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WARPED PRODUCT POINTWISE SEMI-SLANT SUBMANIFOLDS OF NEARLY PARA-KAEHLER MANIFOLDS

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Abstract. In this article, firstly we introduce pointwise slant and pointwise semi-slant submanifolds in nearly para-Kaehler manifolds. We demonstrate that there exist pointwise semi-slant non-trivial warped product submanifold $\mathcal{M}^T \times_k \mathcal{M}^\theta$ in nearly para-Kaehler manifolds by giving an example. We get a characterization, give certain theorems depending on the casual characters and we reach an optimal inequality.

Keywords: Nearly para-Kaehler manifold, pointwise semi-slant submanifold, warped product submanifold
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1. Introduction

Pointwise slant submanifolds were first introduced by F. Etayo in [11] as quasi-slant submanifolds. Such submanifolds have been studied extensively by B.-Y. Chen and O.J. Garay [10]. Then P.Alegre and A.Carriazo studied slant submanifolds in para-Hermitian manifolds and detailed definitions of types of submanifolds in semi-Riemannian setting were provided by them [3, 4].

Warped products emerged in the mathematical and physical subjects before 1969, for example, semi-reducible space, which is utilized for the warped product by Kruchkovich in 1957 [19]. It has been succesfully used in the general theory of relativity, string theory and black holes. On the other hand, warped product manifolds was introduced and studied by R.L. Bishop and B. O'Neill [9]. Later, many authors researched on warped product and submanifolds [1, 2, 5, 7, 8, 12, 14, 15, 20]

B. Sahin studied warped product pointwise semi-slant submanifolds in Kaehler manifolds [23]. He researched that there exist of the second form $\mathcal{M}^T \times_k \mathcal{M}^\theta$ in Kaehler manifold $\bar{\mathcal{M}}$. Also he found a characterization, theorem, interesting results, inequality and he obtained examples of such submanifolds. Later, S. Ayaz and Y. Gündüzalp studied warped product pointwise hemi-slant submanifolds whose ambient spaces are nearly para-Kaehler manifolds [6].

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Nearly Kaehler manifolds were studied by Tachibana in [25]. For example, S^6 (six dimensional sphere) is a example of nearly Kaehler non-Kaehler manifold.

Every nearly para-Kaehler manifold isn't a para-Kaehler. But Every para-Kaehler manifold is a nearly para-Kaehler [24]. So, we give some examples for both nearly para-Kaehler and para-Kaehler manifold and we research pointwise semi-slant warped product submanifolds in nearly para-Kaehler manifolds in this paper.

This article is organized as follows. In section 2, we recall some fundamental notins for the paper. In section 3, we introduce pointwise semi-slant submanifolds of nearly para-Kaehler manifold and give some examples. In section 4, we introduce pointwise semi-slant non-trivial warped product submanifolds in nearly para-Kaehler manifold. We also provide an example. In section 5, we obtain an inequality in terms of the second fundamental form.

2. Preliminaries

Let $(\bar{\mathcal{M}}, \breve{g})$ be a $\mathtt{2n}$-dimensional semi-Riemannian manifold. If there is a tensor field $\mathcal{P}$ of type $(\mathtt{1},\mathtt{1})$ on $\bar{\mathcal{M}}$, such that

$$\breve{g}(\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{Y}_b) = -\breve{g}(\mathcal{X}_a, \mathcal{Y}_b), \quad \mathcal{P}^2\mathcal{X}_a = \mathcal{X}_a \tag{2.1}$$

for any vector fields $\mathcal{X}_a, \mathcal{Y}_b$ on $\bar{\mathcal{M}}$, it is said a para-Hermitian manifold. In addition, it is called to be para-Kaehler manifold, if it satisfies $\bar{\nabla}\mathcal{P} = \mathtt{0}$ identically [17].

Let $\mathcal{T}\bar{\mathcal{M}}$ be the tangent bundle of $\bar{\mathcal{M}}$ and $\bar{\nabla}$, the covariant differential operator on $\bar{\mathcal{M}}$ with respect to $\breve{g}$. If

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{X}_a = 0 \tag{2.2}$$

for any $\mathcal{T}\bar{\mathcal{M}}$, then an almost para Hermitian manifold is called nearly para-Kaehler structure. Equation (2) is equivalent to

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b + (\bar{\nabla}_{\mathcal{Y}_b}\mathcal{P})\mathcal{X}_a = 0 \tag{2.3}$$

for any vector fields $\mathcal{X}_a, \mathcal{Y}_b$ on $\bar{\mathcal{M}}$

Let $\mathcal{M}$ be a submanifold of $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$. The Gauss and Weingarten equations are

$$\bar{\nabla}_{\mathcal{X}_\mathtt{a}}\mathcal{Y}_\mathtt{b} = \nabla_{\mathcal{X}_\mathtt{a}}\mathcal{Y}_\mathtt{b} + \breve{h}(\mathcal{X}_\mathtt{a}, \mathcal{Y}_\mathtt{b}), \tag{2.4}$$

$$\bar{\nabla}_{\mathcal{X}_\mathtt{a}}\mathcal{V}_c = -A_{\mathcal{V}_c}\mathcal{X}_\mathtt{a} + \nabla^{\perp}_{\mathcal{X}_\mathtt{a}}\mathcal{V}_c, \tag{2.5}$$

for $\mathcal{X}_\mathtt{a}, \mathcal{Y}_\mathtt{b} \in \Gamma(\mathcal{T}\mathcal{M})$ and $\mathcal{V}_c \in \Gamma(\mathcal{T}\mathcal{M}^{\perp})$, that $\breve{h}$ is the second fundamental form of $\mathcal{M}$, $A_{\mathcal{V}_c}$ is the Weingarten endomorphism with $\mathcal{V}_c$ and $\nabla^{\perp}$ is the normal connection. $A_{\mathcal{V}_c}$ and $\breve{h}$ are related by

$$\breve{g}(A_{\mathcal{V}_c}\mathcal{X}_a, \mathcal{Y}_b) = \breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{V}_c), \tag{2.6}$$

here $\breve{g}$ states the semi-Riemannian metric on $\mathcal{M}$. For any tangent vector field $\mathcal{X}_a$, we denote

$$\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a + S\mathcal{X}_a, \tag{2.7}$$

that $R\mathcal{X}_a$ is the tangential part of $\mathcal{P}\mathcal{X}_a$ and $S\mathcal{X}_a$ is the normal part.
For any normal vector field $\mathcal{V}_c$,

$$\mathcal{P}\mathcal{V}_c = r\mathcal{V}_c + s\mathcal{V}_c, \tag{2.8}$$

that $r\mathcal{V}_c$ and $s\mathcal{V}_c$ are the tangential and normal vectors of $\mathcal{P}\mathcal{V}_c$.

Now, denote by $\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b$ and $\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b$ the tangential and normal parts of $(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b$, i.e.,

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b = \mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, \tag{2.9}$$

for any $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{TM}_x)$. Using of (2.7), (2.8), (2.9) and the Weingarten and Gauss formulas, we obtain

$$\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b = (\bar{\nabla}_{\mathcal{X}_a}\mathcal{R})\mathcal{Y}_b - \mathcal{A}_{\mathcal{S}\mathcal{Y}_b}\mathcal{X}_a - r\breve{h}(\mathcal{X}_a, \mathcal{Y}_b) \tag{2.10}$$

and

$$\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b = (\bar{\nabla}_{\mathcal{X}_a}\mathcal{S})\mathcal{Y}_b - \breve{h}(\mathcal{X}_a, \mathcal{R}\mathcal{Y}_b) - s\breve{h}(\mathcal{X}_a, \mathcal{Y}_b). \tag{2.11}$$

Similarly, for any $\mathcal{V}_c \in \mathcal{T}^{\perp}\mathcal{M}$, denote the tangential and normal parts of $(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b$ by $\mathcal{G}_{\mathcal{V}_c}\mathcal{Y}_b$ and $\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b$ respectively, we get

$$\mathcal{G}_{\mathcal{X}_a}\mathcal{V}_c = (\bar{\nabla}_{\mathcal{X}_a}r)\mathcal{V}_c + \mathcal{R}\mathcal{A}_{\mathcal{V}_c}\mathcal{X}_a - \mathcal{A}_{s\mathcal{V}_c}\mathcal{X}_a \tag{2.12}$$

and

$$\mathcal{U}_{\mathcal{X}_a}\mathcal{V}_c = (\bar{\nabla}_{\mathcal{X}_a}s)\mathcal{V}_c + \breve{h}(r\mathcal{V}_c, \mathcal{X}_a) + \mathcal{S}\mathcal{A}_{\mathcal{V}_c}\mathcal{X}_a \tag{2.13}$$

where the covariant derivative of $\mathcal{R}, \mathcal{S}, r, s$ are defined by

$$(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R})\mathcal{Y}_b = \nabla_{\mathcal{X}_a}\mathcal{R}\mathcal{Y}_b - \mathcal{R}\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \quad (\bar{\nabla}_{\mathcal{X}_a}\mathcal{S})\mathcal{Y}_b = \nabla^{\perp}_{\mathcal{X}_a}\mathcal{S}\mathcal{Y}_b - \mathcal{S}\nabla_{\mathcal{X}_a}\mathcal{Y}_b,$$

$$(\bar{\nabla}_{\mathcal{X}_a}r)\mathcal{V}_c = \nabla_{\mathcal{X}_a}r\mathcal{V}_c - r\nabla^{\perp}_{\mathcal{X}_a}\mathcal{V}_c, \quad (\bar{\nabla}_{\mathcal{X}_a}s)\mathcal{V}_c = \nabla^{\perp}_{\mathcal{X}_a}s\mathcal{V}_c - s\nabla^{\perp}_{\mathcal{X}_a}\mathcal{V}_c$$

for any $\mathcal{X}_a, \mathcal{Y}_b \in \mathcal{TM}$ and $\mathcal{V}_c \in \Gamma(\mathcal{TM}^{\perp})$

