CALCULATION

*Volume 2, Issue 2, 2026, Pages:***102–111**

E-ISSN: 3062–2107

www.simadp.com/calculation

SOME PROPERTIES OF GEODESICS AND F-GEODESICS ON TANGENT BUNDLE WITH GRADIENT SASAKI METRIC

ABDERRAHIM ZAGANE *

Abstract. In this paper, we investigate the properties of geodesics and F-geodesics on the tangent bundle equipped with the gradient Sasaki metric. First, we establish necessary and sufficient conditions for a curve to be a geodesic. We then study the behavior of F-geodesics and F-planar curves on the tangent bundle with respect to the induced Levi-Civita connection. Our theoretical results are supported by explicit examples that illustrate the behavior of these curves.

Keywords: Tangent bundle, gradient Sasaki metric, geodesics, F-geodesics, F-planar curves.

2020 Mathematics Subject Classification: 53C22, 58E10, 53C05, 53A04.

1. Introduction

Natural Riemannian metrics on the tangent bundle of a Riemannian manifold are typically constructed using the Levi–Civita connection of the base manifold. Among these constructions, the Sasaki metric [10] occupies a central position and has been rigorously investigated across numerous geometric contexts. However, due to the inherent rigidity of the Sasaki metric, research has increasingly shifted toward various natural deformations. Prominent examples include the Cheeger–Gromoll metric [8], the Berger-type deformed Sasaki metric [3, 13], and the gradient Sasaki metric [4]. To date, the geometry of tangent bundles remains a highly active and fertile domain within modern differential geometry.

The characterization of geodesics on tangent bundles has attracted substantial attention, particularly concerning the analysis of oblique (non-vertical) geodesics and their projections onto the base manifold. Sasaki [11] and Sato [12] provided a comprehensive description of the curves and associated vector fields that generate non-vertical geodesics on the tangent bundle and the unit tangent bundle, respectively. Their findings established that the projected curves exhibit constant geodesic curvatures (i.e., constant Frenet curvatures). Subsequently, Nagy [9] extended these results to the case of locally symmetric base manifolds. Furthermore, Yampolsky [13] conducted analogous investigations for the tangent and unit tangent bundles

Received: 2026.05.16 *Accepted: 2026.06.10*

* Corresponding author

Abderrahim ZAGANE ⋄ zaganeabr2018@gmail.com ⋄ https://orcid.org/0000-0001-9339-3787

endowed with a Berger-type deformed Sasaki metric over Kähler manifolds, considering both locally symmetric spaces and manifolds of constant holomorphic sectional curvature.

In parallel, the exploration of F-planar curves and F-geodesics has emerged as a significant research frontier. These structures are of particular interest as F-planar curves generalize both magnetic curves and standard geodesics [6, 7]. It is essential to distinguish the concept of F-geodesics, as introduced in [5], which constitutes a related but formally distinct framework from that of F-planar curves. Recently, several studies have focused on the behavior of magnetic curves, F-planar curves, and F-geodesics within the geometry of tangent and unit tangent bundles (see, e.g., [1, 2, 16, 17, 18]). These contributions have substantially broadened the theoretical understanding of these geometric configurations.

The primary objective of the present paper is to investigate several problems concerning geodesics and F-geodesics on the tangent bundle endowed with the gradient Sasaki metric. The manuscript is organized as follows.

In Section 2, we recall some notions and results concerning tangent bundles with the gradient Sasaki metric .

Section 3 is devoted to the derivation of the necessary and sufficient conditions for a curve to constitute a geodesic on the tangent bundle with respect to the Levi–Civita connection of the gradient Sasaki metric. We also elucidate several specific properties of these geodesic trajectories.

In the final section, we analyze the behavior of F-geodesics and F-planar curves on the tangent bundle. To conclude, we provide illustrative examples of geodesics and F-geodesics to support the theoretical framework developed in the preceding sections.

Our work aims to deepen the understanding of Riemannian and pseudo-Riemannian structures on tangent bundles, effectively extending classical results associated with the Sasaki metric and its natural deformations to the case of the gradient Sasaki metric.

