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CLAIRAUT CONFORMAL HEMI-SLANT SUBMERSIONS FROM KÄHLER MANIFOLDS

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Abstract. In this paper, we introduce and study Clairaut conformal hemi-slant submersions from Kähler manifolds onto Riemannian manifolds. This class of maps combines the geometry of Clairaut conformal submersions with the hemi-slant decomposition of the vertical distribution in the almost Hermitian manifolds. We first establish a characterization theorem for Clairaut conformal hemi-slant submersions in terms of the geodesic behavior on the total manifold, the mean curvature of the fibers, and the behavior of the dilation along the fibers. We then derive equivalent formulations of the Clairaut condition adapted to the slant and anti-invariant components of the vertical distribution and obtain refined decompositions of the Clairaut relation and the harmonicity condition with respect to the hemi-slant splitting. Furthermore, we investigate the stability of the Clairaut conformal hemi-slant structure under conformal deformations of the total metric. We also study curvature properties of such submersions and obtain vertical sectional, scalar and Ricci curvature decomposition formulas compatible with the hemi-slant structure. In particular, the vertical curvature is decomposed into its slant, anti-invariant and mixed components, revealing the geometric influence of the hemi-slant splitting on the Clairaut and harmonic structures. Finally, we provide an explicit nontrivial example illustrating the theory.

Keywords: Kähler manifold, conformal hemi-slant submersion, Clairaut conformal submersion, Clairaut conformal hemi-slant submersion.

2020 Mathematics Subject Classification: 53C15, 53C43, 53C55, 53D15.

1. Introduction

The study of smooth maps preserving geometric structures between Riemannian manifolds has long occupied a central position in differential geometry. Among such mappings, Riemannian submersions, introduced by O'Neill [23], provide a fundamental mechanism for relating the geometry of a manifold to that of its quotient through the interaction of vertical and horizontal distributions. O'Neill's tensorial formalism and curvature identities [23] remain the basic tools in the subject and have been systematically developed in standard

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references such as [9, 24]. These mappings play an important role not only in global differential geometry but also in the study of foliations, fiber bundles and geometric reduction procedures.

A natural extension of Riemannian submersions is obtained by replacing the isometric condition on the horizontal distribution with a conformal one. This leads to the notion of horizontally conformal, or simply conformal, submersions. Such mappings arise naturally in the theory of harmonic morphisms and have been extensively studied from both geometric and analytic viewpoints [10]. In particular, the monograph of [5] shows that horizontally conformal submersions form the natural geometric background for harmonic morphisms between Riemannian manifolds. The geometry of conformal submersions is strongly influenced by the dilation function, whose behavior governs the interaction between the horizontal and vertical distributions and plays a decisive role in harmonicity and curvature questions [2, 3, 11, 12, 14].

On the other hand, submersion theory in the presence of an almost Hermitian structure has generated several important classes of geometrically distinguished mappings. The interaction between the almost complex structure and the vertical distribution gives rise to invariant, anti-invariant, slant, semi-invariant and semi-slant geometries. Anti-invariant, semi-invariant, slant and semi-slant constructions in almost Hermitian geometry were developed in the context of submersions and submanifolds by several authors, and their modern formulations in submersion theory may be traced through the works of Watson, Şahin, Park and Prasad, and others [34, 30, 31, 32, 25, 1, 29, 13]. In this setting, the vertical distribution carries additional geometric information inherited from the ambient complex structure, and this interaction has proved to be highly effective in the study of integrability, harmonicity and curvature properties.

Within this framework, hemi-slant submersions form a particularly natural class. Introduced in the almost Hermitian manifolds by [16], hemi-slant submersions are characterized by the orthogonal decomposition of the vertical distribution into a slant component and an anti-invariant component. This decomposition provides a natural common generalization of anti-invariant and slant geometries and yields a flexible setting in which the complex structure interacts nontrivially with the vertical bundle. The geometry of hemi-slant submersions and related distributions has subsequently been developed further in several directions, including hemi-slant submersions, conformal versions and harmonic aspects [33, 16, 17, 27]. In particular, conformal variants of complex-compatible submersions have been studied extensively and shown to produce rich geometric structures.

