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TRIANGULAR NUMBERS AND CENTERED SQUARE NUMBERS HIDDEN IN PYTHAGOREAN RUNS

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Abstract.

The Pythagorean theorem, which asserts that in a right triangle, the sum of the squares of the legs is equal to the square of the hypotenuse and is mathematically expressed as $a^2 + b^2 + c^2$ can be generalized to equations with 5, 7, or more variables. If we seek to find t consecutive numbers that satisfy such equations, which can be extended infinitely by increasing the number of variables, and observe the equality of sums of squares for each case, we encounter what are known as Pythagorean runs. In this study, it was observed that within Pythagorean runs, which can become increasingly complex as we increase the number of variables, there exists a strikingly unique solution set when we restrict ourselves to finding consecutive integers.

By examining the consecutive integers that form these Pythagorean runs, new findings have emerged. Specifically, Pythagorean runs were analyzed using triangular numbers and centered square numbers. A hypothesis was formulated, positing that there is a unique solution involving consecutive integers for Pythagorean runs with figurate numbers. This hypothesis has been proven using both inductive and geometric proof methods.

Keywords: Consecutive Numbers, Pythagorean Runs, Triangular Numbers, Centered Square Numbers.
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1. Introduction

In a right triangle, the Pythagorean theorem states that the square of the hypotenuse (the longest side) is always equal to the sum of the squares of the other two sides. This fundamental principle can be algebraically expressed as $a^2 + b^2 = c^2$. By extending this principle with more variables, we can formulate new equations involving sums of squares with a structure similar to the Pythagorean Theorem:

$a^2 + b^2 = c^2$, involving 3 variables.
$a^2 + b^2 + c^2 = d^2 + e^2$, involving 5 variables.
$a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2$, involving 7 variables.
$a^2 + b^2 + c^2 + d^2 + e^2 = f^2 + g^2 + h^2 + j^2$, involving 9 variables, and so forth.

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An infinite number of integers satisfy these equations, which initially appear as sums of squares. But what if we restrict these integers to be consecutive?

For the Pythagorean Theorem, which is the first of these equations, many different integer combinations like (7, 24, 25), (8, 15, 17), and (9, 40, 41) come to mind. However, when seeking consecutive integers that satisfy the Pythagorean Theorem, there is only one solution: (3, 4, 5). This can be observed as follows: $3^2 + 4^2 = 5^2$.

In such equations, regardless of the number of variables added, only one set of consecutive integers forms a solution, just as with the Pythagorean Theorem. For example:

For the equation $a^2 + b^2 + c^2 = d^2 + e^2$, the consecutive numbers are: $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.

For the equation $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2$, the consecutive numbers are: $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$.

The sequences of consecutive integers that satisfy such equations extend as follows:

$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$

$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$

$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$

According to the literature, these perfect sequences can continue indefinitely, and for any equation with one term, a unique solution of consecutive integers exists. These sequences, which appear as sums of squares of consecutive integers, are known in the literature as "Pythagorean runs", and various intriguing studies have explored their properties.

This study began by gathering the proofs for special cases of equations with three, five and seven variables, with the aim of discovering new insights into consecutive integers that satisfy equations with more variables, and to compare these findings with existing literature.

The discovery of the relationship between triangular numbers and centered square numbers, which are well-known for their fascinating properties in number theory, and their connection to consecutive integers forming solutions to the studied equations, constitutes the original aspect of this research.

At the conclusion of the study, an original algorithm was developed to extend these perfect sequences indefinitely using figurate numbers.

2. Material and Method

2.1. **The Pythagorean Theorem.** The Pythagorean Theorem, which has been proven and widely known for centuries, from ancient Egypt to the present day, has captivated the attention of many, including renowned mathematicians such as Euclid, Archimedes and Sabit bin Qurra, as well as the 20th president of the United States, James A. Garfield, who famously provided a simple proof using a trapezoid. The Pythagorean Theorem, which describes the geometric relationship in right triangles, states that in a right triangle, the square of the hypotenuse (the side opposite to the right angle) is equal to the sum of the squares of the other two legs (the sides adjacent to the right angle). This relationship, expressed algebraically as $a^2 + b^2 = c^2$, where a and b are the legs of the right triangle, is often treated as a fundamental mathematical exercise [1]. Kindly refer to Figure 1 for further reference.

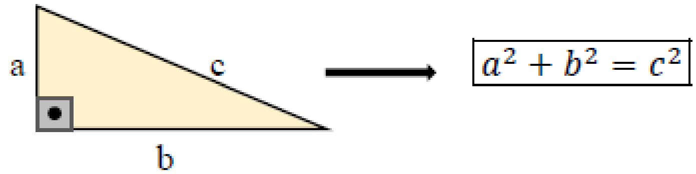

FIGURE 1. Pythagorean Theorem

Definition 2.1.1: A triple (a, b, c) is called a Pythagorean triple if it satisfies the equation $a^2 + b^2 = c^2$, ,where a, b and c positive integers [2, p. 4].

The right triangle with side lengths 3, 4, and 5 provides the first integer solution to the Pythagorean Theorem, forming what is known as the Pythagorean triple (3, 4, 5), as it satisfies the equation $3^2 + 4^2 = 5^2$. Pythagorean triples can be extended to other integer combinations, such as (5, 12, 13), where $5^2 + 12^2 = 13^2$. These examples demonstrate the recurring relationship between integer side lengths in right triangles governed by the Pythagorean theorem.

Definition 2.1.2: A Pythagorean triple (a, b, c) is called a primitive Pythagorean triple if it satisfies the equation gcd(a, b, c) = 1 [2, p. 4].