2. Tangent bundle with gradient Sasaki metric

Let TM be the tangent bundle of an n-dimensional Riemannian manifold (M^n, g), with $\pi : TM \to M$ the natural projection. If $(U, x^j), j = 1, \dots, n$ is a local chart on M, then it induces a local chart $(\pi^{-1}(U), x^j, \xi^j), j = 1, \dots, n$ on TM. Here each ξ^j corresponds to the coordinate of a tangent vector ξ in the direction $\frac{\partial}{\partial x^j}$.

We recall the standard notion of vertical and horizontal lifts on the tangent bundle. For a vector field $Z = Z^j \frac{\partial}{\partial x^j}$ on M, its *vertical lift* ${}^VZ$ and *horizontal lift* ${}^HZ$ are defined on TM by

$$
{}^VZ_{(p,\xi)} = Z^j \frac{\partial}{\partial \xi^j}\Big|_{(p,\xi)}, \quad {}^HZ_{(p,\xi)} = Z^j \big(\frac{\partial}{\partial x^j} - \xi^i \Gamma^k_{ji} \frac{\partial}{\partial \xi^j}\big)\Big|_{(p,\xi)}.
$$

Equivalently,

$$
d\pi({}^HZ) = Z, \quad d\pi({}^VZ) = 0,
$$

for all vector field Z on M, where $(p, \xi) \in TM$.

Consider a smooth strictly positive function $f : M \to \mathbb{R}_+^*$. We define the gradient Sasaki metric on TM, denoted by g^f, as follows:

$$g^f({}^HX, {}^HY) = g(X, Y),$$
$$g^f({}^VX, {}^HY) = g^f({}^HX, {}^VY) = 0,$$
$$g^f({}^VX, {}^VY) = g(X, Y) + X(f)Y(f),$$

for all vector fields X, Y on M.

Theorem 2.1. [4] *Let (M^n, g) be a Riemannian manifold and (TM, g^f) its tangent bundle endowed with the gradient Sasaki metric. The Levi–Civita connection ∇^f of (TM, g^f) satisfies*

$$\begin{aligned}
\nabla^f_{{}^HX}{}^HY =& {}^H(\nabla_X Y) - \frac{1}{2}{}^V(\mathrm{R}(X, Y)\xi),\\
\nabla^f_{{}^HX}{}^VY =& \frac{1}{2}{}^H(\mathrm{R}(\xi, Y)X) + \frac{1}{2}Y(f){}^H(\mathrm{R}(\xi, \mathrm{grad} f)X) + \frac{1}{2}Y(f){}^V(\nabla_X \mathrm{grad} f)\\
&+ (\nabla_X Y)^V + \frac{1}{2\alpha}\big(\mathrm{Hess}_f(X, Y) - \frac{1}{2}X(\alpha)Y(f)\big)^V(\mathrm{grad} f),\\
\nabla^f_{{}^VX}{}^HY =& \frac{1}{2}{}^H(\mathrm{R}(\xi, X)Y) + \frac{1}{2}X(f){}^H(\mathrm{R}(\xi, \mathrm{grad} f)Y) + \frac{1}{2}X(f){}^V(\nabla_Y \mathrm{grad} f)\\
&+ \frac{1}{2\alpha}\big(\mathrm{Hess}_f(X, Y) - \frac{1}{2}Y(\alpha)X(f)\big)^V(\mathrm{grad} f),\\
\nabla^f_{{}^VX}{}^VY =& -\frac{1}{2}X(f){}^H(\nabla_Y \mathrm{grad} f) - \frac{1}{2}Y(f){}^H(\nabla_X \mathrm{grad} f),
\end{aligned}$$

for all vector fields X, Y on M, where $\alpha = 1 + |\mathrm{grad} f|^2$ and Hess_f *is the Hessian of f with respect to g.*

3. Geodesics on tangent bundle with the gradient Sasaki metric

Let $\Gamma = (\gamma(t), \xi(t))$ be a naturally parameterized curve on the tangent bundle TM (i.e. t is an arc length parameter on Γ), where γ is a curve on M and ξ is a vector field along this curve.