A different and equally influential direction in submersion theory originates in Clairaut's classical theorem for geodesics on surfaces of revolution. The extension of this idea to submersion theory was initiated by Bishop [7], who introduced Clairaut submersions and showed that the constancy of a suitable angular quantity along geodesics imposes strong geometric restrictions on the fibers. Since then, Clairaut-type conditions have become an important tool in understanding the relationship between the geodesic flow of the total manifold and the geometry of the submersion. In particular, Clairaut conditions encode a subtle compatibility between the geometry of the fibers and the behavior of geodesics.

In recent years, Clairaut-type structures have been studied intensively in broader settings, especially for conformal submersions. Meena and coauthors developed the theory of Clairaut conformal submersions in a systematic way and obtained several characterization theorems in terms of totally umbilical fibers, mean curvature vector fields and dilation functions [19, 20, 22]. Their results show that the classical Clairaut relation extends naturally to the conformal category and remains closely tied to the geometry of the vertical foliation. Related developments for Clairaut-type conditions, conformal submersions and conformal maps in different geometric settings may also be found in [4, 21, 28, 26], where further curvature and structural consequences were investigated.

Although conformal submersions, Clairaut submersions and hemi-slant geometries have each been studied extensively, their simultaneous interaction has not yet been systematically examined. In particular, the effect of the hemi-slant splitting of the vertical distribution on the Clairaut condition remains essentially unexplored. This raises several natural questions: how does the decomposition

$$\ker \pi_* = D_\theta \oplus D^\perp$$

influence the Clairaut relation, how does the conformal factor interact with the slant and anti-invariant parts, and what additional curvature information can be extracted from this refined geometric structure?

The aim of the present paper is to initiate a systematic study of Clairaut conformal hemi-slant submersions from Kähler manifolds onto Riemannian manifolds. We introduce this class of maps and establish a characterization of the Clairaut condition in terms of the geodesic behavior on the total manifold, the geometry of the fibers, the behavior of the dilation function, and the hemi-slant decomposition of the vertical distribution. We then derive equivalent formulations of the Clairaut relation adapted to the slant and anti-invariant components of the vertical bundle and show that the hemi-slant splitting yields refined decompositions of both the curvature and harmonicity structures associated with the submersion. In particular, we obtain vertical sectional, scalar and Ricci curvature identities compatible with the decomposition of the vertical distribution into its slant, anti-invariant and mixed components, together with harmonicity criteria reflecting the contribution of each part of the hemi-slant splitting. These results show that the hemi-slant structure does not merely coexist with the Clairaut condition, but contributes directly to its geometric, curvature and harmonic properties.

Finally, we construct an explicit nontrivial example showing that the class introduced here is nonempty and geometrically natural.

2. Preliminaries

Let (M^{2m}, g, J) be a Kähler manifold and (N^n, g_N) a Riemannian manifold. Then

$$J^2 = -I, \tag{2.1}$$

$$g(JE_1, JE_2) = g(E_1, E_2), \qquad g(JE_1, E_2) = -g(E_1, JE_2), \tag{2.2}$$

and

$$(\nabla_{E_1} J)E_2 = 0 \tag{2.3}$$

for all $E_1, E_2 \in \Gamma(TM)$, where ∇ denotes the Levi-Civita connection of g [6, 15].

Let $\pi : (M, g, J) \to (N, g_N)$ be a smooth submersion. We write

$$TM = \ker \pi_* \oplus (\ker \pi_*)^\perp \tag{2.4}$$

and denote the vertical and horizontal projections by $\mathcal{V}$ and $\mathcal{H}$, respectively. The map π is called a horizontally conformal submersion if there exists a positive function λ on M such that

$$g_N(\pi_* X_1, \pi_* X_2) = \lambda^2 g(X_1, X_2) \tag{2.5}$$

for all horizontal vector fields $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$ [11, 5, 19]. The function λ is called the dilation of π. In particular, π is a Riemannian submersion whenever $\lambda \equiv 1$ [23, 9].

A vector field E on M is called projectable if there exists a vector field $\widetilde{E}$ on N such that $\pi_* E = \widetilde{E} \circ \pi$ [9, 19]. A horizontal projectable vector field is called basic. For every vector field $\widetilde{Z}$ on N, there exists a unique basic vector field Z on M such that $\pi_* Z = \widetilde{Z} \circ \pi$ [9, 5].