For the proporties of $\mathcal{G}$ and $\mathcal{U}$ we refer [18], which we express here for later use.

$$(m_1) \quad (a) \quad \mathcal{G}_{\mathcal{X}_a+\mathcal{Y}_b}\mathcal{W}_c = \mathcal{G}_{\mathcal{X}_a}\mathcal{W}_c + \mathcal{G}_{\mathcal{Y}_b}\mathcal{W}_c$$
$$(b) \quad \mathcal{U}_{\mathcal{X}_a+\mathcal{Y}_b}\mathcal{W}_c = \mathcal{U}_{\mathcal{X}_a}\mathcal{W}_c + \mathcal{U}_{\mathcal{Y}_b}\mathcal{W}_c$$

$$(m_2) \quad (a) \quad \mathcal{G}_{\mathcal{X}_a}(\mathcal{Y}_b + \mathcal{W}_c) = \mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{G}_{\mathcal{X}_a}\mathcal{W}_c$$
$$(b) \quad \mathcal{U}_{\mathcal{X}_a}(\mathcal{Y}_b + \mathcal{W}_c) = \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{W}_c$$

$$(m_3) \quad (a) \quad \breve{g}(\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{W}_c) = -\breve{g}(\mathcal{Y}_b, \mathcal{A}_{\mathcal{X}_a}\mathcal{W}_c)$$
$$(b) \quad \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{V}_f) = -\breve{g}(\mathcal{Y}_b, \mathcal{G}_{\mathcal{X}_a}\mathcal{V}_f)$$

$$(m_4) \quad \mathcal{G}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b = -\mathcal{P}(\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b)$$

for any $\mathcal{X}_a, \mathcal{Y}_b, \mathcal{W}_c \in \Gamma(\mathcal{TM}_x)$ and $V_f \in \Gamma(\mathcal{TM}_b^{\perp})$

On a nearly para-Kaehler manifold $\bar{\mathcal{M}}_x$. by equations (2.2) and (2.9), we get

$$(a)\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{G}_{\mathcal{Y}_b}\mathcal{X}_a = 0 \quad (b)\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b + \mathcal{U}_{\mathcal{Y}_b}\mathcal{X}_a = 0 \tag{2.14}$$

for any $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{TM}_x)$

The mean curvature vector field is defined by

$$H = \frac{1}{n} trace\breve{h}. \tag{2.15}$$

We now introduce the following notions in a nearly para-Kaehler manifolds.

Definition 2.1. *We call that a submanifold $\mathcal{M}$ in a nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$ is pointwise slant, if for all timelike or spacelike tangent vector field $\mathcal{X}_a$, the ratio $\breve{g}(R\mathcal{X}_a, R\mathcal{X}_a)/\breve{g}(\mathcal{P}\mathcal{X}_a, P\mathcal{X}_a)$ is a function. Moreover, a submanifold $\mathcal{M}$ of nearly para-Kaehler manifold $\bar{\mathcal{M}}$ is said pointwise slant* [13], *if at each point $\mathtt{p} \in \mathcal{M}$, the Wirtinger angle $\mathcal{P}X$ between $\theta(X)$ and $\mathcal{T}_p\mathcal{M}$ is dependent of the choice of the non-zero $X \in \mathcal{T}_p\mathcal{M}$. In this instance, the Wirtinger angle causes a real-valued function $\theta : \mathcal{TM} - \{0\} \to \mathtt{R}$ which is said slant function or the Wirtinger function.*

It is easy to see that a pointwise slant submanifold in nearly para-Kaehler manifold is said slant, if its Wirtinger function α is globally constant. Also we notice that all slant submanifolds are pointwise slant submanifolds.

If $\mathcal{M}$ is a para-complex (para-holomorphic) submanifold, in that case, $\mathcal{P}\mathcal{X}_a = R\mathcal{X}_a$ and the above ratio is equal to 1. Moreover if $\mathcal{M}$ is totaly real (anti-invariant), then $R = 0$, so $\mathcal{P}\mathcal{X}_a = S\mathcal{X}_a$ and the above ratio equals 0. Hence, both totally real and para-complex submanifolds are the private situations of pointwise slant submanifolds. Neither totally real nor para-complex pointwise slant submanifold can be called a proper pointwise slant. These manifolds are proper manifolds.

Definition 2.2. *Let $\mathcal{M}$ be a proper pointwise slant submanifold in nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$. We call that it is of*
type-1 if for any space-like or time-like vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is time-like or space-like, and $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} > 1$.
type-2 if for any space-like or time-like vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is time-like or space-like, and $\frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} < 1$.

Similar to the way of P. Alegre and A. Carriazo used [4], the following theorem and results are obtained.

Theorem 2.1. *Let $\mathcal{M}$ be a pointwise slant submanifold in nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$. So,*
***(a)** $\mathcal{M}$ is pointwise slant submanifold of type-1 if and only if for any spacelike or timelike vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike or spacelike, also arise a function $\mu \in (1, +\infty)$. Therefore,*

$$R^2 = \mu Id. \tag{2.16}$$

If θ indicates the slant function of $\mathcal{M}$, $\mu = \cosh^2\theta$.
***(b)** $\mathcal{M}$ is pointwise slant submanifold of type-2 if and only if for any spacelike or timelike vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike or spacelike, also arise a function $\mu \in (0, 1)$. Therefore,*

$$R^2 = \mu Id. \tag{2.17}$$

If θ indicates the slant function of $\mathcal{M}$, $\mu = \cos^2\theta$.

Proof. Firstly, if $\mathcal{M}$ is the pointwise slant submanifold of type-1 for any spacelike tangent vector field $\mathcal{X}_a$, $R\mathcal{X}_a$ is timelike and by the equation of (2.1), $\mathcal{P}\mathcal{X}_a$ is too. Furthermore, they supply $|R\mathcal{X}_a|/|\mathcal{P}\mathcal{X}_a| > 1$. Therefore, it follows that the slant function θ. Because of,

$$\cosh\theta = \frac{|R\mathcal{X}_a|}{|\mathcal{P}\mathcal{X}_a|} = \frac{\sqrt{-\breve{g}(R\mathcal{X}_a, R\mathcal{X}_a))}}{\sqrt{-\breve{g}(\mathcal{P}\mathcal{X}_a, \mathcal{P}\mathcal{X}_a)}}. \tag{2.18}$$

Using (2.1) and (2.18), we have

$$\breve{g}(R^2\mathcal{X}_a, \mathcal{X}_a) = \cosh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{X}_a).$$

Thus, we get $R^2\mathcal{X}_a = \mathcal{X}_a I$. So, from (2.18), we get $\mu = \cosh^2\theta$.

In a similar method for any timelike tangent vector field $\mathcal{Z}$, now, $R\mathcal{Z}$ and $\mathcal{P}\mathcal{Z}$ are spacelike and therefore, instead of (2.18) we get

$$\cosh\theta = \frac{|R\mathcal{Z}|}{|\mathcal{P}\mathcal{Z}|} = \frac{\sqrt{\breve{g}(R\mathcal{Z}, R\mathcal{Z}))}}{\sqrt{\breve{g}(\mathcal{P}\mathcal{Z}, \mathcal{P}\mathcal{Z})}}.$$

Because of $R^2\mathcal{Z} = \mu\mathcal{Z}$, for any spacelike and timelike $\mathcal{Z}$ it further provides for lightlike vector fields and therefore we get $R^2 = \mu Id$. The converse is (a) direct calculation.
Similarly, we have (b).

Finally, for both pointwise slant submanifolds of type-1 and type-2, if $\mathcal{X}_a$ is space-like, in that case, $\mathcal{P}\mathcal{X}_a$ is timelike. Thus, all pointwise slant submanifold of type-1 and type-2 should be a neutral semi-Riemann manifold.

□

Using (2.1),(2.7) and Theorem 2.1, we obtain the following corollary.

Corollary 2.4. Let $\mathcal{M}$ be a pointwise slant submanifold of a nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$ with the slant function θ. For any non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{T}\mathcal{M})$, we obtain:
If $\mathcal{M}$ is of type-1, then

$$\breve{g}(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cosh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b), \;\; \breve{g}(S\mathcal{X}_a, S\mathcal{Y}_b) = \sinh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b). \tag{2.19}$$

If $\mathcal{M}$ is of type-2, then

$$\breve{g}(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cos^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b), \;\; \breve{g}(S\mathcal{X}_a, S\mathcal{Y}_b) = -\sin^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b). \tag{2.20}$$

Using (2.1),(2.7),(2.8) and Theorem 2.1, we get the following corollary.

Corollary 2.1. *Let $\mathcal{M}$ be a pointwise slant submanifold in nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$. $\mathcal{M}$ is a pointwise slant submanifold of*
**type-1 if and only if $rS\mathcal{X}_a = -\sinh^2\theta\mathcal{X}_a$ and $SR\mathcal{X}_a = -sS\mathcal{X}_a$ for all timelike (spacelike) vector field $\mathcal{X}_a$.*
**type-2 if and only if $rS\mathcal{X}_a = \sin^2\theta\mathcal{X}_a$ and $SR\mathcal{X}_a = -sS\mathcal{X}_a$ for all timelike (spacelike) vector field $\mathcal{X}_a$.*

3. Pointwise Semi-Slant Submanifolds in Nearly Para-Kaehler Manifolds

In this section, we introduce and study pointwise semi-slant submanifolds in nearly para-Kaehler manifold. Also we give some examples.

Definition 3.1. *A semi-Riemannian submanifold $\mathcal{M}$ of a nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$ is called pointwise semi-slant submanifold, if there are two orthogonal distributions D^T , D^θ on $\mathcal{M}$ at the point $q \in \mathcal{M}$ such that the following conditions are satisfied.*
1) $\mathcal{TM} = \mathcal{D}^T \oplus \mathcal{D}^\theta$;
2) $\mathcal{D}^T$ is an invariant (para-holomorphic) distribution, $\mathcal{P}\mathcal{D}^T = \mathcal{D}^T$;
3) $\mathcal{D}^\theta$ is a pointwise slant distribution.

Then, we say the θ is the semi-slant function with the pointwise slant distribution $\mathcal{D}^\theta$. The invariant distribution $\mathcal{D}^T$ of a pointwise semi-slant submanifold is a pointwise slant distribution with slant function $\theta = 0$.
In the above definition, if we suppose that the dimensions $a = dim\mathcal{D}^T$ and $b = dim\mathcal{D}^\theta$, then we get
*) If $a = 0$ and θ is globally constant, $\mathcal{M}$ is a slant submanifold.
*) If $a = 0$, $\mathcal{M}$ is a pointwise slant submanifold.
*) If $b = 0$, $\mathcal{M}$ is an invariant submanifold.
*) If $a = 0$ and $\theta = \frac{\pi}{2}$, $\mathcal{M}$ is an anti-invariant submanifold.
*) If $a \neq 0$ and θ is constant on $\mathcal{M}$, $\mathcal{M}$ is a semi-slant submanifold.
*) If $a \neq 0$, $b \neq 0$ and $\theta = \frac{\pi}{2}$, $\mathcal{M}$ is a semi invariant submanifold.
A pointwise semi-slant submanifold $\mathcal{M}$ is called proper if its semi-slant function satisfies $\theta \neq 0, \frac{\pi}{2}$, also θ is nonconstant on $\mathcal{M}$.

Remark 3.1. *Pointwise slant submanifold is a generalization of slant submanifold.*

Using (1),(5),(6), Theorem 2.1 and Remark 3.1, we have the following result.

Corollary 3.1. *Let $\mathcal{M}$ be a pointwise semi-slant submanifold in nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$ with semi-slant function θ. Then, for any non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{D}^\theta)$, we obtain*
If $\mathcal{M}$ is of type-1, then

$$\breve{g}(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cosh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b), \ \ \breve{g}(S\mathcal{X}_a, S\mathcal{Y}_b) = \sinh^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b). \tag{3.21}$$

If $\mathcal{M}$ is of type-2, then

$$\breve{g}(R\mathcal{X}_a, R\mathcal{Y}_b) = -\cos^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b), \ \breve{g}(S\mathcal{X}_a, S\mathcal{Y}_b) = -\sin^2\theta\breve{g}(\mathcal{X}_a, \mathcal{Y}_b). \tag{3.22}$$

Now, we give two lemmas for using next section.

Lemma 3.1. *Let $\mathcal{M}$ be a proper pointwise semi-slant type-1-2 submanifold whose ambient spaces are nearly para-Kaehler manifold $(\bar{\mathcal{M}}, \mathcal{P}, \breve{g})$. $\mathcal{D}^\theta$ is slant distribution and $(\mathcal{D}^{\mathrm{T}})$ is holomorphic distribution. Then we get*
1) (for type-1)

$$\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{Z}) = -csch^2\theta\{\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{P}\mathcal{Y}_b), S\mathcal{Z}) - \breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Y}_b), SR\mathcal{Z}) - \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b, S\mathcal{Z})\} \tag{3.23}$$

2) (for type-2)

$$\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}) = csc^2\theta\{\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),S\mathcal{Z}) - \breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Y}_b),SR\mathcal{Z}) - \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z})\} \tag{3.24}$$

for any non-null vector fields $\mathcal{X}_a,\mathcal{Y}_b \in \Gamma(\mathcal{D}^T)$, $\mathcal{Z} \in \Gamma(\mathcal{D}^\theta)$.

Proof. **1)** (for type-1)

$$\begin{aligned}\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}) &= -\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{P}\mathcal{Z})\\ &= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b,\mathcal{P}\mathcal{Z}) + \breve{g}((\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})\mathcal{Y}_b,\mathcal{P}\mathcal{Z})\end{aligned}$$

By using (7),(8) and (9), we get

$$\begin{aligned}\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}) &= \breve{g}(\mathcal{P}\mathcal{Y}_b,\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{Z}) + \breve{g}(\mathcal{G}_{\mathcal{X}_a}\mathcal{Y}_b,R\mathcal{Z}) + \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z})\\ &= \breve{g}(\mathcal{Y}_b,\mathcal{P}\bar{\nabla}_{\mathcal{X}_a}R\mathcal{Z}) + \breve{g}(\mathcal{P}\mathcal{Y}_b,\bar{\nabla}_{\mathcal{X}_a}S\mathcal{Z}) - \breve{g}(\mathcal{Y}_b,\mathcal{G}_{\mathcal{X}_a}R\mathcal{Z})\\ &+ \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z})\\ &= \breve{g}(\mathcal{Y}_b,\bar{\nabla}_{\mathcal{X}_a}R^2\mathcal{Z}) - \breve{g}(\mathcal{Y}_b,\bar{\nabla}_{\mathcal{X}_a}SR\mathcal{Z}) + \breve{g}(\mathcal{Y}_b,(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P})S\mathcal{Z})\\ &- \breve{g}(\mathcal{P}\mathcal{Y}_b,A_{S\mathcal{Z}}\mathcal{X}_a) - \breve{g}(\mathcal{Y}_b,\mathcal{G}_{\mathcal{X}_a}R\mathcal{Z}) + \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z}).\end{aligned}$$

By using (9),(4),(5),(6),(16) and (17) we get

$$\begin{aligned}\breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}) &= -cosh^2\theta\breve{g}(\mathcal{Y}_b,\bar{\nabla}_{\mathcal{X}_a}\mathcal{Z}) + \breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Y}_b),SR\mathcal{Z})\\ &- \breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),S\mathcal{Z}) + \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z})\\ &= cosh^2\theta\breve{g}_1(\nabla_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{Z}) + \breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Y}_b),SR\mathcal{Z})\\ &- \breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),S\mathcal{Z}) + \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,S\mathcal{Z}).\end{aligned}$$

From the above relation, we get (1) and using similar method, we obtain (2). □

Also, we find the following result.

Corollary 3.2. *Let $\mathcal{M}$ be a proper pointwise semi-slant type-1,2 submanifold in nearly para-Kaehler manifold $\bar{\mathcal{M}}$. Holomorphic distribution $\mathcal{D}^{\mathrm{T}}$ defines a totally geodesic foliation if and only if*

$$-A_{S\mathcal{Z}}\mathcal{P}\mathcal{X}_a + A_{SR\mathcal{Z}}\mathcal{X}_a + \mathcal{U}_{\mathcal{X}_a}S\mathcal{Z} \in \mathcal{D}^\theta$$

for any non-null vector fields $\mathcal{X}_a \in \Gamma(\mathcal{D}^T)$ *and* $\mathcal{Z} \in \Gamma(\mathcal{D}^\theta)$.

Proof. By using (23), (24) and m_3 (b), we get corollary. □

Lemma 3.2. *Let $\mathcal{M}$ be a proper pointwise semi-slant type-1-2 submanifold in nearly para-Kaehler manifold $(\bar{\mathcal{M}},\mathcal{P},\breve{g})$. The distribution $\mathcal{D}^{\mathrm{T}}$ is holomorphic distribution and distribution $\mathcal{D}^\theta$ is slant distribution. Then we get*
1) (for type-1)

$$\begin{aligned}-\sinh^2\theta\breve{g}([\mathcal{Z},W],\mathcal{X}_a) &= \breve{g}(A_{S\mathcal{Z}}\mathcal{P}\mathcal{X}_a - A_{SR\mathcal{Z}}\mathcal{X}_a,W) - \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Z},SW)\\ &+ \breve{g}(\mathcal{U}_{\mathcal{X}_a}W,S\mathcal{Z}) - \breve{g}(A_{SW}\mathcal{P}\mathcal{X}_a - A_{SRW}\mathcal{X}_a,\mathcal{Z}),\end{aligned}$$

2)(for type-2)

$$\begin{aligned}\sin^2\theta\breve{g}([\mathcal{Z},W],\mathcal{X}_a) &= \breve{g}(A_{S\mathcal{Z}}\mathcal{P}\mathcal{X}_a - A_{SR\mathcal{Z}}\mathcal{X}_a, W) - \breve{g}(\mathcal{U}_{\mathcal{X}_a}\mathcal{Z}, SW)\\ &+ \breve{g}(\mathcal{U}_{\mathcal{X}_a}W, S\mathcal{Z}) - \breve{g}(A_{SW}\mathcal{P}\mathcal{X}_a - A_{SRW}\mathcal{X}_a, \mathcal{Z}),\end{aligned}$$

for any non-null vector fields $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{D}^T)$ *and* $\mathcal{Z}, W \in \Gamma(\mathcal{D}^\theta)$.

Proof. 1)(for type-1)

$$\breve{g}([\mathcal{Z},\mathcal{W}],\mathcal{X}_a) = -\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a) + \breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{W}}\mathcal{Z},\mathcal{X}_a) \tag{3.25}$$

By using the two terms in the right hand side of (25), we obtain

$$\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a) = \breve{g}(\bar{\nabla}_{\mathcal{Z}}\mathcal{P}\mathcal{W},\mathcal{X}_a) - \breve{g}((\bar{\nabla}_{\mathcal{Z}}\mathcal{P})\mathcal{W},\mathcal{X}_a)$$

By using (7),(8) and (9), we have

$$\begin{aligned}\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a) &= \breve{g}(\bar{\nabla}_{\mathcal{Z}}R\mathcal{W},\mathcal{X}_a) + \breve{g}(\bar{\nabla}_{\mathcal{Z}}S\mathcal{W},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a)\\ &= -\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{Z}}R\mathcal{W},\mathcal{P}\mathcal{X}_a) - \breve{g}(A_{S\mathcal{W}}\mathcal{Z},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a)\\ &= cosh^2\theta\breve{g}(\bar{\nabla}_{\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a) - \breve{g}(\bar{\nabla}_{\mathcal{Z}}SR\mathcal{W},\mathcal{P}\mathcal{X}_a)\\ &+ \breve{g}((\bar{\nabla}_{\mathcal{Z}}\mathcal{P})R\mathcal{W},\mathcal{P}\mathcal{X}_a) - \breve{g}(A_{S\mathcal{W}}\mathcal{Z},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a)\\ &= cosh^2\theta\breve{g}(\bar{\nabla}_{\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a) + \breve{g}(A_{SR\mathcal{W}}\mathcal{Z},\mathcal{P}\mathcal{X}_a)\\ &+ \breve{g}(\mathcal{G}_{\mathcal{Z}}R\mathcal{W},\mathcal{P}\mathcal{X}_a) - \breve{g}(A_{S\mathcal{W}}\mathcal{Z},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a).\end{aligned} \tag{3.26}$$

Interchanging $\mathcal{W}$ and $\mathcal{Z}$ in (26). We have

$$\begin{aligned}\breve{g}(\mathcal{P}\bar{\nabla}_{\mathcal{W}}\mathcal{Z},\mathcal{X}_a) &= cosh^2\theta\breve{g}(\bar{\nabla}_{\mathcal{W}}\mathcal{Z},\mathcal{P}\mathcal{X}_a) + \breve{g}(A_{SR\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a)\\ &+ \breve{g}(\mathcal{G}_{\mathcal{W}}R\mathcal{Z},\mathcal{P}\mathcal{X}_a) - \breve{g}(A_{S\mathcal{Z}}\mathcal{W},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{W}}\mathcal{Z},\mathcal{X}_a).\end{aligned} \tag{3.27}$$

By using (25),(26) and (27), we get

$$\begin{aligned}-sinh^2\theta\breve{g}([\mathcal{Z},\mathcal{W}],\mathcal{X}_a) &= -\breve{g}(A_{SR\mathcal{W}}\mathcal{Z},\mathcal{P}\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{Z}}R\mathcal{W},\mathcal{P}\mathcal{X}_a)\\ &+ \breve{g}(A_{S\mathcal{W}}\mathcal{Z},\mathcal{X}_a) + \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{X}_a)\\ &+ \breve{g}(A_{SR\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a) + \breve{g}(\mathcal{G}_{\mathcal{W}}R\mathcal{Z},\mathcal{P}\mathcal{X}_a)\\ &- \breve{g}(A_{S\mathcal{Z}}\mathcal{W},\mathcal{X}_a) - \breve{g}(\mathcal{G}_{\mathcal{W}}\mathcal{Z},\mathcal{X}_a).\end{aligned}$$

By using the symmetry property of the shape operator and interchanging $\mathcal{X}$ and $\mathcal{P}\mathcal{X}_a$ for any $\mathcal{X}_a \in \mathcal{D}^T$, we get

$$\begin{aligned}-sinh^2\theta\breve{g}_1([\mathcal{Z},\mathcal{W}],\mathcal{X}_a) &= -\breve{g}(A_{SR\mathcal{W}}\mathcal{X}_a,\mathcal{Z}) - \breve{g}(\mathcal{G}_{\mathcal{Z}}R\mathcal{W},\mathcal{X}_a)\\ &- \breve{g}(A_{S\mathcal{W}}\mathcal{P}\mathcal{X}_a,\mathcal{Z}) - \breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a)\\ &+ \breve{g}(A_{SR\mathcal{Z}}\mathcal{X}_a,\mathcal{W}) + \breve{g}(\mathcal{G}_{\mathcal{W}}R\mathcal{Z},\mathcal{X}_a)\\ &+ \breve{g}(A_{S\mathcal{Z}}\mathcal{P}\mathcal{X}_a,\mathcal{W}) + \breve{g}(\mathcal{G}_{\mathcal{W}}\mathcal{Z},\mathcal{P}\mathcal{X}_a).\end{aligned} \tag{3.28}$$

Also, by using m_4 and m_3 (b), we find

$$\breve{g}(\mathcal{G}_{\mathcal{W}}\mathcal{Z},\mathcal{P}\mathcal{X}_a)-\breve{g}(\mathcal{G}_{\mathcal{Z}}\mathcal{W},\mathcal{P}\mathcal{X}_a) = -\breve{g}(\mathcal{G}_{\mathcal{W}}R\mathcal{Z},\mathcal{X}_a)+\breve{g}(\mathcal{U}_{\mathcal{W}}\mathcal{X}_a,S\mathcal{Z}) + \breve{g}(\mathcal{G}_{\mathcal{Z}}R\mathcal{W},\mathcal{X}_a)-\breve{g}(\mathcal{U}_{\mathcal{Z}}\mathcal{X}_a,S\mathcal{W}). \quad (3.29)$$

By using (2.14) and from (3.28), (3.29), we get proof.
Also, for type-2 the proof is obtained using the same method. □

4. Geometry of Pointwise Semi-Slant Warped Product Submanifolds in Nearly Para-Kaehler Manifolds

Let $(\mathcal{L},\breve{g}_1)$ and $(\mathcal{E},\breve{g}_2)$ be two semi-Riemannian submanifold, $k:\mathcal{L}\to(0,\infty)$ and $q:\mathcal{L}\times\mathcal{E}\to\mathcal{L}$, $a:\mathcal{L}\times\mathcal{E}\to\mathcal{E}$ the projection maps obtained by $q(t,p)=t$, $a(t,p)=p$ for all $(t,p)\in\mathcal{L}\times\mathcal{E}$. The warped product $\mathcal{M}=\mathcal{L}\times_k\mathcal{E}$ is the manifold $\mathcal{L}\times\mathcal{E}$ with the semi-Riemannian constructure. In that case,

$$\breve{g}(\mathcal{X}_a,\mathcal{Y}_b)=\breve{g}_1(q_*\mathcal{X}_a,q_*\mathcal{Y}_b)+(k\circ q)^2\breve{g}_2(q_*\mathcal{X}_a,q_*\mathcal{Y}_b)$$

for every $\mathcal{X}_a$ and $\mathcal{Y}_b$ of $\mathcal{M}$ where * denotes the tangent map [9]. The function k is called the warping function. Especially, if the warping function is constant, $\mathcal{M}$ is called to be trivial.

For $\mathcal{X}_a$, $\mathcal{Y}_b$ on $\mathcal{L}$ and $\mathcal{V}_c$, $\mathcal{W}_d$ vector fields on $\mathcal{E}$. Later, using Lemma 7.3 of [9], we obtain

$$\nabla_{\mathcal{X}_a}\mathcal{V}_c=\nabla_{\mathcal{V}_c}\mathcal{X}_a=\mathcal{V}_c(lnk) \quad (4.30)$$

where ∇ is the Levi-Civita connection on $\mathcal{K}$.