We denote

$$\gamma' = \frac{d\gamma}{dt}, \qquad \xi' = \nabla_{\gamma'}\xi, \qquad \Gamma' = \frac{d\Gamma}{dt}.$$

Then, according to [15], we have

$$\Gamma' = {}^H\gamma' + {}^V\xi'. \tag{3.1}$$

Furthermore, we set

$$\gamma'' = \nabla_{\gamma'}\gamma', \qquad \xi'' = \nabla_{\gamma'}\xi'.$$

Note that $(')$ denotes the covariant derivative along γ with respect to parameter t.

The curve $\gamma = \pi \circ \Gamma$ is called the projection (or projected curve) of Γ onto M, where $\pi : TM \to M$ denotes the canonical bundle projection.

A curve $\Gamma = (\gamma(t), \xi(t))$ on TM is said to be the horizontal lift of the curve γ to TM if and only if $\xi' = 0$ [14].

A curve $\Gamma = (\gamma(t), \gamma'(t))$ is called a natural lift of the curve γ to TM[14].

Theorem 3.1. *Let (M^n, g) be a Riemannian manifold and (TM, g^f) its tangent bundle endowed with the gradient Sasaki metric. Let $\Gamma = (\gamma(t), \xi(t))$ be a curve on TM. Then Γ is* *a geodesic if and only if*

$$\begin{cases} \gamma'' = \mathrm{R}(\xi',\xi)\gamma' - g(\xi',\operatorname{grad} f)\mathrm{R}(\xi,\operatorname{grad} f)\gamma' + g(\xi',\operatorname{grad} f)\nabla_{\xi'}\operatorname{grad} f, \\ \xi'' = -g(\xi',\operatorname{grad} f)\nabla_{\gamma'}\operatorname{grad} f - \dfrac{1}{\alpha}\mathrm{Hess}_f(\gamma',\xi')\operatorname{grad} f \\ \qquad + \dfrac{1}{\alpha}\mathrm{Hess}_f(\gamma',\operatorname{grad} f) g(\xi',\operatorname{grad} f)\operatorname{grad} f. \end{cases} \tag{3.2}$$

Proof. From (3.1) and Theorem 2.1, we obtain

$$\begin{aligned} \nabla^f_{\Gamma'}\Gamma' &= \nabla^f_{({}^H\gamma' + {}^V\xi')}({}^H\gamma' + {}^V\xi') \\ &= \nabla^f_{{}^H\gamma'}{}^H\gamma' + \nabla^f_{{}^H\gamma'}{}^V\xi' + \nabla^f_{{}^V\xi'}{}^H\gamma' + \nabla^f_{{}^V\xi'}{}^V\xi' \\ &= {}^H\gamma'' + {}^H(\mathrm{R}(\xi,\xi')\gamma') + \xi'(f){}^H(\mathrm{R}(\xi,\operatorname{grad} f)\gamma') + {}^V\xi'' + \xi'(f){}^V(\nabla_{\gamma'}\operatorname{grad} f) \\ &\quad + \frac{1}{\alpha}\big(\mathrm{Hess}_f(\gamma',\xi') - \frac{1}{2}\gamma'(\alpha)\xi'(f)\big)^V(\operatorname{grad} f) - \xi'(f){}^H(\nabla_{\xi'}\operatorname{grad} f) \\ &= {}^H\big(\gamma'' + \mathrm{R}(\xi,\xi')\gamma' + g(\xi',\operatorname{grad} f)\mathrm{R}(\xi,\operatorname{grad} f)\gamma' - g(\xi',\operatorname{grad} f)\big(\nabla_{\xi'}\operatorname{grad} f\big)\big) \\ &\quad + {}^V\big(\xi'' + g(\xi',\operatorname{grad} f)\nabla_{\gamma'}\operatorname{grad} f + \frac{1}{\alpha}\mathrm{Hess}_f(\gamma',\xi')\operatorname{grad} f \\ &\quad - \frac{1}{2\alpha}\gamma'(\alpha)g(\xi',\operatorname{grad} f)\operatorname{grad} f\big) \\ &= {}^H\big(\gamma'' + \mathrm{R}(\xi,\xi')\gamma' + g(\xi',\operatorname{grad} f)\mathrm{R}(\xi,\operatorname{grad} f)\gamma' - g(\xi',\operatorname{grad} f)\big(\nabla_{\xi'}\operatorname{grad} f\big)\big) \\ &\quad + {}^V\big(\xi'' + g(\xi',\operatorname{grad} f)\nabla_{\gamma'}\operatorname{grad} f + \frac{1}{\alpha}\mathrm{Hess}_f(\gamma',\xi')\operatorname{grad} f \\ &\quad - \frac{1}{\alpha}\mathrm{Hess}_f(\gamma',\operatorname{grad} f)g(\xi',\operatorname{grad} f)\operatorname{grad} f\big). \end{aligned} \tag{3.3}$$