Following O'Neill [23], we define tensor fields $\mathcal{A}$ and $\mathcal{T}$ by

$$\mathcal{A}_{E_1} E_2 = \mathcal{H} \nabla_{\mathcal{H} E_1} \mathcal{V} E_2 + \mathcal{V} \nabla_{\mathcal{H} E_1} \mathcal{H} E_2, \tag{2.6}$$

$$\mathcal{T}_{E_1} E_2 = \mathcal{H} \nabla_{\mathcal{V} E_1} \mathcal{V} E_2 + \mathcal{V} \nabla_{\mathcal{V} E_1} \mathcal{H} E_2 \tag{2.7}$$

for all $E_1, E_2 \in \Gamma(TM)$. Hence, for $U_1, U_2 \in \Gamma(\ker \pi_*)$ and $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$, we have

$$\nabla_{U_1} U_2 = \mathcal{T}_{U_1} U_2 + \widehat{\nabla}_{U_1} U_2, \tag{2.8}$$

$$\nabla_{U_1} X_1 = \mathcal{H} \nabla_{U_1} X_1 + \mathcal{T}_{U_1} X_1, \tag{2.9}$$

$$\nabla_{X_1} U_1 = \mathcal{A}_{X_1} U_1 + \mathcal{V} \nabla_{X_1} U_1, \tag{2.10}$$

$$\nabla_{X_1} X_2 = \mathcal{H} \nabla_{X_1} X_2 + \mathcal{A}_{X_1} X_2, \tag{2.11}$$

where $\widehat{\nabla}_{U_1} U_2 = \mathcal{V} \nabla_{U_1} U_2$ [23, 9, 19]. Moreover, for each $p \in M$, $U \in \ker \pi_{*p}$ and $X \in (\ker \pi_{*p})^\perp$, the endomorphisms $\mathcal{T}_U$ and $\mathcal{A}_X$ are skew-symmetric, that is,

$$g(\mathcal{A}_X E_1, E_2) = -g(E_1, \mathcal{A}_X E_2), \qquad g(\mathcal{T}_U E_1, E_2) = -g(E_1, \mathcal{T}_U E_2) \tag{2.12}$$

for all $E_1, E_2 \in T_p M$ [23, 9].

The fibers of π are said to be totally umbilical if there exists a horizontal vector field H such that

$$\mathcal{T}_{U_1} U_2 = g(U_1, U_2) H \tag{2.13}$$

for all $U_1, U_2 \in \Gamma(\ker \pi_*)$ [5, 19]. The vector field H is the mean curvature vector field of the fibers and is given by

$$(2m - n) H = \sum_{i=1}^{2m-n} \mathcal{T}_{U_i} U_i, \tag{2.14}$$

where $\{U_i\}_{i=1}^{2m-n}$ is a local orthonormal frame of $\ker \pi_*$ [9, 19]. In particular, the fibers are minimal if and only if $H = 0$ [5, 9].

For a conformal submersion, the tensor $\mathcal{A}$ satisfies

$$\mathcal{A}_{X_1} X_2 = \frac{1}{2} \left\{ \mathcal{V}[X_1, X_2] - \lambda^2 g(X_1, X_2) \nabla^{\mathcal{V}} \left(\frac{1}{\lambda^2} \right) \right\} \tag{2.15}$$

for all $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$, where $\nabla^{\mathcal{V}}(1/\lambda^2) = \mathcal{V}\big(\mathrm{grad}(1/\lambda^2)\big)$ [19]. In particular, $(\ker \pi_*)^\perp$ is totally geodesic if and only if λ is constant along the fibers.

The second fundamental form of π is defined by

$$(\nabla \pi_*)(E_1, E_2) = \nabla^{\pi}_{E_1} \pi_* E_2 - \pi_*(\nabla_{E_1} E_2), \tag{2.16}$$

where ∇^{π} denotes the pull-back connection [5, 11]. If X_1, X_2 are basic horizontal vector fields, then

$$\begin{aligned} \pi_*(\mathcal{H}\nabla_{X_1} X_2) =& \nabla^N_{\pi_* X_1} \pi_* X_2 + \frac{\lambda^2}{2} \left\{ X_1 \left(\frac{1}{\lambda^2} \right) \pi_* X_2 + X_2 \left(\frac{1}{\lambda^2} \right) \pi_* X_1 \right. \\ & \left. - g(X_1, X_2) \pi_* \left(\operatorname{grad}^{\mathcal{H}} \frac{1}{\lambda^2} \right) \right\}. \end{aligned} \tag{2.17}$$