Many primitive Pythagorean triples can be derived, such as (3, 4, 5), (16, 63 ve 65), (21, 20 ve 29), (55, 48 ve 73), (65, 72 ve 97),(1155, 1292 ve 1733), (20737, 23184 ve 31105) [2].

These triples consist of positive integers that satisfy the equation $a^2 + b^2 = c^2$ and have no common divisor greater than 1, thereby representing primitive solutions to the Pythagorean theorem.

Proposition 2.1.1: Given that a, b and $c \in Z^+$, the only primitive Pythagorean triple consisting of consecutive integers that satisfy the equation $a^2 + b^2 = c^2$ is (3, 4, 5).

Proof 2.1.1: Let us express a and c in the form of b such that they are consecutive integers. Thus, we have $c = b + 1$ and $a = b - 1$. Now let us substitute these into the equation and solve for equality:

$$a^2 + b^2 = c^2 \tag{1}$$

$$(b-1)^2 + b^2 = (b+1)^2$$

$$(b-1)^2 + b^2 - (b+1)^2 = 0$$

$$(b-1+b+1).(b-1-b-1) + b^2 = 0$$

$$2b.(-2) + b^2 = 0$$

$$b^2 - 4b = 0$$

$$b.(b-4) = 0$$

From here, we find $b = 0$ or $b = 4$. However, since $b \in Z^+$, b cannot be 0. Therefore b must be 4. In this case, $a = 3, c = 5$. Substituting these results into the equation $a^2 + b^2 = c^2$, we obtain the equality $3^2 + 4^2 = 5^2$. As can be understood from this direct proof, the numbers 3, 4 and 5 have a distinct significance as the only consecutive integers that solve the equation $a^2 + b^2 = c^2$.

Let us examine Figure 2 for the visual geometric proof found in literature. The proof by Michael Boardman, which resperents $3^2 + 4^2 = 5^2$ by diving it into squares, is illustrated in Figure 2:

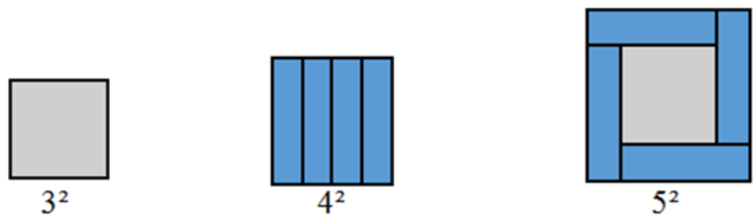

FIGURE 2. Boardman's dissection of squares for $3^2 + 4^2 = 5^2$ [3]

Boardman sliced the 4-square into four congruent rectangles that fit against the four sides of the 3-square to produce the 5-square [3, p.323]. As illustrated in Figure 2, this transformation demonstrates how a square can be restructured into a larger square by breaking it down and incorporating it with smaller square. By dividing the 4-square into four equal parts and placing them around the sides of 3-square, as shown in the figure, we produce a 5-square. This geometric process leads to the vertification of the equation $3^2 + 4^2 = 5^2$.

2.2. **Extending the Pythagorean Equation With Additional Variables While Preserving Its Fundamental Principle.**

The problem is to determine, if possible, two consecutive integers the sum of whose squares equals the sum of the squares of three conse- cutive integers; three consecutive integers, the sum of whose squares equals the sum of the squares of four consecutive integers; and so on [4, p.155].

The problem Alfred mentioned is to expand the equation $a^2 + b^2 = c^2$ by adding more variables while preserving its fundamental principle, thereby creating new equations that maintain a structure similar to the Pythagorean Theorem:

$a^2 + b^2 = c^2$, with 3 variables,

$a^2 + b^2 + c^2 = d^2 + e^2$, with 5 variables,

$a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2$, with 7 variables,

$a^2 + b^2 + c^2 + d^2 + e^2 = f^2 + g^2 + h^2 + j^2$, with 9 variables,

$a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + = g^2 + h^2 + j^2 + k^2 + m^2$, with 11 variables, and so on.

Let us proceed by analyzing the equation with five variables and attempt to find consecutive numbers that satisfy the equation $a^2 + b^2 + c^2 = d^2 + e^2$.

For example; $13^2 + 14^2 = 10^2 + 11^2 + 12^2$

$133^2 + 134^2 = 108^2 + 109^2 + 110^2$

$1321^2 + 1322^2 = 1078^2 + 1079^2 + 1080^2$ " [4, p.155].

When we examine the individual examples in this study, we observe that only the first example consists entirely of consecutive numbers. This observation supports the second proposition that we aim to prove in our research.

Proposition 2.2.1: The only solution set composed of consecutive integers for the five-variable equation $a^2 + b^2 + c^2 = d^2 + e^2$, where $a, b, c, d, e \in Z^+, is (10, 11, 12, 13, 14)$.

Proof 2.2.1: The fact that the numbers a, b, c, d and e in the equation $a^2+b^2+c^2 = d^2+e^2$ are consecutive allows us to express these numbers in terms of a single variable. Let the middle number be $c = x$. In this case $a = x - 2, b = x - 1, d = x + 1$ and $e = x + 2$. Now, substituting these expressions into the equation, we can proceed to solve for equality.

$$(x - 2)^2 + (x - 1)^2 + x^2 = (x + 1)^2 + (x + 2)^2$$
$$x^2 - 4x + 4 + x^2 - 2x + 1 + x^2 = x^2 + 2x + 1 + x^2 + 4x + 4$$
$$3x^2 - 6x + 5 = 2x^2 + 6x + 5$$
$$x^2 - 12x = 0$$
$$x.(x - 12) = 0$$

Thus, x = 0 or x = 12 are the solutions.