Theorem 4.1. *Let $\bar{\mathcal{M}}$ be a nearly para-Kaehler manifold. Then, there don't exist pointwise semi-slant non-trivial warped product type 1-2 submanifolds $\mathcal{M}=\mathcal{M}^\theta\times_k\mathcal{M}^T$ in nearly para-Kaehler manifold $\bar{\mathcal{M}}$.*

Proof. For type-1, using (3.22), (2.1) (2.2), (2.5), (2.6) and (2.7), we get

$$\begin{aligned}\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{Y}_b) &= \breve{g}(\nabla_{\mathcal{X}_a}\mathcal{V}_c,\mathcal{Y}_b)=\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{V}_c,\mathcal{Y}_b)\\ &= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{P}\mathcal{V}_c,\mathcal{P}\mathcal{Y}_b)\\ &= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{R}^2\mathcal{V}_c+\mathcal{S}\mathcal{R}\mathcal{V}_c,\mathcal{Y}_b)+\breve{g}(A_{\mathcal{S}\mathcal{V}_c}\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b).\end{aligned}$$

From (Theorem 3.3.) we obtain

$$\begin{aligned}\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{Y}_b) &= \breve{g}(\bar{\nabla}_{\mathcal{X}_a}\cosh^2\theta\mathcal{V}_c,\mathcal{Y}_b)+\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{R}\mathcal{V}_c,\mathcal{Y}_b)+\breve{g}(A_{\mathcal{S}\mathcal{V}_c}\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b)\\ &= \sinh 2\theta\mathcal{X}_a(\theta)\breve{g}(\mathcal{V}_c,\mathcal{Y}_b)\cosh^2\theta\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{V}_c,\mathcal{Y}_b)\\ &- \breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Y}_b),\mathcal{S}\mathcal{R}\mathcal{V}_c)+\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),\mathcal{S}\mathcal{V}_c).\end{aligned}$$

Since $\mathcal{D}^\theta$ and $\mathcal{D}^T$ are orthogonal, using (3.22), we get

$$-\sinh^2\theta\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{Y}_b)=-\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{Y}_b),\mathcal{S}\mathcal{R}\mathcal{V}_c)+\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),\mathcal{S}\mathcal{V}_c).$$

In above equation interchanging $\mathcal{X}_a$ and $\mathcal{Y}_b$, we get

$$-\sinh^2\theta\mathcal{V}_c(lnk)\breve{g}(\mathcal{Y}_b,\mathcal{X}_a) = -\breve{g}(\breve{h}(\mathcal{Y}_b,\mathcal{X}_a),\mathcal{S}\mathcal{R}\mathcal{V}_c) + \breve{g}(\breve{h}(\mathcal{Y}_b,\mathcal{P}\mathcal{X}_a),\mathcal{S}\mathcal{V}_c).$$

If we substract the last two equations from each other, we have

$$\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),\mathcal{S}\mathcal{V}_c) = \breve{g}(\breve{h}(\mathcal{Y}_b,\mathcal{P}\mathcal{X}_a),\mathcal{S}\mathcal{V}_c) \tag{4.31}$$

$$\begin{aligned}\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),\mathcal{S}\mathcal{V}_c) &= \breve{g}(\bar{\nabla}_{\mathcal{X}a}\mathcal{P}\mathcal{Y}_b,\mathcal{S}\mathcal{V}_c)\\ &= \breve{g}(P\bar{\nabla}_{Xa}\mathcal{Y}_b,\mathcal{P}\mathcal{V}_c) - \breve{g}(P\bar{\nabla}_{Xa}\mathcal{Y}_b,\mathcal{R}\mathcal{V}_c)\\ &- \breve{g}(\nabla_{Xa}\mathcal{V}_c,\mathcal{Y}_b) + \breve{g}(\nabla_{Xa}\mathcal{R}\mathcal{V}_c,\mathcal{P}\mathcal{Y}_b).\end{aligned}$$

Using (3.22), we get

$$\breve{g}(\breve{h}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b),\mathcal{S}\mathcal{V}_c) = \mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{Y}_b) + \mathcal{R}\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b). \tag{4.32}$$

Using (23), (1), Theorem 2.1 and for $\mathcal{V}_c = \mathcal{R}\mathcal{V}_c$, $\mathcal{X}_a = \mathcal{P}\mathcal{X}_a$ we have

$$\begin{aligned}0 &= \mathcal{R}\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b)\\ &= \mathcal{R}^2\mathcal{V}_c(lnk)\breve{g}(\mathcal{P}\mathcal{X}_a,\mathcal{P}\mathcal{Y}_b)\\ &= -\cosh^2\theta\mathcal{V}_c(lnk)\breve{g}(\mathcal{X}_a,\mathcal{Y}_b).\end{aligned}$$

Because of $\mathcal{V}_c(lnk) = 0$, lnk is constant. Proof is complete. Also for type-2, we use in a similar way. ☐

Remark 4.1. *We express that Theorem (4.1) is a generalization of Theorem (3.1) in* [22] *and Theorem (4.1) in* [23].

It is clear from the above theorem that there don't exist pointwise semi slant non-trivial warped product submanifolds of the first form $\mathcal{M} = \mathcal{M}^\theta \times_k \mathcal{M}^T$ in nearly para-Kaehler manifolds. Conversely, we demonstrate that there exist of the second form $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ in this part.

Now we write an example with related to the second form $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$.

Let $\mathcal{M}$ be a semi-Riemannian submanifold of $\bar{K}^{20}_{10}$ described by the immersion $\psi : \mathcal{M} \to \bar{K}^{20}_{10}$ with the cartesian coordinates $(x_1, ..., x_{20})$ and the almost para-complex structure
$\mathcal{P}(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_{j-2}}$ $j = (3, 4, 7, 8, 11, 12, 15, 16, 19, 20)$ and
$\mathcal{P}(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{i+2}}$ $i = (1, 2, 5, 6, 9, 10, 13, 14, 17, 18)$. Let $\bar{K}^{20}_{10}$ be a semi-Riemannian space of signature $(+,+,-,-,+,+,-,-,+,+,-,-,+,-,-,+,+,-,+,-)$ with the canonical basis $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{20}})$.

Example 4.1. *$\mathcal{M}$ be defined by the immersion ψ as follows*

$$\begin{aligned}\psi(a, b, c, d) &= (a\sin c, a\cos c, b\sin c, b\cos c, a\sin d, a\cos d, b\sin d,\\ &\quad b\cos d, x, 2a, y, 2b, \sqrt{2}d, \sqrt{2}c, c, d, \sqrt{3}c, \sqrt{3}d, x, y)\end{aligned}$$

$$\psi_{\mathtt{a}} = \sin \mathtt{c}\frac{\partial}{\partial x_1} + \cos \mathtt{c}\frac{\partial}{\partial x_2} + \sin d\frac{\partial}{\partial x_5} + \cos d\frac{\partial}{\partial x_6} + 2\frac{\partial}{\partial x_{10}}$$

$$\psi_{\mathtt{b}} = \sin \mathtt{c}\frac{\partial}{\partial x_3} + \cos \mathtt{c}\frac{\partial}{\partial x_4} + \sin d\frac{\partial}{\partial x_7} + \cos d\frac{\partial}{\partial x_8} + 2\frac{\partial}{\partial x_{12}}$$

$$\psi_{\mathtt{c}} = \mathtt{a}\cos \mathtt{c}\frac{\partial}{\partial x_1} - \mathtt{a}\sin \mathtt{c}\frac{\partial}{\partial x_2} + b\cos \mathtt{c}\frac{\partial}{\partial x_3} - b\sin \mathtt{c}\frac{\partial}{\partial x_4} + \sqrt{2}\frac{\partial}{\partial x_{14}} + \frac{\partial}{\partial x_{15}} + \sqrt{3}\frac{\partial}{\partial x_{17}}$$

$$\psi_{d} = \mathtt{a}\cos d\frac{\partial}{\partial x_5} - \mathtt{a}\sin d\frac{\partial}{\partial x_6} + \mathtt{b}\cos d\frac{\partial}{\partial x_7} - \mathtt{b}\sin d\frac{\partial}{\partial x_8} + \sqrt{2}\frac{\partial}{\partial x_{13}} + \frac{\partial}{\partial x_{16}} + \sqrt{3}\frac{\partial}{\partial x_{18}}$$

defines a pointwise semi-slant submanifold $\mathcal{M}$ with type-1,2 in $(\bar{K}^{20}_{10}, \mathcal{P}, \breve{g})$ para-complex manifold with $\mu = \mathcal{R}^2 = \frac{8}{(\mathtt{a}^2-\mathtt{b}^2)(\mathtt{b}^2-\mathtt{a}^2+6))}$ Actually $D^\theta = span\{\psi_{\mathtt{c}}, \psi_{\mathtt{d}}\}$ is pointwise slant distribution and $\mathcal{D}^T = span\{\psi_a, \psi_{\mathtt{b}}\}$ is invariant distribution.
So, we get that $\mathcal{D}^T$ and D^θ distributions are integrable. The induced metric tensor $\breve{g}_{\mathcal{M}}$ on $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ is given by
$\breve{g}_{\mathcal{M}} = 6({d_a}^2 - {d_b}^2) + (\mathtt{a}^2 - \mathtt{b}^2)({d_c}^2 + {d_d}^2)$. Thus,
**) if $0 < (\mathtt{a}^2 - \mathtt{b}^2) < 2$ or $6 > (\mathtt{a}^2 - \mathtt{b}^2) > 4$, $\mathcal{M}$ is a pointwise semi-slant non-trivial warped product type-1 submanifold in nearly para-Kaehler manifold $\bar{K}^{20}_{10}$ with warping function $k = \sqrt{(\mathtt{a}^2 - \mathtt{b}^2)}$.*
**) if $2 < (\mathtt{a}^2 - \mathtt{b}^2) < 4$ $\mathcal{M}$ is a pointwise semi-slant non-trivial warped product type-2 submanifold in nearly para-Kaehler manifold $\bar{K}^{20}_{10}$ with warping function $k = \sqrt{(\mathtt{a}^2 - \mathtt{b}^2)}$*

We now give below lemmas for later use.

Lemma 4.1. *Let $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ be a pointwise semi-slant non-trivial warped product type-1,2 submanifold in nearly para-Kaehler manifold $\bar{\mathcal{M}}$. In that case, we get*

(i) $\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{V}_c) = 0$

(ii) $\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{Z}) = (\mathcal{X}_a lnk) cosh^2\theta ||\mathcal{Z}||^2$ (for type-1)

$\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{Z}) = (\mathcal{X}_a lnk) cos^2\theta ||\mathcal{Z}||^2$ (for type-2)

(iii) $\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{Z}) = -(\mathcal{P}\mathcal{X}_a lnk) cosh^2\theta ||\mathcal{Z}||^2$ (for type-1)

$\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{Z}) = -(\mathcal{P}\mathcal{X}_a lnk) cos^2\theta ||\mathcal{Z}||^2$ (for type-2)

for $\mathcal{X}_a, \mathcal{Y}_b \in \Gamma(\mathcal{D}^T)$ and $\mathcal{V}_c, \mathcal{Z} \in \Gamma(\mathcal{D}^\theta)$.

Proof. Using (2.7), (2.1) and (2.2) we get

$$\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{V}_c) = \breve{g}(\nabla_{\mathcal{X}_a}\mathcal{P}\mathcal{Y}_b, \mathcal{V}_c) + \breve{g}(\nabla_{\mathcal{X}_a}\mathcal{Y}_b, \mathcal{R}\mathcal{V}_c).$$

From (30) and because of $\mathcal{P}\mathcal{Y}_b$ with $\mathcal{V}_c$ and $\mathcal{R}\mathcal{V}_c$ with $\mathcal{Y}_b$ orthogonality, we obtain

$$\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Y}_b), \mathcal{S}\mathcal{V}_c) = \mathcal{X}_a(lnk)\breve{g}(\mathcal{P}\mathcal{Y}_b, \mathcal{V}_c) + \mathcal{X}_a(lnk)\breve{g}(\mathcal{Y}_b, \mathcal{R}\mathcal{V}_c) = 0.$$

Proof is complete and we get proof of equation (ii) and (iii) with smilar way.

If we interchange $\mathcal{Z}$ by $\mathcal{R}\mathcal{Z}$ in (ii) and (iii), we obtain

$$\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a, \mathcal{R}\mathcal{Z}), \mathcal{S}\mathcal{R}\mathcal{Z}) = (\mathcal{X}_a lnk) cosh^2\theta ||\mathcal{Z}||^2 (type-1), \tag{4.33}$$

$$\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a, \mathcal{R}\mathcal{Z}), \mathcal{S}\mathcal{R}\mathcal{Z}) = (\mathcal{X}_a lnk) cos^2\theta ||\mathcal{Z}||^2 (type-2) \tag{4.34}$$

and

$$\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{R}\mathcal{Z}), \mathcal{S}\mathcal{R}\mathcal{Z}) = -(\mathcal{P}\mathcal{X}_a lnk) cosh^2\theta ||\mathcal{Z}||^2 (type-1), \tag{4.35}$$

$$\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{R}\mathcal{Z}), \mathcal{S}\mathcal{R}\mathcal{Z}) = -(\mathcal{P}\mathcal{X}_a lnk) cos^2\theta ||\mathcal{Z}||^2 (type-2). \tag{4.36}$$

□

Now, using the above lemma, we get the following results.

Corollary 4.1. *Let $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ be pointwise proper semi-slant warped product type-1,2 submanifold in nearly para-Kaehler manifold $\bar{\mathcal{M}}$. In that case, we obtain*

$$\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{R}\mathcal{Z}), \mathcal{S}\mathcal{Z}) = -\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{R}\mathcal{Z}) = -\frac{1}{3}(\mathcal{X}_a lnk) cosh^2\theta ||\mathcal{Z}||^2 (type-1) \tag{4.37}$$

and

$$\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{R}\mathcal{Z}), \mathcal{S}\mathcal{Z}) = -\breve{g}(\breve{h}(\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{R}\mathcal{Z}) = -\frac{1}{3}(\mathcal{X}_a lnk) cos^2\theta ||\mathcal{Z}||^2 (type-2) \tag{4.38}$$

for $\mathcal{X}_a \in \Gamma\mathcal{D}^T$ and $\mathcal{V}_c, \mathcal{Z} \in \Gamma\mathcal{D}^\theta$.

If we replace $\mathcal{X}_a$ by $\mathcal{P}\mathcal{X}_a$ in (37) and (38), we get

$$\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a, \mathcal{R}\mathcal{Z}), \mathcal{S}\mathcal{Z}) = -\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{R}\mathcal{Z}) = -\frac{1}{3}(\mathcal{P}\mathcal{X}_a lnk) cosh^2\theta ||\mathcal{Z}||^2 (type-1) \tag{4.39}$$

and

$$\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a, \mathcal{R}\mathcal{Z}), \mathcal{S}\mathcal{Z}) = -\breve{g}(\breve{h}(\mathcal{P}\mathcal{X}_a, \mathcal{Z}), \mathcal{S}\mathcal{R}\mathcal{Z}) = -\frac{1}{3}(\mathcal{P}\mathcal{X}_a lnk) cos^2\theta ||\mathcal{Z}||^2 (type-2). \tag{4.40}$$

Theorem 4.2. *Let $\mathcal{M}$ be a pointwise semi-slant type-1,2 submanifold of nearly para-Kaehler manifold $\bar{\mathcal{M}}$. In that case, $\mathcal{M}$ is locally a non-trivial warped product submanifold $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$, such that, $\mathcal{M}^T$ is a holomorphic submanifold and $\mathcal{M}^\theta$ is a pointwise slant submanifold in $\bar{\mathcal{M}}$ If the following situation is satisfied*
for type-1

$$\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{Z}}\mathcal{X}_a - \mathcal{A}_{\mathcal{S}\mathcal{Z}}\mathcal{P}\mathcal{X}_a = (1 - \frac{1}{3}\cosh^2\theta)\mathcal{X}_a(\gamma)\mathcal{Z} \tag{4.41}$$

for type-2

$$\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{Z}}\mathcal{X}_a - \mathcal{A}_{\mathcal{S}\mathcal{Z}}\mathcal{P}\mathcal{X}_a = (1 - \frac{1}{3}\cos^2\theta)\mathcal{X}_a(\gamma)\mathcal{Z} \tag{4.42}$$

where $\gamma = lnk$ is a function on $\mathcal{M}$ so that $\mathcal{Z}(\gamma) = 0$ for any $\mathcal{X}_a \in \Gamma(\mathcal{D}^T)$, $\mathcal{Z} \in \Gamma(\mathcal{D}^\theta)$.

Proof. Let $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ be a proper pointwise semi-slant non-trivial warped product type-1 submanifold in nearly para-Kaehler manifolds $\bar{\mathcal{M}}$. In that case, from (2.1), (2.5), (2.7) and Lemma 4.4, we get

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{Z}}\mathcal{P}\mathcal{X}_a, \mathcal{Y}_b) = 0 \tag{4.43}$$

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{Z}}\mathcal{P}\mathcal{X}_a,\mathcal{Z})=(\mathcal{X}_a\gamma)||\mathcal{Z}||^2 \tag{4.44}$$

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{Z}}\mathcal{X}_a,\mathcal{Z})=\frac{1}{3}(\mathcal{X}_a\gamma)\cosh^2\theta||\mathcal{Z}||^2 \quad (type-1) \tag{4.45}$$

$\mathcal{V}_c,\mathcal{Z}\in\Gamma(\mathcal{D}^\theta)$ and $\mathcal{X}_a,\mathcal{Y}_b\in\Gamma(\mathcal{D}^T)$ which specifies that $\mathcal{A}_{\mathcal{S}\mathcal{Z}}\mathcal{P}\mathcal{X}_a$ with related to $\mathcal{D}^\theta$. On the contrary, accept that $\mathcal{M}$ is a pointwise semi-slant type-1 submanifold of nearly para-Kaehler manifold $\bar{\mathcal{M}}$ and using (44) and (45), we get

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{Z}}\mathcal{X}_a-\mathcal{A}_{\mathcal{S}\mathcal{Z}}\mathcal{P}\mathcal{X}_a)=(1-\frac{1}{3}\cosh^2\theta)\mathcal{X}_a(\gamma)\mathcal{Z}. \tag{4.46}$$

So, we get (4.41). Then from Lemma 3.1 (2), $\mathcal{D}^\theta$ is integrable and from Lemma 3.2 (1), $\mathcal{D}^T$ is totally geodesic. Let $\mathcal{M}^\theta$ be the integral manifold of $\mathcal{D}^\theta$. Because of Weingarten operator $A_{\mathcal{N}}$ is self-adjoint and using (2.1),(2.2),(2.5) and (2.7) we have

$$\begin{aligned}\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{V}_c}\mathcal{X}_a-\mathcal{A}_{\mathcal{S}\mathcal{V}_c}\mathcal{P}\mathcal{X}_a,\mathcal{Z}) &= -\breve{g}(\bar{\nabla}_{\mathcal{X}_a}\mathcal{S}\mathcal{R}\mathcal{V}_c,\mathcal{Z})-\breve{g}(\bar{\nabla}_{\mathcal{P}\mathcal{X}_a}\mathcal{S}\mathcal{V}_c,\mathcal{Z})\\ &+ \mathcal{U}_{\mathcal{X}_a}\mathcal{Y}_b,\mathcal{S}\mathcal{Z}\\ &= -\breve{g}(\mathcal{X}_a,\bar{\nabla}_{\mathcal{Z}}\mathcal{P}\mathcal{S}\mathcal{V}_c)\\ &= -\breve{g}(\mathcal{X}_a,\nabla_{\mathcal{Z}}\mathcal{R}^2\mathcal{V}_c)-\breve{g}(\mathcal{X}_a,\nabla_{\mathcal{Z}}\mathcal{V}_c).\end{aligned}$$

Using (2.18) for type-1 we get

$$\begin{aligned}\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{V}_c}\mathcal{X}_a-\mathcal{A}_{\mathcal{S}\mathcal{V}_c}\mathcal{P}\mathcal{X}_a,\mathcal{Z}) &= 2\cosh\theta\sinh\theta\mathcal{Z}(\theta)\breve{g}(\mathcal{X}_a,\mathcal{V}_c)\\ &+ (-1+\cosh^2\theta)\breve{g}(\mathcal{X}_a,\nabla_{\mathcal{Z}}\mathcal{V}_c)\\ &= \sinh^2\theta\breve{g}(\mathcal{X}_a,\nabla_{\mathcal{Z}}\mathcal{V}_c).\end{aligned}$$

Using (2.2) we get

$$\breve{g}(\mathcal{A}_{\mathcal{S}\mathcal{R}\mathcal{V}_c}\mathcal{X}_a-\mathcal{A}_{\mathcal{S}\mathcal{V}_c}\mathcal{P}\mathcal{X}_a,\mathcal{Z})=\sinh^2\theta(\mathcal{X}_a,\breve{h}_\theta(\mathcal{V}_c,\mathcal{Z})). \tag{4.47}$$

Then (4.46) indicate that

$$\breve{h}_\theta(\mathcal{V}_c,\mathcal{Z}) = (\frac{1}{3}+\frac{2}{3}cosech^2\theta)\nabla_\gamma\breve{g}(\mathcal{V}_c,\mathcal{Z})$$

which indicate that $\mathcal{M}^\theta$ is a totaly umbilical submanifold in $\mathcal{M}$ with the mean curvature vector field $(\frac{1}{3}+\frac{2}{3}cosech^2\theta)\nabla_\gamma$, where ∇_γ is the gradient of γ.
Conversely, by direct calculations, we have

$$\begin{aligned}\breve{g}(\nabla_{\mathcal{V}_c}\nabla_\gamma,\mathcal{X}_a) &= [\mathcal{V}_c\breve{g}(\nabla_\gamma,\mathcal{X}_{\mathtt{a}})-\breve{g}(\nabla_\gamma,\nabla_{\mathcal{V}_c}\mathcal{X}_a)]\\ &= [\mathcal{V}_c(X_a(\gamma))-[\mathcal{V}_c,X_{\mathtt{a}}]\gamma-\breve{g}(\nabla_\gamma,\nabla_{\mathcal{X}_a}\mathcal{V}_c)]\\ &= [\mathcal{V}_c,\mathcal{X}_a]\gamma+\mathcal{X}_a(\mathcal{V}_c(\gamma))[\mathcal{V}_c(X_a(\gamma))-[\mathcal{V}_c,X_a]\gamma-\breve{g}(\nabla_\gamma,\nabla_{\mathcal{X}_a}\mathcal{V}_c)]\\ &= [\mathcal{X}_a(\mathcal{V}_c(\gamma))[\mathcal{V}_c(\mathcal{X}_a(\gamma))-\breve{g}(\nabla_\gamma,\nabla_{\mathcal{X}_a}\mathcal{V}_c)].\end{aligned}$$

Because of $\mathcal{V}_c(\gamma)=0$, we get

$$\breve{g}(\nabla_{\mathcal{V}_c}\nabla_\gamma,\mathcal{X}_a) = \breve{g}(\nabla_\gamma,\nabla_{\mathcal{X}_a}\mathcal{V}_c).$$

Conversely, since $\nabla_\gamma\in\Gamma(T\mathcal{M}^T)$ and $\mathcal{M}^T$ is totally geodesic in $\mathcal{M}$, it shows that $\nabla_{\mathcal{X}_a}\mathcal{V}_c\in\Gamma(T\mathcal{M}^\theta)$ for $\mathcal{V}_c\in\Gamma(\mathcal{D}^\theta)$, $\mathcal{X}_a\in\Gamma(\mathcal{D}^T)$. So, $\breve{g}(\nabla_{\mathcal{V}_c}\nabla_\gamma,\mathcal{X}_a)=0$. Then the sphrecial situation is also accomplished, that is $\mathcal{M}^\theta$ is an extrinsic sphere in $\mathcal{M}$. So, proof is complete. Using a similar way, the result is also obtained for type-2. □

5. An Optimal Inequality

We first indicate an orthonormal frame. Let $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ be a $(m + \mathtt{n})$ dimensional pointwise semi-slant warped product submanifold a $(\mathtt{m} + 2n)$-dimensional $\bar{\mathcal{M}}$ nearly para-Kaehler manifold. We give orthonormal frames according to type-1 and type-2. Firstly for type-1, we indicate the orthonormal frames respectively;
$\{\mathtt{E}_1, .., \mathtt{E}_m, \bar{\mathtt{E}}_1, ..., \bar{\mathtt{E}}_n, \mathtt{E}_1^*, ..., \mathtt{E}_n^*\}$ of $\bar{\mathcal{M}}$ so that, restricted to $\mathcal{M}$, $\{\mathtt{E}_1, .., E_\mathtt{m}, \bar{E}_1, ..., \bar{\mathtt{E}}_\mathtt{n}\}$ are tangent to $\mathcal{M}$. So $\{E_1, .., \mathtt{E}_\mathtt{m}, \bar{\mathtt{E}}_1, ..., \bar{E}_\mathtt{n}\}$ form an orthonormal frame of $\mathcal{M}$. We can indicate $\{E_1, .., \mathtt{E}_\mathtt{m}, \bar{\mathtt{E}}_1, ..., \bar{\mathtt{E}}_\mathtt{m}\}$ in such a way that $\{E_1, .., \mathtt{E}_\mathtt{m}\}$ form an orthonormal frame of $\mathcal{D}^T$ and $\{\bar{E}_1, ..., \bar{E}_n\}$ form an orthonormal frame of $\mathcal{D}^\theta$, where $dim(\mathcal{D}^T) = m$ and $dim(\mathcal{D}^\theta) = n$. We can indicate $\{\mathtt{E}_1^*, ..., \mathtt{E}_\mathtt{n}^*\}$ as an orthonormal frame of $\mathcal{S}(\mathcal{D}^\theta)$. In that case, $n = 2p$ and orthonormal frames are $\{\bar{E}_1, ..., \bar{\mathtt{E}}_{2\mathtt{p}}\}$ of $(\mathcal{D}^\theta)$ and $\{E_1^*, ..., E_{2p}^*\}$ of $\mathcal{S}(\mathcal{D}^\theta)$.

$$\bar{\mathtt{E}}_2 = sech\theta\mathcal{R}\bar{\mathtt{E}}_1, ..., \bar{\mathtt{E}}_{2p} = sech\theta\mathcal{R}\bar{\mathtt{E}}_{2p-1}, \quad (type-1)$$

$$\bar{\mathtt{E}}_1^* = csch\theta\mathcal{S}\bar{\mathtt{E}}_1, ..., \bar{\mathtt{E}}_{2p}^* = csch\theta\mathcal{S}\bar{\mathtt{E}}_{2p}, \quad (type-1)$$

We assume that
* on $\mathcal{D}^T$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,m}$, where $m = dim(\mathcal{D}^T)$; also, supposed that $\breve{g}(\mathtt{E}_v, \mathtt{E}_v) = 1$.
* on $\mathcal{D}^\theta$: orthonormal basis $\{\mathtt{E}_w^*\}_{w=1,...,n}$. $n = dim(\mathcal{D}^\theta)$ also $\breve{g}(\mathtt{E}_w^*, \mathtt{E}_w^*) = \mp 1$.
* on $\mathcal{PD}^T$: orthonormal basis $\{\mathtt{E}_v\}_{v=1,...,m}$, where $d_1 = dim\mathcal{P}(\mathcal{D}^T)$ also $\breve{g}(\mathcal{P}\mathtt{E}_v, \mathcal{P}\mathtt{E}_v) = -1$.
* on $\mathcal{SD}^\theta$: orthonormal basis $\{\mathtt{E}_w^*\}_{w=1,...,n}$, where $n = dim\mathcal{S}(\mathcal{D}^\theta)$ also $\breve{g}(\mathtt{E}_w^*, \mathtt{E}_w^*) = \mp 1$.