and consequently

$$(\nabla \pi_*)(X_1, X_2) = -\frac{\lambda^2}{2} \left\{ X_1 \left(\frac{1}{\lambda^2} \right) \widetilde{X}_2 + X_2 \left(\frac{1}{\lambda^2} \right) \widetilde{X}_1 - g(X_1, X_2) \pi_* \left(\operatorname{grad}^{\mathcal{H}} \frac{1}{\lambda^2} \right) \right\}, \tag{2.18}$$

where $\widetilde{X}_i$ denotes the π-related vector field on N [19].

For later use, we also record

$$(\nabla \pi_*)(U_1, U_2) = -\pi_*(\mathcal{T}_{U_1} U_2), \tag{2.19}$$

$$(\nabla \pi_*)(X_1, U_1) = -\pi_*(\nabla_{X_1} U_1) = -\pi_*(\mathcal{A}_{X_1} U_1) \tag{2.20}$$

for $X_1 \in \Gamma((\ker \pi_*)^{\perp})$ and $U_1, U_2 \in \Gamma(\ker \pi_*)$ [17].

Now we recall the notion of conformal hemi-slant submersion in the almost Hermitian manifolds [16]. A horizontally conformal submersion $\pi : (M, g, J) \to (N, g_N)$ is called a conformal hemi-slant submersion if the vertical distribution $\ker \pi_*$ admits two orthogonal complementary distributions D_θ and $D^{\perp}$ such that

$$\ker \pi_* = D_\theta \oplus D^{\perp}, \tag{2.21}$$

where D_θ is a slant distribution with slant angle θ and $D^{\perp}$ is anti-invariant, that is,

$$J(D^{\perp}) \subset (\ker \pi_*)^{\perp}. \tag{2.22}$$

The angle θ is called the hemi-slant angle. We say that π is proper if $D^{\perp} \neq \{0\}$ and $\theta \notin \{0, \frac{\pi}{2}\}$ [16].

For each $U \in \Gamma(\ker \pi_*)$, we write

$$U = PU + QU, \tag{2.23}$$

where $PU \in \Gamma(D_\theta)$ and $QU \in \Gamma(D^{\perp})$. Moreover, for $U \in \Gamma(\ker \pi_*)$ and $X \in \Gamma((\ker \pi_*)^{\perp})$, we set

$$JU = \phi U + \omega U, \tag{2.24}$$

$$JX = BX + CX, \tag{2.25}$$

where ϕU, $BX \in \Gamma(\ker \pi_*)$ and ωU, $CX \in \Gamma((\ker \pi_*)^{\perp})$ [16, 30, 31].

The horizontal distribution decomposes as

$$(\ker \pi_*)^{\perp} = \omega D_\theta \oplus JD^{\perp} \oplus \mu, \tag{2.26}$$

where μ is the orthogonal complement of $\omega D_\theta \oplus JD^{\perp}$ in $(\ker \pi_*)^{\perp}$; moreover, μ is J-invariant [16].

The following elementary relations will be used repeatedly:

$$\phi(D_\theta) = D_\theta, \qquad \phi(D^\perp) = \{0\}, \qquad B(\omega D_\theta) = D_\theta, \qquad B(JD^\perp) = D^\perp. \tag{2.27}$$

Also, from (2.1), one obtains

$$\phi^2 U + B\omega U = -U, \tag{2.28}$$

$$\omega\phi U + C\omega U = 0, \tag{2.29}$$

$$\phi BX + BCX = 0, \tag{2.30}$$

$$\omega BX + C^2 X = -X \tag{2.31}$$

for all $U \in \Gamma(\ker \pi_*)$ and $X \in \Gamma((\ker \pi_*)^\perp)$ [16].

For the slant part, one has

$$\phi^2 W = -(\cos^2\theta)\, W \tag{2.32}$$

for all $W \in \Gamma(D_\theta)$ [16, 31]. Equivalently,

$$g(\phi W_1, \phi W_2) = \cos^2\theta\, g(W_1, W_2), \qquad g(\omega W_1, \omega W_2) = \sin^2\theta\, g(W_1, W_2) \tag{2.33}$$

for all $W_1, W_2 \in \Gamma(D_\theta)$ [16].