Since $c \in Z^+$, c cannot be 0 $(c \neq 0)$, so we conclude that $c = 12$. Therefore $a = 10, b = 11, d = 12, e = 13$. Substituting these values into the equation $a^2 + b^2 + c^2 = d^2 + e^2$ we obtain: $10^2 + 11^2 + 12^2 = 13^2 + 14^2$.

Let us examine Figure 3 for the visual geometric proof referenced in the literature. The proof by Michael Boardman, which decomposes $10^2 + 11^2 + 12^2 = 13^2 + 14^2$ into squares, is illustrated in Figure 3:

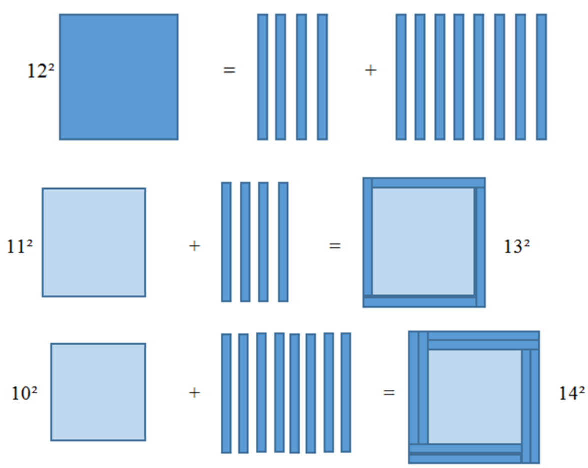

FIGURE 3. Boardman's dissection of squares for $10^2 + 11^2 + 12^2 = 13^2 + 14^2$ [5]

The geometric proof in Figure 3 demonstrates that if we divide a square with side length 12 into 12 parts and distribute them evenly along the edges of the 11-unit and 10-unit squares, we can transform these squares into 13-unit and 14-unit squares. This confirms the equation $10^2 + 11^2 + 12^2 = 13^2 + 14^2$ [5]. From this, it can be inferred that the middle number, 12, plays a crucial role in the geometric proof of the equation.

Let us now find the consecutive numbers that satisfy the seven-variable equation $a^2 + b^2 + c^2 + d^2 = e^2 + f^2 + g^2$. Refer to Figure 4 for the geometric proof found in the literature. The proof by Michael Boardman, which divides the equation $21^2+22^2+23^2+24^2 = 25^2+26^2+27^2$ into squares, is illustrated in Figure 4:

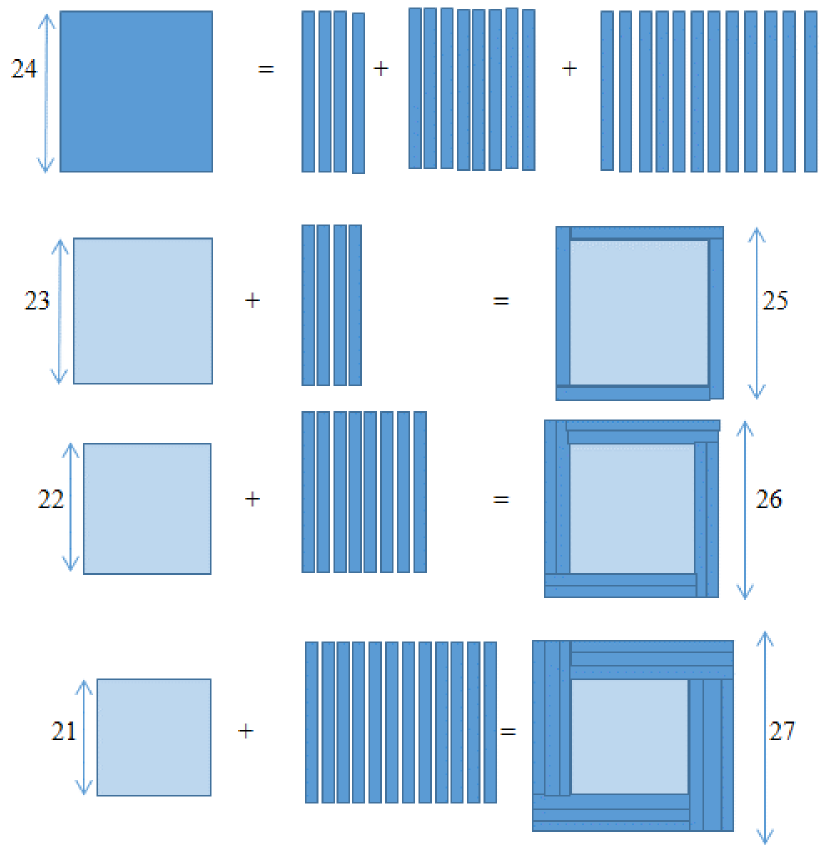

FIGURE 4. Boardman's dissection of squares for $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$ [6]

In Figure 4, a 24-unit square is divided into 24 equal parts, which are subsequently grouped as $24 = 4 + 8 + 12$. By arranging these groups neatly along the edges of the squares with side lengths of 23,22 and 21 units, respectively, we can transform them into squares with side lengths of 25, 26 and 27 units [6]. The geometric proof presented in Figure 4 illustrates that the middle number,24, is essential in establishing the equation $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$.

At this juncture, let us enumerate the equations for the sum of consecutive squares that we have examined in detail, involving three, five, and seven variable:

$$3^2 + 4^2 = 5^2$$

$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

Boardman termed these identities Pythagorean runs, because they involve consecutive positive integers, just like $3^2 + 4^2 = 5^2$, the simplest of the Pythagorean triples [5, p.21] .

If $T_n = 1 + 2 + ... + n, (4T_n - n)^2 + ... + (4T_n)^2 = (4T_n + 1)^2 + ... + (4T_n + n)^2$ [6].

Thanks to Boardman, we can now readily construct any equation that satisfies the conditions we are seeking:

For $n = 3$, with $T_3 = 6$, we have $(4.6 - 3)^2 + ... + (4.6)^2 = (4.6 + 1)^2 + ... + (4.6 + 3)^2$.

This results in the equation: $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$

For $n = 4$, with $T_4 = 10$, we have $36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$.

For $n = 5$, with $T_5 = 15$, we have $55^2+56^2+57^2+58^2+59^2+60^2 = 61^2+62^2+63^2+64^2+65^2$.

Boardman's research demostrates that if we establish the appropriate equations for each $n \geq 1$ according to the equality $(4T_n - n)^2 + ... + (4T_n)^2 = (4T_n + 1)^2 + ... + (4T_n + n)^2$, where $T_n = 1 + 2 + ... + n$, there exists a specific rhythmic arrangement that is evident not only horizontally but also vertically.

Refer to Figure 5 for the Pythagorean runs composed of consecutive squares:

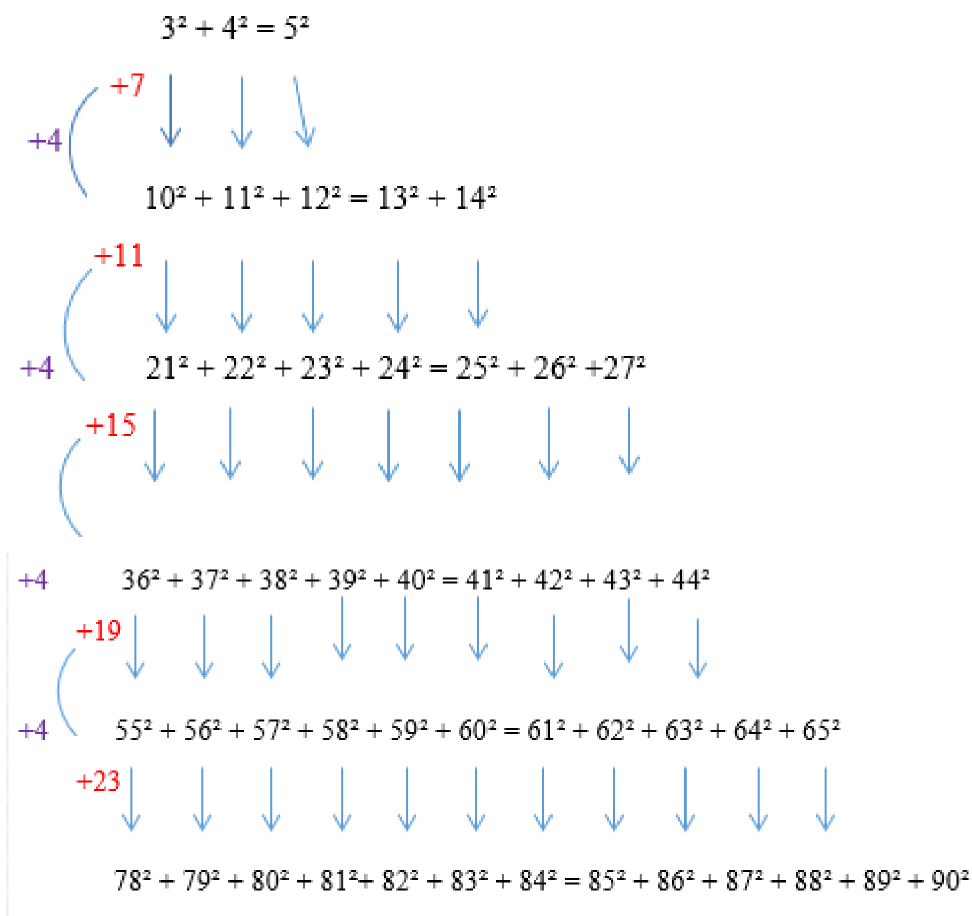

FIGURE 5. Pythagorean Runs of Consecutive Numbers

These Pythagorean sequences, with their astonishing and captivating mathematical essence, represent merely the tip of the iceberg. The technique of determining the identities of such equations by consecutive sums of squares was first discovered by Georges Dostor. [7, p. 44]. "Are there other relationships hidden within Pythagorean runs waiting to be uncovered?" has engaged mathematicians for quite some time and continues to be a pertinent question among them. The findings, proofs, and relationships we have compiled thus far are of considerable significance to our research, guiding us to explore how these non-coincidental equations can be solved with greater ease, under what conditions they hold validity, and concentrating on their mathematical properties to unveil the profound mathematical structure inherent in these equations.

2.3. **Figured Numbers Hidden in Consecutive Pythagorean Runs.**

Upon a meticulous examination of the equations presented in Figure 5, it becomes evident that there exist certain critical prerequisites within the unique solutions that facilitate the indefinite continuation of the equations found in the Pythagorean runs of consecutive squares.

Condition 2.3.1: The initial terms in all Pythagorean runs are triangular numbers.

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$
$$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$$
$$105^2+106^2+107^2+108^2+109^2+110^2+111^2+112^2 = 113^2+114^2+115^2+116^2+117^2+118^2+119^2$$

The numbers $3, 10, 21, 36, 55, 78, 105$ are known as triangular numbers. Consequently, it has been found that triangular numbers are embedded within the solution sets of the equations analyzed in our research.

These numbers are termed triangular numbers because they can be represented by arranging equal-diameter spheres in the shape of an equilateral triangle. Each triangular number is generated by adding an additional row to the preceding triangular number, meaning that successive row contains one more unit than the previous one. Therefore, when a series of equal-diameter spheres is organized in the configuration of an equilateral triangle, triangular numbers are produced. [8, p. 9]. To further elucidate triangular numbers, let us consider Figure 6:

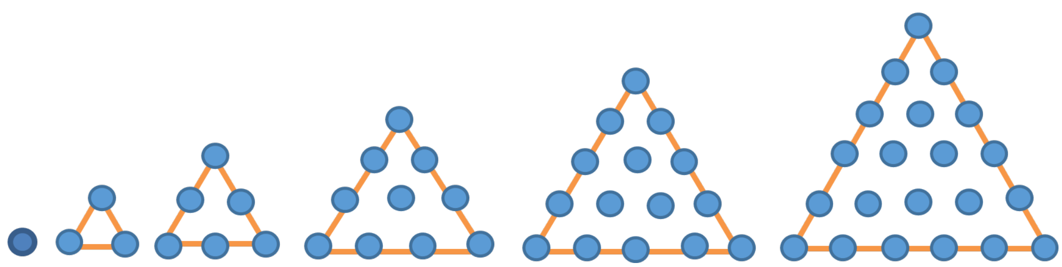

FIGURE 6. The representation of triangular numbers in an equilateral triangle figure

Starting from a point, add to it two points, so that to obtain an equilateral triangle. Six-points equilateral triangle can be obtained from three-points triangle by adding to it three points; adding to it four points gives ten-points triangle, etc. So, by adding to a point two, three, four etc. points, then organizing the points in the form of an equilateral triangle and counting the number of points in each such triangle, one can obtain the numbers1,3,6,10,15,21,28,36,45,55 which are called triangular number [9, p. 58]. Based on this, we can extend the sequence of triangular numbers indefinitely as follows: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 405, 435, and so on.

If we closely examine the initial terms of the Pythagorean runs, we observe the following equations:

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$
$$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$$

From the triangular numbers, a discernible pattern emerges, wherein the numbers 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, and so forth alternate between being present and absent. Specifically, among the triangular numbers, we find the following sequence: 1 is absent,3 is present, 6 is absent, 10 is present, 15 is absent, 21 is present, 28 is absent, 36 is present, 45 is absent, 55 is present, 66 is absent, 78 is present, 91 is absent, and so on. This conclusion indicates that within the triangular number sequence 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 405, 435, 465, the numbers highlighted in red are particularly significant. Consequently, these red triangular numbers also constitute the initial terms of consecutive Pythagorean runs. The pattern observed with triangular numbers is indeed intriguing and may lead to valuable mathematical insights.

Condition 2.3.2: In the Pythagorean runs, the initial terms on the right side of the equations are centered square numbers.

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$
$$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$$

Thus, within the solution sets of the Pythagorean runs we have examined, it has been revealed that the centered square numbers, which can be arranged into a centered square configuration, are present.

Centered Square Number is a centered polygonal number consisting of a central dot with four dots around it, and then additional dots in the gaps between adjacent dots [10]. To enhance our understanding of centered square numbers, let us consider Figure 7:

If we enumerate the centered square numbers, 1, 5, 13, 25, 41, 61, 85, 113, 145, 181, 221, 265, 313, 365, 421, 481, 545, 613, 685, 761, 841, 925... [11].

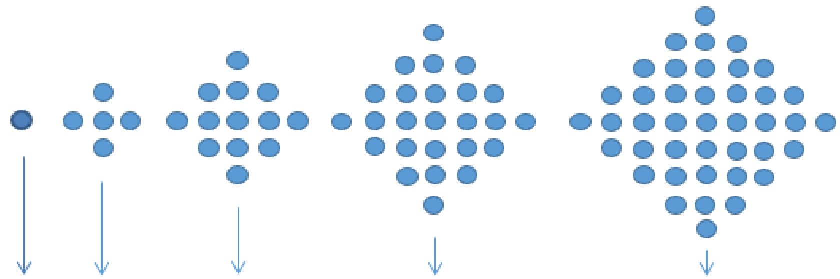

FIGURE 7. Arranging centered square numbers in a centered square figure

Centered square numbers are the sum of two consecutive square numbers and are congruent to 1 (mod 4) and the general term is $[n^2 + (n+1)^2]$ [10].

Examples of obtaining some centered square numbers include:

For $n = 1, 1^2 + 2^2 = 5$

For $n = 2, 2^2 + 3^2 = 13$

For $n = 3, 3^2 + 4^2 = 25$

For $n = 4, 4^2 + 5^2 = 41$

For $n = 5, 5^2 + 6^2 = 61$

As can be observed; all centered square numbers, with the exception of 1, constitute the first terms on the right side of the equations in the Pythagorean runs, respectively. However, some triangular numbers are included in the Pythagorean runs.

Building upon this, the original Proposition 2.3.1 is developed. Figure 8 was discovered as the original non-verbal proof to validate this proposition. In Figure 8, the objective is to derive the triangular numbers relevant to the Pythagorean runs from the figures of centered square numbers.

Proposition 2.3.1: In the equations of Pythagorean runs, for $\forall\, n \in N^+$, the first term on the right side of the equation is a centered square number, expressible as $[n^2+(n+1)^2]$. The triangular number derived from these square numbers, which can be represented using the formula $[(n+n+1).n]$, yields the first term on the left side of the equation.

Proof 2.3.1: The innovative approach we seek to explore involves deriving the triangular configuration illustrated in Figure 6 from the centered square numbers depicted in Figure 7, utilizing the geometric proof method. To facilitate comprehension, let us examine Figure 8:

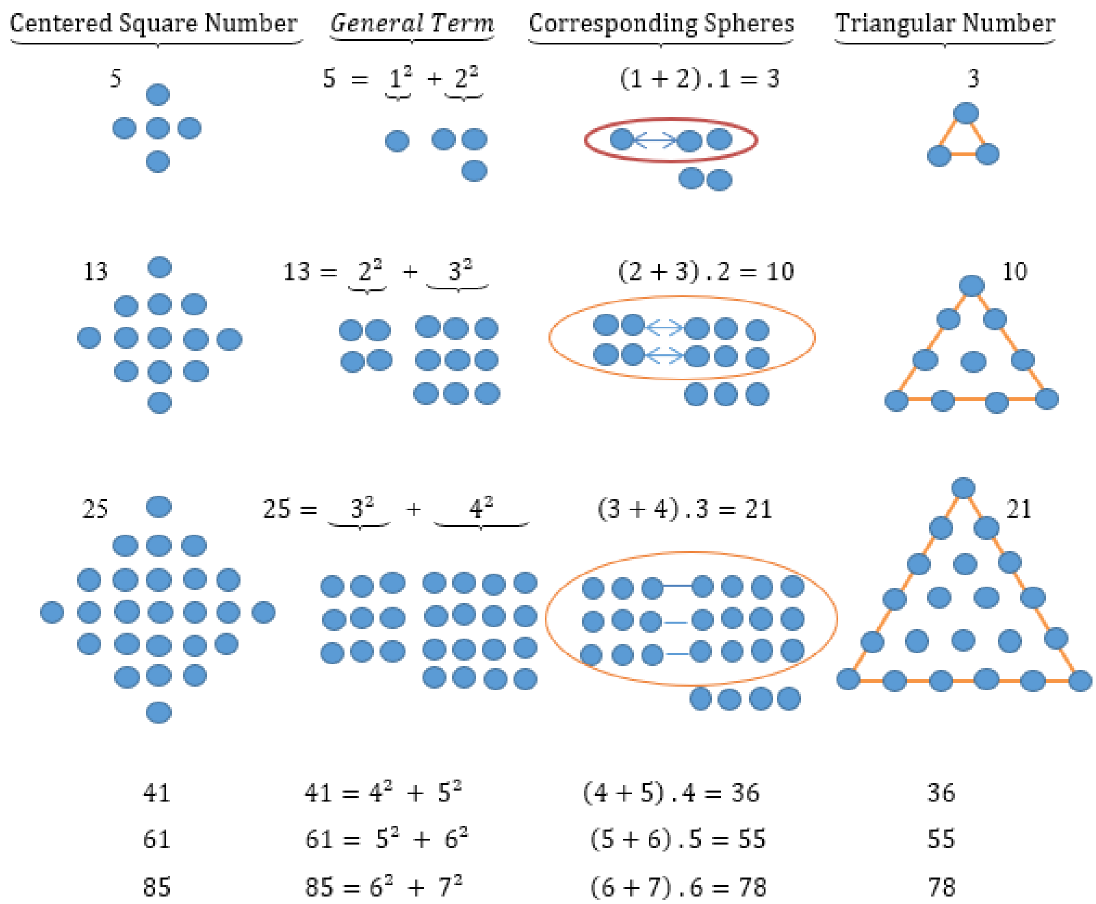

FIGURE 8. Geometric proof. Derivation of triangular numbers belonging to Pythagorean runs from centered square numbers

The original geometric proof presented in Figure 8 illustrates that certain triangular numbers can be derived from all centered square numbers expressible by the general term $[n^2 + (n+1)^2]$ within Pythagorean runs. Furthermore, these triangular numbers can be represented by the specific term $[(n + n + 1).n]$. Rearranging the equations yields;

For the centered square numbers, $n^2 + (n+1)^2 = 2n^2 + 2n + 1$,

For the triangular numbers, $(n + n + 1).n = 2n^2 + n$.

Condition 2.3.3: In equations associated with Pythagorean runs, for $\forall\ n \in N^+$, the first term on the left side of the equation is a triangular number expressible as $(2n^2 + n)$, while the first term on the right side is a centered square number expressible as $(2n^2 + 2n + 1)$.

$$3^2 + 4^2 = 5^2$$
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$
$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$
$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$
$$55^2 + 56^2 + 57^2 + 58^2 + 59^2 + 60^2 = 61^2 + 62^2 + 63^2 + 64^2 + 65^2$$
$$78^2 + 79^2 + 80^2 + 81^2 + 82^2 + 83^2 + 84^2 = 85^2 + 86^2 + 87^2 + 88^2 + 89^2 + 90^2$$

Conditions 2.3.1, 2.3.2, and 2.3.3, along with the original geometric proof presented in Figure 8, collectively provide highly significant validations for Pythagorean runs that can be extended indefinitely. No studies have been identified in the literature that establish a connection between the Pythagorean runs and centered square numbers or triangular numbers, highlighting the originality of this research. As a result of the investigation, a novel model has been developed encompassing all centered square numbers and triangular numbers hidden within the Pythagorean runs. Proposition 2.3.2, an original proposition pertaining to Pythagorean runs, was formulated using Figure 8 and Condition 2.3.3, and was subsequently proven through the method of induction, which is one of the recognized mathematical proof techniques.

Proposition 2.3.2: The Pythagorean runs, for $\forall\ n \in N^+$, are represented by consecutive terms expressed in the form $(2n^2+n)^2+ ... = (2n^2+2n+1)^2+ ...$, where $(2n^2+2n+1)$ denotes a centered square number and $(2n^2+n)$ signifies a triangular number. The total number of consecutive terms in the equation is $(2n+1)$, with the left side containing one additional term compared to the right side.

Proof 2.3.2: For $n = 1$, the first equation S_1 is : $(2.1^2+1)^2+ ... = (2.1^2+2.1+1)^2+ ...$ and the total number of terms is $2.1+1 = 3$. Therefore, the three-term equation with consecutive terms is $3^2+4^2 = 5^2$.

For $n = 2$, the second equation S_2 is : $(2.2^2+2)^2+ ... = (2.2^2+2.2+1)^2+ ...$ and the total number of terms is $2.2+1 = 5$.

$$10^2+ ... = 13^2+ ...$$

The equation $10^2+11^2+12^2 = 13^2+14^2$ consists of five consecutive terms.

Let the k-th equation S_k be true for $n = k$.

Assume S_k has consecutive terms in the form $(2k^2+k)^2+ ... = (2k^2+2k+1)^2+ ...$ and let the total number of consecutive terms in the equation be $(2k+1)$ with the left side containing one additional term compared to the right side. Thus, we can express S_k as follows:

$S_k : (2k^2+k)^2+ ... = (2k^2+2k+1)^2+ ...$

To explicitly illustrate the consecutive terms of the proposition S_k , we have:

$S_k : (2k^2+k)^2+(2k^2+k+1)^2+(2k^2+k+2)^2+ ... +(2k^2+2k)^2 = (2k^2+2k+1)^2+(2k^2+2k+2)^2+ ... +(2k^2+3k)^2$

To express the sum of the consecutive terms on both sides of this equation using the summation symbol Σ we introduce the variable z:

Thus, S_k can be rewritten as:

$$S_k : \Sigma_{z=0}^{k}(2k^2+k+z)^2 = \Sigma_{z=1}^{k}(2k^2+2k+z)^2$$
$$S_k : \Sigma_{z=0}^{k}(2k^2+k+z)^2 = \Sigma_{z=1}^{k}(2k^2+2k+z)^2$$
$$(2k^2+k)^2+\Sigma_{z=1}^{k}(2k^2+2k+z)^2 = \Sigma_{z=1}^{k}(2k^2+k+z)^2$$
$$(2k^2+k)^2 = \Sigma_{z=1}^{k}(2k^2+2k+z)^2-\Sigma_{z=1}^{k}(2k^2+k+z)^2$$

To utilize the difference of squares identity $a^2-b^2 = (a-b).(a+b)$ we will arrange similar terms side by side as follows:

$$(2k^2+k)^2 = \underbrace{(2k^2+2k+1)^2 - (2k^2+k+1)^2}_{for\ z=1} + \underbrace{(2k^2+2k+2)^2 - (2k^2+k+2)^2}_{for\ z=2} + \ldots$$

$$\ldots + \underbrace{(2k^2+3k)^2 - (2k^2+2k)^2}_{for\ z=k}.$$

Utilizing the difference of squares identity $a^2 - b^2 = (a-b).(a+b)$, we can derive the following:

$$(2k^2+k)^2 = k.(4k^2+3k+2) + k.(4k^2+3k+4) + \ldots + k.(4k^2+3k+2k)$$

$$(2k^2+k)^2 = k.[4k^2+3k+2 + 4k^2+3k+4 + \ldots + 4k^2+3k+2k]$$

$$(2k^2+k)^2 = k.[\underbrace{4k^2 + \ldots + 4k^2}_{\text{There are } k \text{ terms}} + \underbrace{3k + \cdots + 3k}_{\text{There are } k \text{ terms}} + \underbrace{2 + 4 + \ldots + 2k}_{\text{There are } k \text{ terms}}]$$

$$(2k^2+k)^2 = k.[k.4k^2 + k.3k + \underbrace{2 + 4 + \ldots + 2k}_{\sum_{z=1}^{k} 2z}]$$

$$k^2.(2k+1)^2 = k.[4k^3 + 3k^2 + \sum_{z=1}^{k} 2z]$$

$$k^2.(2k+1)^2 = k.[4k^3 + 3k^2 + \sum_{z=1}^{k} 2z],\ k \neq 0$$

$$k.(2k+1)^2 = 4k^3 + 3k^2 + \sum_{z=1}^{k} 2z$$

$$k.(4k^2+4k+1) = 4k^3 + 3k^2 + \sum_{z=1}^{k} 2z$$

$$4k^3 + 4k^2 + k = 4k^3 + 3k^2 + \sum_{z=1}^{k} 2z$$

$$4k^3 + 4k^2 + k = 4k^3 + 3k^2 + \sum_{z=1}^{k} 2z$$

$$4k^2 + k = 3k^2 + \sum_{z=1}^{k} 2z$$

$$4k^2 + k - 3k^2 = \sum_{z=1}^{k} 2z$$

The equality $k^2 + k = \Sigma_{z=1}^{k} 2z$ ...(1) is established. This equality holds true and is a consequence of the equation accepted as valid for $n = k$.

