvature is introduced, an example is given, and a special expression is given for the curvature tensor field of Kaehler potent manifolds with constant potent section curvature. We present the following definition.

Definition 4.1. *Let (M, F, g) be a Kaehler potent manifold. For $X, Y \in \chi(ImF(M))$ the potent sectional curvature is defined by*

$$K(X \wedge FY) = \frac{g(R(X, FY)FY, X)}{g(X, X)g(FY, FY) - g(X, FY)^2}. \tag{4.13}$$

If the potent sectional curvature is constant for arbitrary vector fields X and Y and arbitrary point $p \in M$, then the Kaehler potent manifold is said to have constant sectional curvature, or simply a potent space form.

Example 4.1. *Consider half space $H = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \mid x^1 > 0\}$ endowed with the Riemannian metric*

$$g = \frac{1}{K}\left(\frac{dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3}{(x^1)^2}\right) \tag{4.14}$$

for $K \in \mathbb{R}$. We define an idempotent map by $Fe_1 = e_1$, $Fe_2 = e_2$ and $Fe_3 = 0$ where $e_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, 3$. Since $[e_i, e_j] = 0$ we have

$$\nabla_{e_1} e_1 = -\frac{1}{x^1} e_1,\ \nabla_{e_2} e_2 = \frac{1}{x^1} e_1, \nabla_{e_1} e_2 = -\frac{1}{x^1} e_2, \nabla_{e_2} e_1 = -\frac{1}{x^1} e_2.$$

Thus we get

$$g(R(e_1, Pe_2)Pe_2, e_1) = -\frac{1}{K}\frac{1}{(x^1)^4}, g(R(e_2, Pe_1)Pe_1, e_2) = -\frac{1}{K}\frac{1}{(x^1)^4}.$$

Hence we obtain

$$K(e_1 \wedge Pe_2) = K(e_2 \wedge Pe_1) = -K.$$

Thus (H, g, F) is a potent space form.

We now obtain a special expression of the curvature tensor field of a potent space form.

Theorem 4.1. *Let $(M(c), g, F)$ be a potent space form. Then we have*

$$R(X, FY)FZ = c\{g(Y, FZ)FX - g(FX, Z)FY\} \tag{4.15}$$

$X, Y, Z \in \chi(ImF(M))$.

Proof. Since M is a potent space form, we have

$$g(R(X, FY)FY, X) = c\{g(X, X)g(Y, FY) - g(X, FY)^2\}. \tag{4.16}$$

Replacing Y by $Y + Z$ in (4.16), we get

$$g(R(X, FY)X, FZ) = c\{-g(X, X)g(Y, FZ) + g(X, FY)g(X, FZ)\} \tag{4.17}$$

Using (2.3) and (3.11) we derive

$$R(X, FY)FX = c\{-g(X, X)FY + g(X, FY)FX\}. \tag{4.18}$$

Substituting X by $X+Z$ in (4.18) we obtain

$$\begin{aligned} R(X,FY)FZ+R(Z,FY)FX &= c\{-2g(X,Z)FY+g(X,FY)FZ \\ &\quad +g(Z,FY)FX\}. \end{aligned} \tag{4.19}$$

Replacing X by $X+Z$ in (4.16) we get

$$g(R(X,FY)FY,Z)=c\{g(X,Z)g(Y,FY)-g(X,FY)g(Z,FY)\}. \tag{4.20}$$

Hence we have

$$R(X,FY)FY=c\{g(Y,FY)X-g(X,FY)FY\}. \tag{4.21}$$

If $Y+Z$ is written instead of Y in (4.21) we arrive at

$$\begin{aligned} 2R(X,FY)FZ-R(Z,FY)X &= c\{2g(Y,FZ)X-g(X,FY)FZ \\ &\quad -g(X,FZ)FY\}. \end{aligned} \tag{4.22}$$

Substituting FX instead of X in (4.22) and using (2.1), (2.3) and (3.12) we get

$$\begin{aligned} 2R(X,FY)FZ-R(Z,FY)FX &= c\{2g(Y,FZ)FX-g(X,FY)FZ \\ &\quad -g(X,FZ)FY\}. \end{aligned} \tag{4.23}$$

Thus from (4.21) and (4.23) we conclude that

$$3R(X,FY)FZ=c\{3g(Y,FZ)FX-g(X,FZ)FY-2g(X,Z)FY\}. \tag{4.24}$$

Finally substituting FX instead of X in (4.24), and using (3.12) and (2.4) we obtain

$$R(X,FY)FZ=c\{g(Y,FZ)FX-g(FX,Z)FY\}$$

which is (4.16). □

5. Concluding Remarks

This paper presents a new class of manifolds. This manifold class is quite different from the manifold classes in the literature. In the paper, examples of the existence of such manifolds are presented, their properties are examined and their sectional curvatures are investigated. When it is seen that the classical sectional curvatures do not work, a new sectional curvature notion is introduced, an example is given and accordingly the curvature tensor field of the manifold is specifically expressed. As can be seen, this presented manifold class shows that it will have a rich geometry. Therefore, we invite researchers to explore this manifold class.

In the first stage, we especially propose the following research problems.

Open Problem 1. As it is well known, submanifolds of manifolds endowed with extra structures on them offer a very rich research [4]. Therefore, the investigation of submanifolds (such as invariant, anti-invariant, semi-invariant, slant) that will be defined depending on the potent structure of an almost potent manifold may produce interesting results.

Open Problem 2. Harmonic maps between manifolds are one of the most important research topics in differential geometry. For example, it is well known that holomorphic maps between two Kaehler manifolds are harmonic [1]. It would be an interesting research problem to investigate the harmonicity of the map defined between a Kaehler potent manifold (potent maps may be defined similarly to holomorphic maps).

Open Problem 3. Riemannian submersions defined on manifolds with a special structure have interesting geometric properties [8], [24]. Therefore, studying the geometric properties of a Riemannian submersion defined on a Kaehler potent manifold will produce rich research results.

Open Problem 4. Special curves on a manifold begin with Nomizu and Yano's definition of the notion of a circle on a manifold [17]. After this concept, helices and similar concepts were also defined [12],[15], [22]. The geometry of special curves on a Kaehler potent manifold will produce interesting geometric results.

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(B. Şahin) Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkiye