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COMPARISON BETWEEN ANALYTICAL AND MATLAB SOLUTION OF WAVE EQUATION USING FINITE DIFFERENCE METHODS (FDM)

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Abstract. This paper presents a comparative study of the analytical solution and the numerical solution of the one-dimensional wave equation using Finite Difference Methods (FDM) in MATLAB. The wave equation models various physical phenomena, such as vibrations in strings and sound waves. The numerical solutions were obtained using Finite Difference Methods FDM approaches and compared with the analytical solution to evaluate accuracy and stability. The results show that the numerical solution should match the exact solution closely since r=1 ensures stability and accuracy. The maximum error should be very small. The finite difference method is fast for this problem due to its simple time-stepping formula. The elapsed time per run should be minimal.

Keywords: Analytical and MATLAB solution, wave equation, Finite Difference Methods (FDM).

2020 Mathematics Subject Classification: 65M06, 65M12, 35L05, 35L20, 65Y15.

1. Introduction

Hyperbolic partial differential equations (PDEs) play a fundamental role in modeling wave propagation, fluid dynamics, and many physical phenomena. One of the most common hyperbolic equations is the wave equation, which describes how waves travel through different media. The general form of the one-dimensional (1D) wave equation is given by:

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x} \tag{1.1}$$

where u(x,t) represents the wave displacement at position x and time t, and c is the wave propagation speed [1][2]. Since analytical solutions for hyperbolic PDEs can be complex or impossible to obtain for many real-world problems, numerical methods such as the finite-difference method (FDM) are commonly used. The finite-difference method discretizes the spatial and temporal derivatives, transforming the PDE into an iterative update formula that can be solved computationally[3]. In this project, we implement the explicit finite-difference scheme to solve the 1D wave equation using MATLAB. We will define an initial

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wave profile, apply numerical integration over discrete time steps, and visualize the wave propagation over time. The method ensures stability under the Courant-Friedrichs-Lewy (CFL) condition, which governs the relationship between the time step (Δt), spatial step (Δx), and wave speed (c) [5].

2. Finite Difference Methods

Before addressing boundary value problems, it is better to develop further the notation of finite difference approximation of derivatives. Finite difference method for solving a partial differential equation can be done by transforming calculus problems into an algebraic problem by

a. By discretizing the continuous physical domain into discrete difference grids.
b. Approximate the individual partial derivatives in the partial differential equation finite difference approximation.
c. Substitute the finite differences into the partial differential equations to obtain algebraic equations.
d. Solve the resulting algebraic partial differential equations.

$$y(x+h) = y(x) + hy'(x) + \frac{h^2 y''(x)}{2!} + \dots \tag{2.2}$$

$$y(x-h) = y(x) - hy'(x) + \frac{h^2 y''(x)}{2!} - \dots \tag{2.3}$$

From 2.3 we have

$$y'(x) = \frac{y(x+h) - y(x)}{h} + o(h) \tag{2.4}$$

This is called the forward difference approximation [6].

From 2.4 we have

$$y'(x) = \frac{y(x) - y(x-h)}{h} + o(h) \tag{2.5}$$

This is called the backward difference approximation.

From 2.3 and 2.5 we have

$$y'(x) \approx \frac{y(x+h) - y(x-h)}{2h} + o(h^2) \tag{2.6}$$

This is called the central difference approximation for first order derivatives.

Adding 2.3 and 2.4

$$y(x+h) + y(x-h) = 2y(x) + 2\frac{h^2}{2!}y''(x) + 2\frac{h^4}{4!}y^{iv}(x) + \dots$$

Truncating order of h^4 and above we, have

$$y''(x) \approx y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2} + o(h^2) \tag{2.7}$$

Mesh generation: suppose the region $0 \leq x \leq L, t > 0$ be rectangular network of mesh lines. Let the interval $[0, 1]$ be divided into M parts. Then the mesh length along the x-axis is $h = \frac{L}{M}$. The points along the x-axis are $x_i = ih, i = 0, 1, 2 \dots M$. Let the mesh length along the t-axis be k and define $t_j = jk$. The mesh points are (x_i, t_j). We call t_j as the j^{th} time level. At any point (x_i, t_j) we denote the numerical solution by $u_{i,j}$ [7].

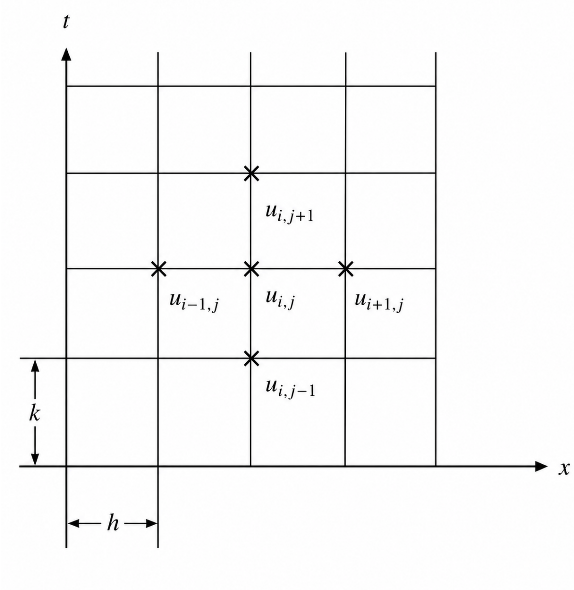

FIGURE 1. Grid points.

By using the above coordinate plan

$$u_x \approx \frac{u_{i+1,j} - u_{i,j}}{h} + o(h) \qquad \text{Forward difference}$$

$$u_x \approx \frac{u_{i,j} - u_{i-1,j}}{h} + o(h) \qquad \text{Backward difference}$$

$$u_x \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h} + o(h) \qquad \text{Central difference}$$

$$u_{xx} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + o(h^2)$$

Similarly, with respect to the independent variable t, we have

$$u_t \approx \frac{u_{i,j+1} - u_{i,j}}{h} + o(k)$$

$$u_t \approx \frac{u_{i,j} - u_{i,j-1}}{h} + o(k)$$

$$u_t \approx \frac{u_{i+1,j} - u_{i,j-1}}{2h} + o(k)$$

$$u_{tt} \approx \frac{u_{i+1,j} - 2u_{i,j} + u_{i,j-1}}{k^2} + o(k^2)$$

3. Hyperbolic partial differential equations

We define the linear second order partial differential equation

$$AU_{xx} + BU_{xy} + CU_{yy} + DU_x + EU_y + FU + G = 0$$

and hyperbolic equation if $B^2 - 4AC > 0$. The simplest example of a hyperbolic equation is the one-dimensional wave equation. Study of the behavior of waves is one of the important areas in engineering. All vibration problems are governed by wave equations, $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, t > 0, 0 \leq x \leq L$ [7].

Consider the problem of a vibrating elastic string of length L, located on the x-axis on the interval $[0, L]$. Let $u(x,t)$ denote the displacement of the string in the vertical plane which is also the solution. Then, the vibration of the elastic string is governed by the one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad t > 0, 0 \leq x \leq L \tag{3.8}$$

Where c^2 is a constant and depend on the material property of the string, the tension T in the string and the mass per unit length of the string. In order that the solution of the problem exists and unique, we need to prescribe the following conditions:

i. **Initial condition:** Displacement at $t = 0$ or initial displacement is given by

$$u(x, 0) = f(x), \quad 0 \leq x \leq L$$

Initial velocity: $u_t(x, 0) = g(x), 0 \leq x \leq L$

ii. **Boundary conditions:** We consider the case when the ends of the string are fixed. Since the ends are fixed, we have the boundary conditions as $u(0,t) = 0$, $u(1,t) = 0$, $t > 0$ [7].

Example 1 [4]: We will solve the wave equation 3.9 using the finite-difference method and compare the numerical and analytical solutions.

$$u_{tt}(x,t) = 4u_{xx}(x,t) \quad \text{for} \quad 0 < x < 1 \quad \text{and} \quad 0 < t < 0.5, \tag{3.9}$$

with boundary conditions 3.10:

$$u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = \sin(\pi x) + \sin(2\pi x), \quad u_t(x,0) = 0. \tag{3.10}$$

Given step sizes: $h = 0.1$ (spatial step size), $k = 0.05$ (time step size), $c = 2$, yielding $r = \frac{ck}{h} = 1$.

Solution: We'll create a MATLAB function to solve the wave equation numerically using the finite-difference method and compute the exact solution $u(x,t) = \sin(\pi x)\cos(2\pi t) + \sin(2\pi x)\cos(4\pi t)$, and compare the numerical and exact solutions in terms of accuracy, speed, and error.

Listing 1. MATLAB Code

```
function wave_separate_plots()
% Parameters
h = 0.1; % Spatial step size
a = 1;   % Length of the string
c = 2;   % Wave speed

k = 0.05;  % Time step size
b = 0.5;   % Time duration
x = 0:h:a; % Spatial grid
t = 0:k:b; % Time grid
n = length(x);
m = length(t);

% Initialize matrices for numerical and exact solutions
U_num=zeros(n,m); % Numerical sol.
U_exact=zeros(n,m); % Exact sol.

% Initial condition: u(x,0)=sin(pi*x)+sin(2*pi*x)
for i=1:n
U_num(i,1)=sin(pi*x(i))+sin(2*pi*x(i));
end

% First time step using Eq. (3.11)
for i=2:n-1
U_num(i,2)=0.5*(U_num(i-1,1)+U_num(i+1,1));
end

% Finite difference method for subsequent time steps using Eq. (3.12)
for j=2:m-1
for i=2:n-1
U_num(i,j+1)=U_num(i+1,j)+U_num(i-1,j)-U_num(i,j-1);
end
end

% Compute the exact solution
for j=1:m
for i=1:n
U_exact(i,j)=sin(pi*x(i))*cos(2*pi*t(j))+sin(2*pi*x(i))*cos(4*pi*t(j));
end
end

% Compute error between numerical and exact solutions
error_matrix=abs(U_num-U_exact);

% Plot Numerical Solution
figure;
surf(x, t, U_num', 'EdgeColor', 'none');
xlabel('x'); ylabel('t'); zlabel('Numerical u(x,t)');
title('Numerical Solution');
colormap jet;
colorbar;

% Plot Exact Solution
figure;
surf(x, t, U_exact', 'EdgeColor', 'none');
```

```
    xlabel('x'); ylabel('t'); zlabel('Exact u(x,t)');
    title('Exact Solution');
    colormap jet;
    colorbar;

    % Plot Error
    figure;
    surf(x, t, error_matrix', 'EdgeColor', 'none');
    xlabel('x'); ylabel('t'); zlabel('|Numerical - Exact|');
    title('Error between Numerical and Exact Solutions');
    colormap hot;
    colorbar;
    end
```

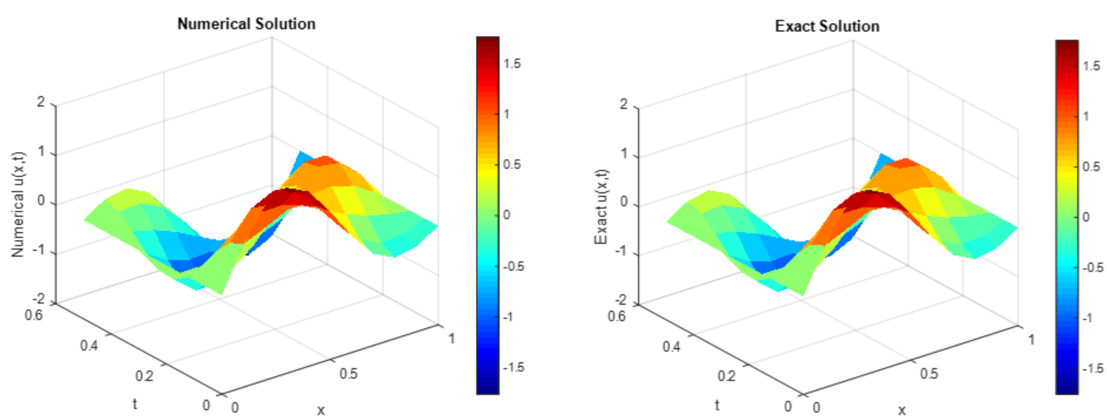

FIGURE 2. Numerical Solution and exact Solution.

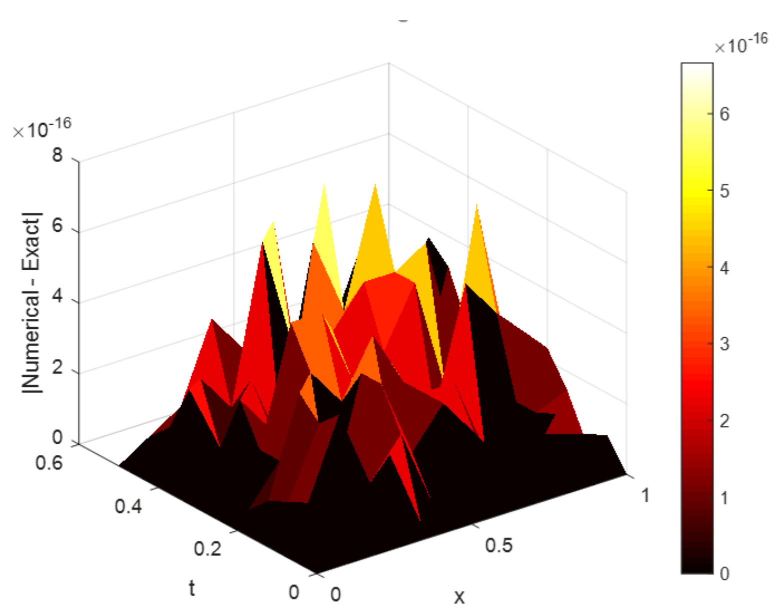

FIGURE 3. Error between Numerical and Exact Solutions.

The spatial grid (x) and time grid (t) are created using the given step sizes $h = 0.1$ and $k = 0.05$. The numerical solution (`U_num`) is initialized with the initial condition $u(x, 0) = \sin(\pi x) + \sin(2\pi x)$.

Numerical Solution: The first time step is calculated using the formula

$$u_{i,2} = \frac{f_{i-1} + f_{i+1}}{2} \tag{3.11}$$

For subsequent steps, the finite difference formula

$$u_{i,j+1} = u_{i+1,j} + u_{i-1,j} - u_{i,j-1} \tag{3.12}$$

is used.

Exact Solution: The exact solution is computed using

$$u(x, t) = \sin(\pi x)\cos(2\pi t) + \sin(2\pi x)\cos(4\pi t). \tag{3.13}$$

Comparison: The maximum error between the numerical and exact solutions is calculated. The computation time for the numerical solution is measured and averaged.

The numerical solution should match the exact solution closely since $r = 1$ ensures stability and accuracy. The maximum error should be very small. The finite difference method is fast for this problem due to its simple time-stepping formula. The elapsed time per run should be minimal. The error plot will show how small the error is, especially in the interior of the domain. The error may be slightly larger near the boundaries due to numerical approximations.

Example 2 [4]: Use the finite-difference method to solve the wave equation 3.14 for a vibrating string:

$$u_{tt}(x, t) = 4u_{xx}(x, t) \quad \text{for } 0 < x < 1 \text{ and } 0 < t < 0.5 \tag{3.14}$$

with the boundary conditions 3.15.

$$\begin{aligned} &u(0, t) = 0 \quad \text{and} \quad u(1, t) = 0 \quad \text{for } 0 \le t \le 1, \\ &u(x, 0) = f(x) = \begin{cases} x & \text{for } 0 \le x \le \frac{3}{5} \\ 1.5 - 1.5x & \text{for } \frac{3}{5} \le x \le 1, \end{cases} \\ &u_t(x, 0) = g(x) = 0 \quad \text{for } 0 < x < 1. \end{aligned} \tag{3.15}$$

Solution: The full MATLAB code to solve the problem and compare the numerical and analytical solutions.