If we put $\nabla^f_{\Gamma'}\Gamma'$ equal to zero, we find (3.2). □

Corollary 3.1. *Let (M^n,g) be a Riemannian manifold and (TM,g^f) its tangent bundle endowed with the gradient Sasaki metric. The natural lift $\Gamma = (\gamma(t),\gamma'(t))$ of any geodesic γ is a geodesic on (TM,g^f).*

Corollary 3.2. *Let (M^n,g) be a Riemannian manifold and (TM,g^f) its tangent bundle endowed with the gradient Sasaki metric. The horizontal lift $\Gamma = (\gamma(t),\xi(t))$ of any geodesic γ is a geodesic on (TM,g^f).*

Theorem 3.2. *Let (M^n,g) be a Riemannian manifold and (TM,g^f) its tangent bundle endowed with the gradient Sasaki metric. Let $\Gamma = (\gamma(t),\xi(t))$ be a geodesic on TM such that γ is a geodesic on M. Then, ξ' is orthogonal to either* $\operatorname{grad} f$ *or* $\nabla_{\gamma'}\operatorname{grad} f$.

Proof. Using the first equation of (3.2), we obtain

$$\begin{aligned} g(\gamma'',\gamma') &= g(\mathrm{R}(\xi',\xi)\gamma',\gamma') - g(\xi',\operatorname{grad} f)g(\mathrm{R}(\xi,\operatorname{grad} f)\gamma',\gamma') \\ &\quad + g(\xi',\operatorname{grad} f)g(\nabla_{\xi'}\operatorname{grad} f,\gamma') \\ &= g(\xi',\operatorname{grad} f)g(\nabla_{\gamma'}\operatorname{grad} f,\xi'). \end{aligned}$$

Since γ is a geodesic, we have $\gamma'' = 0$, and thus

$$g(\xi',\operatorname{grad} f)g(\nabla_{\gamma'}\operatorname{grad} f,\xi') = 0,$$

completing the proof. $\square$

Theorem 3.3. *Let (M^n, g) be a Riemannian manifold and (TM, g^f) its tangent bundle endowed with the gradient Sasaki metric. Let $\Gamma = (\gamma(t), \xi(t))$ be a curve on TM satisfying $\xi' \perp \operatorname{grad} f$. Then Γ is a geodesic on TM if and only if*

$$\begin{cases} \gamma'' = \mathrm{R}(\xi', \xi)\gamma', \\ \xi'' = -\dfrac{1}{\alpha}\mathrm{Hess}_f(\gamma', \xi')\operatorname{grad} f, \end{cases} \tag{3.4}$$

moreover,

$$\begin{cases} |\gamma'| = const, \\ |\xi'| = const. \end{cases}$$

Proof. Since $\xi' \perp \operatorname{grad} f$, it follows that $g(\xi', \operatorname{grad} f) = 0$. Hence, equation (3.2) reduces to

$$\begin{cases} \gamma'' = \mathrm{R}(\xi', \xi)\gamma', \\ \xi'' = -\dfrac{1}{\alpha}\mathrm{Hess}_f(\gamma', \xi')\operatorname{grad} f. \end{cases}$$

Furthermore,

$$(|\gamma'|^2)' = 2g(\gamma'', \gamma') = g(\mathrm{R}(\xi', \xi)\gamma', \gamma') = 0,$$

which implies $|\gamma'| = \mathrm{const}$.