Now, we need to verify whether the proposition S_{k+1} for $n = k+1$ is correct. Specifically, we examine

$S_{k+1} : \Sigma_{z=0}^{k+1}(2(k+1)^2 + k + 1 + z)^2 = \Sigma_{z=1}^{k+1}(2(k+1)^2 + k + 1 + z)^2$, Is this statement valid?

$$\underbrace{[2(k+1)^2 + k + 1]^2}_{for\ z=0} + \sum_{z=1}^{k+1}(2(k+1)^2 + k + 1 + z)^2 = \sum_{z=1}^{k+1}(2(k+1)^2 + 2(k+1) + z)^2$$

$$[2(k+1)^2+k+1]^2] = \Sigma_{z=1}^{k+1}(2(k+1)^2+2(k+1)+z)^2 - \Sigma_{z=1}^{k+1}(2(k+1)^2+k+1+z)^2$$

Utilizing the difference of squares identity $a^2 - b^2 = (a-b).(a+b)$, we can derive the following:

$$[(k+1).(2(k+1)+1)]^2 = \underbrace{[2(k+1)^2 + 2(k+1)+1]^2 - [2(k+1)^2 + (k+1)+1]^2}_{for\ z=1} +$$

$$+\underbrace{[2(k+1)^2 + 2(k+1)+2]^2 - [2(k+1)^2 + (k+1)+2]^2}_{for\ z=2} + \dots$$

$$\dots + \underbrace{[2(k+1)^2 + 3(k+1)]^2 - [2(k+1)^2 + 2(k+1)]^2}_{for\ z=k+1}$$

$$(k+1)^2.[(2(k+1)+1)]^2 = (k+1).[4(k+1)^2 + 3(k+1) + 2 + 4(k+1)^2 + 3(k+1) + 4 + \dots$$

$$\dots + 4(k+1)^2 + 3(k+1) + 2(k+1)]$$

$$(k+1)^2.(2k+3)^2 = (k+1).[(k+1).4(k+1)^2 + (k+1).3(k+1) + \underbrace{2+4+\cdots+2k}_{\sum_{z=1}^{k} 2z} + 2(k+1)]$$

$$\sum_{z=1}^{k} 2z = k^2 + k \quad \dots [1]$$

$$\underbrace{k^2 + k + 2(k+1)}$$

$$k(k+1) + 2(k+1)$$

$$(k+1).(k+2)$$

$$(k+1)^2.(2k+3)^2 = (k+1).[(k+1).4(k+1)^2 + (k+1).3(k+1) + (k+1).(k+2)]$$

$$(k+1)^2.(2k+3)^2 = (k+1).(k+1).[4(k+1)^2 + 3(k+1) + k + 2]$$

$$(k+1)^2.(2k+3)^2 = (k+1)^2.[4(k+1)^2 + 3(k+1) + k + 2]$$

$$\cancel{(k+1)^2}.(2k+3)^2 = \cancel{(k+1)^2}.[4(k+1)^2 + 3(k+1) + k + 2],\ k \neq -1$$

$$(2k+3)^2 = 4(k+1)^2 + 3(k+1) + k + 2$$

$$(2k+3)^2 = 4(k^2+2k+1) + 3(k+1) + k + 2$$

$$(2k+3)^2 = 4k^2 + 8k + 4 + 3k + 3 + k + 2$$

$$(2k+3)^2 = 4k^2 + 12k + 9$$

As seen, according to the method of induction, the result (1) obtained from the assumption S_k, which is accepted to be true for $n = k$, has been substituted into the equation for $n = k+1$ to prove that the proposition S_{k+1} is also valid. In all the equations related to the Pythagorean runs, for $\forall\ n \in N^+$, the first term on the left side of the equation is a triangular number that can be expressed as $(2n^2 + n)$, and the first term on the right side of the equation is a centered square number expressible as $(2n^2 + 2n + 1)$. The relationship between the centered square numbers and triangular numbers, which are embedded in the

Pythagorean runs, was first discovered in this study and was formulated in Proposition 2.3.1, with a visual proof provided in Figure 8. Building on this, Proposition 2.3.2, developed to further address such equations, was proven using the method of mathematical induction.

Table 2.1 demonstrates the mathematical relationship between the triangular numbers, which appear as the first terms on the left side of the Pythagorean sequence equations, and the centered square numbers, which serve as the first terms on the right side, for the first ten natural numbers

n. Pythagorean runs	$2n^2+n$, Triangular number	$2n^2+2n+1$, Centered square number	$2n+1$, Number of consecutive terms	Equations related to Pythagorean sequences
1	3	5	3	$3^2+4^2=5^2$
2	10	13	5	$10^2+11^2+12^2=13^2+14^2$
3	21	25	7	$21^2+22^2+23^2+24^2=25^2+26^2+27^2$
4	36	41	9	$36^2+37^2+38^2+39^2+40^2=41^2+42^2+43^2+44^2$
5	55	61	11	$55^2+56^2+\cdots+60^2=61^2+\cdots+65^2$
6	78	85	13	$78^2+79^2+\cdots+84^2=85^2+\cdots+90^2$
7	105	113	15	$105^2+106^2+\cdots+112^2=113^2+\cdots+119^2$
8	136	145	17	$136^2+137^2+\cdots+144^2=145^2+\cdots+152^2$
9	171	181	19	$171^2+172^2+\cdots+180^2=181^2+\cdots+189^2$
10	210	221	21	$210^2+211^2+\cdots+220^2=221^2+\cdots+230^2$

TABLE 2.1. Pythagorean runs and related equations.

3. Results and Discussion

As a result of the research, a creative and original Proposition 2.3.1, establishing a significant connection between figurate numbers specifically triangular numbers and centered square numbers and Pythagorean sequences, was developed and proven through the visual proof in Figure 8. Building upon this original visual proof, Proposition 2.3.2 was formulated and subsequently proven using the method of mathematical induction.

4. Conclusion and Suggestions

By investigating the applications of Pythagorean sequences in fields such as physics, engineering, computer science, and others, further research could explore the contributions and practical applications that the developed proposition might offer in areas like wave theory, optics, sound analysis, or artificial intelligence. In this way, it may become possible to uncover real-world applications of the Pythagorean Theorem and similar mathematical equalities.

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