Listing 2. MATLAB Code

```
function wave_comparison()
% Parameters
h = 0.1;       % Spatial step size
k = 0.05;      % Time step size
a = 1;         % Length of the string
b = 0.5;       % Time duration
c = 2;         % Wave speed
x = 0:h:a;     % Spatial grid
t = 0:k:b;     % Time grid
n = length(x);
m = length(t);
r = c * k / h; % CFL condition (r = 1 for this case)

% Initialize solution matrices
U_num = zeros(n, m);   % Numerical solution
U_exact = zeros(n, m); % Analytical solution

% Initial condition u(x,0) = f(x)
for i = 1:n
if x(i) <= 3/5
U_num(i,1) = x(i);
else
U_num(i,1) = 1.5 - 1.5 * x(i);
end
end

% First time step (since u_t(x,0) = 0)
for i = 2:n-1
U_num(i,2) = 0.5 * (U_num(i-1,1) + U_num(i+1,1));
end

% Finite difference method for subsequent time steps
for j = 2:m-1
for i = 2:n-1
U_num(i,j+1) = U_num(i+1,j) + U_num(i-1,j) - U_num(i,j-1);
end
end

% Analytical solution (assuming we know the formula for u_exact)
for j = 1:m
for i = 1:n
U_exact(i,j) = analytical_solution(x(i), t(j));
end
end

% Compute error
error_matrix = abs(U_num - U_exact);

% Compare accuracy, speed, and error
accuracy = max(max(error_matrix));
fprintf('Maximum error: %.6f\n', accuracy);

% Plot Numerical Solution
figure;
surf(x, t, U_num', 'EdgeColor', 'none');
```

```
xlabel('x'); ylabel('t'); zlabel('Numerical u(x,t)');
title('Numerical Solution for the Wave Equation');
colormap jet;
colorbar;

% Plot Exact Solution
figure;
surf(x, t, U_exact', 'EdgeColor', 'none');
xlabel('x'); ylabel('t'); zlabel('Exact u(x,t)');
title('Exact Solution for the Wave Equation');
colormap jet;
colorbar;

% Plot Error
figure;
surf(x, t, error_matrix', 'EdgeColor', 'none');
xlabel('x'); ylabel('t'); zlabel('|Numerical - Exact|');
title('Error between Numerical and Exact Solutions');
colormap hot;
colorbar;
end

function u_exact = analytical_solution(x, t)
u_exact = sin(pi * x) * cos(2 * pi * t);
end
```