Using the second equation of (3.4), we obtain

$$(|\xi'|^2)' = 2g(\xi'', \xi') = -\frac{2}{\alpha}\mathrm{Hess}_f(\gamma', \xi')g(\operatorname{grad} f, \xi').$$

Since $g(\operatorname{grad} f, \xi') = 0$ by hypothesis, hence $|\xi'| = \mathrm{const}$. $\square$

Theorem 3.4. *Let (M^n, g) be a Riemannian manifold and (TM, g^f) its tangent bundle endowed with the gradient Sasaki metric. Let $\Gamma = (\gamma(t), \xi(t))$ be a geodesic on TM and assume that $\nabla \operatorname{grad} f = 0$. Then we have*

$$\begin{cases} \gamma'' = \mathrm{R}(\xi', \xi)\gamma' - g(\xi', \operatorname{grad} f)\mathrm{R}(\xi, \operatorname{grad} f)\gamma', \\ \xi'' = 0, \end{cases}$$

and moreover,

$$\begin{cases} |\gamma'| = const, \\ |\xi'| = const. \end{cases}$$

Remark 3.1. *As a reminder, note that locally we have:*

$$\gamma'' = \sum_{k=1}^{m}(\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{m}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^k)\frac{\partial}{\partial x^k}, \tag{3.5}$$

and

$$\xi' = \sum_{k=1}^{m}(\frac{d\xi^k}{dt} + \sum_{i,j=1}^{m}\frac{d\gamma^j}{dt}\xi^i\Gamma_{ij}^k)\frac{\partial}{\partial x^k}. \tag{3.6}$$

Example 3.1. *Consider the upper half-plane*

$$H = \{(x, y) \in \mathbb{R}^2, y > 0\}$$

equipped withe the Riemannian metric g define by

$$g_{11} = 1\ ,\ g_{22} = y^2\ ,\ g_{12} = g_{21} = 0.$$

The non-vanishing Christoffel symbols of the associated Levi–Civita connection are

$$\Gamma_{22}^2 = \frac{1}{y}.$$

The curve $\gamma(t) = (x(t), y(t))$ is a geodesic curve if and only if $\gamma'' = 0$, hence from (3.5), we have

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{2} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^k(\gamma(t)) = 0 \iff \begin{cases} x'' = 0, \\ y'' + \dfrac{1}{y}(y')^2 = 0. \end{cases}$$

Solving these equations yields

$$\begin{cases} x(t) = c_{1t} + c_2, \\ y(t) = \sqrt{c_3 t + c_4}, \qquad c_3 t + c_4 > 0, \end{cases}$$

where c_i are real constants. Consequently,

$$\gamma(t) = \left(c_{1t} + c_2,\ \sqrt{c_3 t + c_4}\right).$$

From Corollary 3.1, the natural lift $\Gamma_1 = (\gamma(t), \gamma'(t))$ of γ is a geodesic on TH.

2) Let $\Gamma = (\gamma(t), \xi(t))$ be the horizontal lift of γ, where $\xi(t) = (\lambda(t), \mu(t))$. Then ξ satisfies $\xi' = 0$, hence from (3.6), we have

$$\frac{d\xi^k}{dt} + \sum_{i,j=1}^{2} \xi_i \frac{d\gamma^j}{dt}\Gamma_{ij}^k(\gamma(t)) = 0 \iff \begin{cases} \lambda' = 0, \\ \mu' + \dfrac{y'}{y}\mu = 0. \end{cases}$$

Solving these equations, we obtain

$$\begin{cases} \lambda(t) = c_5, \\ \mu(t) = \dfrac{c_6}{\sqrt{c_3 t + c_4}}, \end{cases}$$

where c_i are real constants. Consequently,

$$\xi(t) = \left(c_5,\ \frac{c_6}{\sqrt{c_3 t + c_4}}\right).$$

By Corollary 3.2, the horizontal lift $\Gamma = (\gamma(t), \xi(t))$ of γ is a geodesic on TH.

4. F-geodesics on the tangent bundle with the gradient Sasaki metric

Let (M^m, g) be an m-dimensional Riemannian manifold and let F be a $(1,1)$-tensor field on M.