Theorem 5.1. *Let $\mathcal{M}$ be a $(\mathtt{m} + \mathtt{n})$ dimensional pointwise semi-slant type-1 warped product submanifold $\mathcal{M} = \mathcal{M}^T \times_k \mathcal{M}^\theta$ in nearly para-Kaehler manifold $\bar{\mathcal{M}}^{m+2n}$, where $\mathcal{M}^\theta$ is a proper pointwise slant submanifold and $\mathcal{M}^T$ is a invariant submanifold of $\mathcal{M}$. Assume that $\mathcal{M}^T$ is spacelike. So, we get*
1)

$$||\breve{h}||^2 \leq 4n(csch^2\theta + \frac{1}{9}coth^2\theta)||grad(lnk)||^2, \qquad dim(\mathcal{K}^\theta) = n \tag{5.48}$$

where $grad(lnk)$ is the gradient of lnk.
2) *If the equality sign of (5.48) holds the same way, $\mathcal{M}^\theta$ is totally umbilical and $\mathcal{M}^T$ is totally geodesic in $\bar{\mathcal{M}}$. Also, $\mathcal{M}$ is minimal submanifold of $\bar{\mathcal{M}}$.*

Proof. From description $||\breve{h}_1||^2 = ||\breve{h}_1(D^\theta, D^\theta)||^2 + 2||\breve{h}_1(\mathcal{D}^\theta, \mathcal{D}^T)||^2 + ||\breve{h}_1(\mathcal{D}^T, \mathcal{D}^T)||^2$. We get

$$\begin{aligned} ||\breve{h}||^2 &= \sum_{\mathtt{k}=1}^{\mathtt{m}+2\mathtt{p}} \sum_{v,w=1}^{m} \breve{g}(\breve{h}(\mathtt{E}_v, \mathtt{E}_w), \mathtt{E}_\mathtt{k}^*)^2 + \sum_{k=1}^{m+2\mathtt{p}} \sum_{r,s=1}^{2\mathtt{p}} \breve{g}(\breve{h}(\bar{\mathtt{E}}_r, \bar{\mathtt{E}}_s), E_\mathtt{k}^*)^2 \\ &+ 2\sum_{k=1}^{\mathtt{m}+2p} \sum_{r=1}^{2p} \sum_{v=1}^{\mathtt{m}} \breve{g}(\breve{h}(\mathtt{E}_v, \bar{\mathtt{E}}_r), E_k^*)^2 \end{aligned}$$

where $\{E^*_{\mathtt{k}}\}$ is an orthonormal basis of $\mathcal{TM}^{\perp}$. Accepted the adapted frame, we will indicate above equation as

$$\begin{aligned}||\breve{h}||^2 &= \sum_{a=1}^{2p}\sum_{v,w=1}^{2p}\breve{g}(\breve{h}(\mathtt{E}_v,\mathtt{E}_w), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2+\sum_{a,r,s=1}^{2p}\breve{g}(\breve{h}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_s), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2\\ &+ 2\sum_{v=1}^{m}\sum_{a,r=1}^{2p}\breve{g}(\breve{h}(\bar{\mathtt{E}}_r,\mathtt{E}_v), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2.\end{aligned}$$

By using((37) and (39), we obtain

$$\begin{aligned}||\breve{h}||^2 &= \sum_{a,r,s=1}^{2p}\breve{g}(\breve{h}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_s), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2+2\sum_{v=1}^{m}\sum_{a,r=1}^{2p}(csch\theta)^2[(\mathcal{P}\mathtt{E}_v(lnk))\breve{g}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_a))^2\\ &+ 2\mathcal{P}\mathtt{E}_v(lnk)\breve{g}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_a)\mathtt{E}_v(lnk)(\bar{\mathtt{E}}_r,\mathcal{R}\bar{\mathtt{E}}_a)+(\mathtt{E}_v(lnk)\breve{g}(\bar{\mathtt{E}}_r,\mathcal{R}\bar{\mathtt{E}}_a))^2].\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{v=1}^{m}\sum_{a,r=1}^{2p}\mathcal{P}\mathtt{E}_v(lnk)\breve{g}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_a)\mathtt{E}_v(lnk)\breve{g}(\bar{\mathtt{E}}_r,\mathcal{R}\bar{\mathtt{E}}_a)\\ &= \sum_{v=1}^{m}\sum_{a,r=1}^{2p}\breve{g}(grad(lnk),\mathcal{P}\mathtt{E}_v)\breve{g}(grad(lnk),\mathtt{E}_v)\breve{g}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_a)\breve{g}(\bar{\mathtt{E}}_r,\mathcal{R}\bar{\mathtt{E}}_a)\\ &= -\sum_{a,r=1}^{2p}[\sum_{v=1}^{m}\breve{g}(\breve{g}(grad(lnk),\mathtt{E}_v)\mathtt{E}_v,\mathcal{P}grad(lnk))]\breve{g}_1(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_a)\breve{g}(\bar{\mathtt{E}}_r,\mathcal{R}\bar{\mathtt{E}}_a)=0.\end{aligned}$$

By using (33), (35) and lemma 4.4, the above equation will be simlified as

$$||\breve{h}_1||^2 = \sum_{a,\mathtt{r},s=1}^{2\mathtt{p}}\breve{g}(\breve{h}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_s), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2+4n||grad(lnk)||^2[csch^2\theta+\frac{1}{9}coth^2\theta].$$

So, we get the inequality (48). Also the equality sign of (48) gives, we get

$$\sum_{a=1}^{2p}\sum_{r,s=1}^{2p}\breve{g}(\breve{h}(\bar{\mathtt{E}}_r,\bar{\mathtt{E}}_s), csch\theta\mathcal{S}\bar{\mathtt{E}}_a)^2=0. \tag{5.49}$$

Since $\mathcal{M}^T$ is a totally geodesic in $\mathcal{M}$, (4.46) equation specifies that $\mathcal{M}^T$ is totally geodesic in $\bar{\mathcal{M}}$. Also, (5.49) equation specifies that $\breve{h}$ vanishes on $\mathcal{D}^\theta$. Because of $\mathcal{D}^\theta$ is a spherical distribution in $\mathcal{M}$, we reach that $\mathcal{M}^\theta$ is a totally umbilicial submanifold of $\bar{\mathcal{M}}$. Also, using (4.46) and (5.49), we reach that $\mathcal{M}$ is minimal in $\bar{\mathcal{M}}$. □

Remark 5.1. *If the manifold $\mathcal{M}^\theta$ in the above theorem is timelike, in that case, (48) should be changed by*

$$||\breve{h}||^2 \geq 4n(csch^2\theta+\frac{1}{9}coth^2\theta)||grad(lnk)||^2, \qquad dim(\mathcal{K}^\theta)=n \tag{5.50}$$

where grad(lnk) is the gradient of lnk.
Similarly, if pointwise slant submanifold $\mathcal{M}^\theta$ is type-2 , we achieve.

Theorem 5.2. *Let $\mathcal{M}$ be a $(m+\mathtt{n})$-dimensional pointwise semi-slant warped product submanifold $\mathcal{M}=\mathcal{M}^T\times_k\mathcal{M}^\theta$ in nearly para-Kaehler manifold $\bar{\mathcal{M}}^{m+2n}$. Assume that, $\mathcal{M}^\theta$ is spacelike and timelike, respectively. In that case, (for type-2)*

$$||\breve{h}||^2 \leq 4n(\frac{1}{9}cot^2\theta + csc^2\theta)||grad(lnk)||^2 \tag{5.51}$$

$$(respectively, \quad ||\breve{h}||^2 \geq 4n(\frac{1}{9}cot^2\theta + csc^2\theta)||grad(lnk)||^2) \tag{5.52}$$

where grad(lnk) is the gradient of lnk.

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