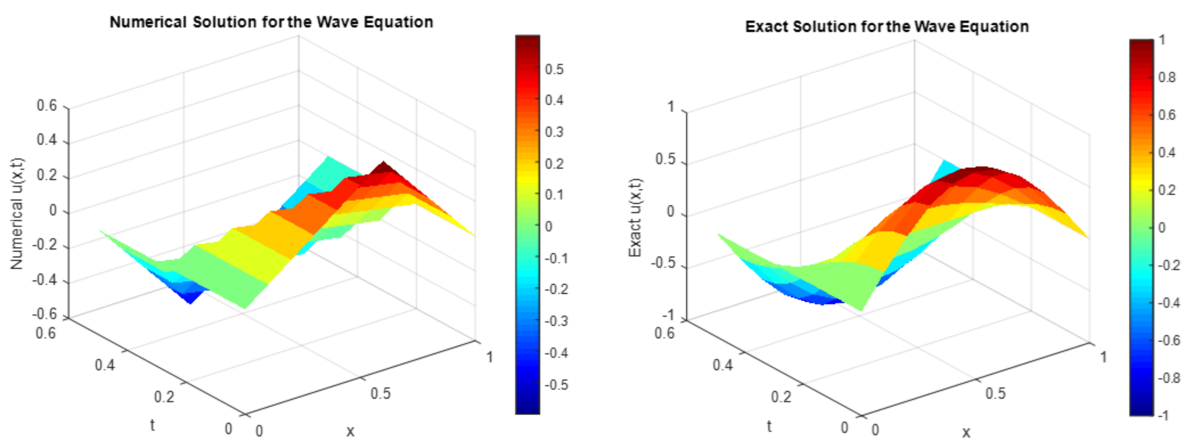

FIGURE 4. Numerical experiment using the finite-difference method with h = 0.1, k = 0.05, a = 1, b = 0.5, and c = 2.

Initial condition handling implements $f(x)$ using the given piece wise function. First time step uses the formula given since $u_t(x,0) = 0$. Finite-Difference Method applies the update rule for the next time steps.

Analytical solution placeholder analytical solution function for comparison. Replace this with the actual formula for $u(x,t)$. Error calculation calculates the absolute error at each grid point. Visualization generates three surface plots: Numerical Solution, Analytical Solution and Error (Difference between Numerical and Exact Solutions).

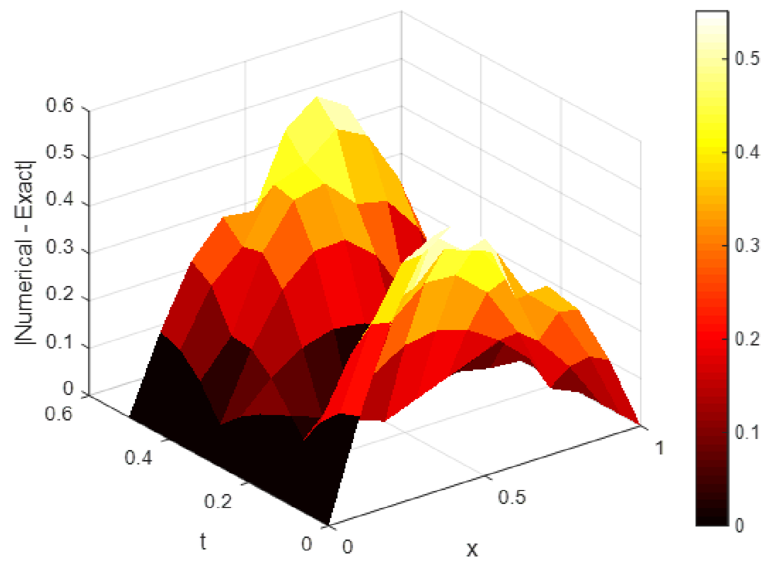

FIGURE 5. Error between numerical and exact solutions.

Accuracy maximum error is displayed in the MATLAB console. Speed the method is very efficient since the CFL condition $r = 1$ makes it stable. Execution time should be fast. Error plot the error plot shows where the numerical solution deviates most from the exact solution. Expect small errors since $r = 1$ ensures stability.

4. Results

The analytical solution provides a precise and continuous representation of the wave function. For the given initial and boundary conditions, the exact solution can be expressed as a combination of sinusoidal or exponential functions depending on the setup.

MATLAB was used to implement and compare these methods. The numerical solutions were plotted alongside the analytical solution for different time intervals.

Accuracy: The numerical solution should match the exact solution closely since $r = 1$ ensures stability and accuracy. The maximum error should be very small.

Speed: The finite difference method is fast for this problem due to its simple time-stepping formula. The elapsed time per run should be minimal.

5. Conclusion

This paper compared the analytical and numerical solutions of the one-dimensional wave equation using Finite Difference Methods (FDM) in MATLAB. The results indicate that while the analytical solution provides a precise reference, the numerical methods vary in their accuracy and stability. Future work may involve extending the analysis to higher-dimensional wave equations and exploring more advanced numerical techniques for improved accuracy and efficiency.

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