A curve $\gamma : I \subset \mathbb{R} \to M$ is called F-planar if its tangent vector field, when subjected to parallel transport along γ, remains confined to the two-dimensional distribution spanned by the tangent vector γ' and its image $F\gamma'$. Analytically, this geometric condition is equivalent to the requirement that the covariant acceleration of γ satisfies

$$\gamma'' = \varrho_1(t)\gamma' + \varrho_2(t)F\gamma',$$

where $\varrho_1(t)$ and $\varrho_2(t)$ are smooth scalar functions of the parameter $t \in I$ [6, 7]. Because this definition relies entirely on the distribution spanned by the vectors rather than the specific timing of the trajectory, F-planarity is an intrinsic, purely geometric property of the curve that is invariant under reparameterization.

A curve γ on M is defined as an F-geodesic if it satisfies:

$$\gamma'' = F\gamma'.$$

Unlike F-planar curves, an F-geodesic represents a dynamic trajectory that dictates a highly specific affine parameterization. While every F-geodesic trivially constitutes an F-planar curve (where $\varrho_1 = 0$ and $\varrho_2 = 1$), the converse is generally false [5].

In the sequel, let $\widetilde{\nabla}$ denote the Levi–Civita connection of the the gradient Sasaki metric on tangent bundle TM with the condition

$$\nabla \mathrm{grad} f = 0,$$

Theorem 4.1. *Let (M^n, g) be a Riemannian manifold, (TM, g^f) its tangent bundle endowed with the gradient Sasaki metric and F be a $(1,1)$-tensor field on M. A curve $\Gamma = (\gamma(t), \xi(t))$ on TM_0^f is an ${}^HF$-planar with respect to $\widetilde{\nabla}$ if and only if the following system holds:*

$$\begin{cases} \gamma'' = \mathrm{R}(\xi', \xi)\gamma' - g(\xi', \mathrm{grad} f)\mathrm{R}(\xi, \mathrm{grad} f)\gamma' + \varrho_1\gamma' + \varrho_2 F\gamma' \\ \\ \xi'' = \varrho_1\xi' + \varrho_2 F\xi' \end{cases} \tag{4.7}$$

where ϱ_1 and ϱ_2 are some functions of the parameter t.

Proof. The curve Γ is ${}^HF$-planar with respect to $\widetilde{\nabla}$ if and only if it satisfies the condition:

$$\widetilde{\nabla}_{\Gamma'}\Gamma' = \varrho_1\Gamma' + \varrho_2{}^HF\Gamma',$$

where ϱ_1 and ϱ_2 are functions of t. Utilizing the decomposition provided in (3.1), we have:

$$\begin{aligned} \widetilde{\nabla}_{\Gamma'}\Gamma' &= \varrho_1({}^H\gamma' + {}^V\xi') + \varrho_2{}^HF({}^H\gamma' + {}^V\xi') \\ &= \varrho_1{}^H\gamma' + \varrho_2{}^HF{}^H\gamma' + \varrho_1{}^V\xi' + \varrho_2{}^HF{}^V\xi' \\ &= {}^H(\varrho_1\gamma' + \varrho_2 F\gamma') + {}^V(\varrho_1\xi' + \varrho_2 F\xi') \\ &= {}^H(\varrho_1\gamma' + \varrho_2 F\gamma') + {}^V(\varrho_1\xi' + \varrho_2 F\xi'). \end{aligned} \tag{4.8}$$

By comparing the horizontal and tangential components of (3.3) with those in (4.8), the system (4.7) follows immediately. □

Corollary 4.1. *Let (M^n, g) be a flat Riemannian manifold, (TM, g^f) its tangent bundle endowed with the gradient Sasaki metric and F be a $(1,1)$-tensor field on M. A curve $\Gamma = (\gamma(t), \xi(t))$ on TM_0^f is an ${}^HF$-planar with respect to $\widetilde{\nabla}$ if and only if the following* *system holds:*

$$\begin{cases} \gamma'' = \varrho_1 \gamma' + \varrho_2 F \gamma' \\ \xi'' = \varrho_1 \xi' + \varrho_2 F \xi' \end{cases}$$

By setting $\varrho_1 = 0$ and $\varrho_2 = 1$ in Theorem 4.1, we obtain the following result for ${}^H F$-geodesics.

Theorem 4.2. *Let (M^n, g) be a Riemannian manifold, (TM, g^f) its tangent bundle endowed with the gradient Sasaki metric and F be a $(1,1)$-tensor field on M. A curve $\Gamma = (\gamma(t), \xi(t))$ on TM_0^f is an ${}^H F$-geodesic with respect to $\widetilde{\nabla}$ if and only if the following system holds:*

$$\begin{cases} \gamma'' = \mathrm{R}(\xi', \xi)\gamma' - g(\xi', \mathrm{grad} f)\mathrm{R}(\xi, \mathrm{grad} f)\gamma' + F\gamma' \\ \xi'' = F\xi' \end{cases}$$

Corollary 4.2. *Let (M^n, g) be a flat Riemannian manifold, (TM, g^f) its tangent bundle endowed with the gradient Sasaki metric and F be a $(1,1)$-tensor field on M. A curve $\Gamma = (\gamma(t), \xi(t))$ on TM_0^f is an ${}^H F$-geodesic with respect to $\widetilde{\nabla}$ if and only if the following system holds:*

$$\begin{cases} \gamma'' = F\gamma' \\ \xi'' = F\xi' \end{cases}$$

Theorem 4.3. *Let (M^n, g) be a Riemannian manifold, (TM, g^f) its tangent bundle endowed with the gradient Sasaki metric and F be a $(1,1)$-tensor field on M. If $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of γ and $\Gamma \in TM_0^f$, then Γ is an ${}^H F$-planar curve (resp., ${}^H F$-geodesic) if and only if γ is an F-planar curve (resp., F-geodesic).*

Proof. Since $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of the curve γ, we have $\xi' = 0$. By Theorem 4.1, system (4.7) is equivalent to

$$\gamma'' = \varrho_1 \gamma' + \varrho_2 F \gamma'.$$

Therefore, Γ is an ${}^H F$-planar curve if and only if γ satisfies

$$\gamma'' = \varrho_1 \gamma' + \varrho_2 F \gamma',$$

that is, if and only if γ is an F-planar curve.

In the particular case where $\varrho_1 = 0$ and $\varrho_2 = 1$, the above equation reduces to

$$\gamma'' = F\gamma',$$

and hence Γ is an ${}^H F$-geodesic if and only if γ is an F-geodesic. □

Example 4.1. *Let $(\mathbb{R}^* \times \mathbb{R}^*, g)$ be a Riemannian manifold and F be a $(1,1)$-tensor field on M, such that*

$$g = \frac{1}{x^2} dx^2 + y^2 dy^2, \quad F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1 = -\frac{1}{x} \quad , \Gamma_{22}^2 = \frac{1}{y}.$$

Let $\Gamma = (\gamma(t), \xi(t))$ *be a horizontal lift of a curve* γ, *such that* $\gamma(t) = (x(t), y(t))$ *and* $\xi(t) = (u(t), v(t))$ *then* $\xi' = 0$, *from* (3.6) *we have,*

$$\begin{cases} u' - \dfrac{x'}{x}u = 0 \\ v' + \dfrac{y'}{y}v = 0 \end{cases} \Leftrightarrow \begin{cases} u(t) = k_1 x(t) \\ v(t) = \dfrac{k_2}{y(t)} \end{cases}$$

where k_1, k_2 *are real constants.*
From Theorem 4.2 Γ *is an* ${}^H F$*-geodesic if and only if* γ *is an* F*-geodesic if and only if* $\gamma'' = F\gamma'$, *from* (3.5) *we have*

$$\begin{cases} x'' - \dfrac{(x')^2}{x} = x' \\ y'' + \dfrac{(y')^2}{y} = -y' \end{cases} \Leftrightarrow \begin{cases} x(t) = c_2 \exp(c_1 e^t) \\ y(t) = \pm\sqrt{c_4 + c_3 e^{-t}} \end{cases}$$

where c_i *are real constants,* $c_4 + c_3 e^{-t} \geq 0$, *and*

$$\begin{cases} u(t) = c_5 \exp(c_1 e^t) \\ v(t) = \dfrac{c_6}{\sqrt{c_4 + c_3 e^{-t}}} \end{cases}$$

where c_i *are real constants, hence*

$$\gamma(t) = (c_2 \exp(c_1 e^t), \pm\sqrt{c_4 + c_3 e^{-t}})$$

and

$$\xi(t) = (c_5 \exp(c_1 e^t), \frac{c_6}{\sqrt{c_4 + c_3 e^{-t}}}).$$

References

[1] Abbassi, M. T. K., & Amri, N. (2020). Natural paracontact magnetic trajectories on unit tangent bundles. *Axioms, 9*, Article 72, 1–15.

[2] Abbassi, M. T. K., & Amri, N. (2023). Geodesics and natural complex magnetic trajectories on tangent bundles. *Turkish Journal of Mathematics, 47*(4), 620–646.

[3] Altunbas, M., Simsek, R., & Gezer, A. (2019). A study concerning Berger-type deformed Sasaki metric on the tangent bundle. *Journal of Mathematical Physics, Analysis, Geometry, 15*(4), 435–447.

[4] Belarbi, L., & Elhendi, H. (2023). On the geometry of the tangent bundle with gradient Sasaki metric. *Arab Journal of Mathematical Sciences, 29*(1), 14–28.

[5] Bejan, C. L., & Druţă-Romaniuc, S. L. (2015). F-geodesics on manifolds. *Filomat, 29*(10), 2367–2379.

[6] Hinterleitner, I., & Mikeš, J. (2007). On F-planar mappings of spaces with affine connections. *Note di Matematica, 27*(1), 111–118.

[7] Mikeš, J., & Sinyukov, N. S. (1983). On quasiplanar mappings of spaces of affine connection. *Soviet Mathematics, 27*(1), 63–70.

[8] Musso, E., & Tricerri, F. (1988). Riemannian metrics on tangent bundles. *Annali di Matematica Pura ed Applicata, 150*(4), 1–20.

[9] Nagy, P. T. (1978). Geodesics on the tangent sphere bundle of a Riemannian manifold. *Geometriae Dedicata, 7*(2), 233–244.

[10] Sasaki, S. (1962). On the differential geometry of tangent bundles of Riemannian manifolds II. *The Tohoku Mathematical Journal, Second Series, 14*(2), 146–155.

[11] Sasaki, S. (1976). Geodesics on the tangent sphere bundle over space forms. *Journal für die Reine und Angewandte Mathematik, 288*, 106–120.

[12] Sato, K. (1978). Geodesics on the tangent bundle over space forms. *Tensor, New Series, 32*(1), 5–10.

[13] Yampolsky, A. (2012). On geodesics of tangent bundle with fiberwise deformed Sasaki metric over Kählerian manifolds. *Journal of Mathematical Physics, Analysis, Geometry, 8*(2), 117–189.

[14] Yano, K., & Ishihara, S. (1973). *Tangent and cotangent bundles*. Marcel Dekker.

[15] Zagane, A., & Djaa, M. (2017). On geodesics of warped Sasaki metric. *Mathematical Sciences and Applications E-Notes, 5*(1), 85–92.

[16] Zagane, A. (2024). Some results on geodesics and F-geodesics in tangent bundle with φ-Sasakian metric over para-Kähler-Norden manifold. *International Journal of Maps in Mathematics, 7*(2), 138–155.

[17] Zagane, A., & Gezer, A. (2024). Geodesic and F-geodesic exploration with a vertical generalized Berger-type deformed Sasaki metric on the tangent and φ-unit tangent bundles. *Filomat, 38*(20), 7143–7159.

[18] Zagane, A. (2025). Geodesicity and F-geodesicity properties of tangent bundle with Berger-type Cheeger-Gromoll metric. *Palestine Journal of Mathematics, 14*(1), 244–261.

(A. Zagane) Department of Mathematics, Faculty of Science and Technology, Ahmed Zabana University-Relizane, Relizane, 48000, Algeria.