Finally, for $f \in C^\infty(M)$, the gradient, divergence and Laplacian are defined by

$$g(\operatorname{grad} f, E) = E(f), \tag{2.34}$$

$$\operatorname{div}(E) = \sum_{i=1}^{2m} g(\nabla_{E_i} E, E_i), \tag{2.35}$$

$$\Delta f = \operatorname{div}(\operatorname{grad} f), \tag{2.36}$$

where $\{E_i\}_{i=1}^{2m}$ is a local orthonormal frame on M [18, 8].

3. Clairaut Conformal Hemi-Slant Submersions

In this section, we investigate Clairaut conformal hemi-slant submersions from Kähler manifolds onto Riemannian manifolds and establish their basic geometric characterizations.

Definition 3.1. *Let $\pi : (M, g, J) \to (N, g_N)$ be a conformal hemi-slant submersion. Then π is called a Clairaut conformal hemi-slant submersion if there exists a positive smooth function r on M such that for every geodesic $\alpha : I \to M$, parametrized by arc length, the quantity*

$$(r \circ \alpha)(t) \sin \omega(t) \tag{3.37}$$

is constant along α, where $\omega(t)$ denotes the angle between the tangent vector $\dot\alpha(t)$ and the horizontal distribution $(\ker \pi_*)^\perp$ at the point $\alpha(t) \in M$.

We observe that Definition 3.1 is independent of the parametrization of the geodesic. Indeed, the angle $\omega(t) \in [0, \pi/2]$ depends only on the direction of the velocity vector field $\dot\alpha(t)$ relative to the horizontal distribution. Hence, in what follows, we may assume that α is parametrized by arc length.

Proposition 3.1. *Let $\pi : (M, g, J) \to (N, g_N)$ be a conformal hemi-slant submersion with dilation λ, and let $\alpha : I \to M$ be a regular curve parametrized by arc length. Suppose that*

$$\dot\alpha = X + U, \tag{3.38}$$

where $X \in \Gamma((\ker \pi_*)^{\perp})$, $U \in \Gamma(\ker \pi_*)$. *Then* α *is a geodesic on* M *if and only if*

$$\mathcal{V}\nabla_X U + \mathcal{T}_U X + \widehat{\nabla}_U U - \frac{\lambda^2}{2}\, g(X,X)\, \nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right) = 0 \tag{3.39}$$

and

$$\mathcal{H}\nabla_X X + 2\mathcal{A}_X U + \mathcal{T}_U U = 0. \tag{3.40}$$

Proof. Using (3.38), we compute

$$\nabla_{\dot\alpha}\dot\alpha = \nabla_X X + \nabla_X U + \nabla_U X + \nabla_U U. \tag{3.41}$$

By (2.11), (2.10), (2.9), and (2.8), we have

$$\begin{aligned}\nabla_{\dot\alpha}\dot\alpha = &\big(\mathcal{A}_X X + \mathcal{V}\nabla_X U + \mathcal{T}_U X + \widehat{\nabla}_U U\big)\\ &+ \big(\mathcal{H}\nabla_X X + \mathcal{A}_X U + \mathcal{H}\nabla_U X + \mathcal{T}_U U\big).\end{aligned} \tag{3.42}$$

Since α is geodesic if and only if $\nabla_{\dot\alpha}\dot\alpha = 0$, the vertical and horizontal components of (3.42) must vanish. For the vertical component, using (2.15) with $X_1 = X_2 = X$ and $[X, X] = 0$, we get

$$\mathcal{A}_X X = -\frac{\lambda^2}{2}\, g(X,X)\, \nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right). \tag{3.43}$$

Substituting (3.43) into the vertical component of (3.42), we obtain (3.39). For the horizontal component, by (2.9), we have $\mathcal{H}\nabla_U X = \mathcal{A}_X U$. Thus the horizontal component of (3.42) becomes (3.40). This completes the proof. □

Theorem 3.1. *Let* $\pi : (M, g, J) \to (N, g_N)$ *be a conformal hemi-slant submersion with connected fibers and dilation* λ. *Then the following assertions are equivalent:*

(1) π *is a Clairaut conformal hemi-slant submersion with* $r = e^f$.

(2) *For every geodesic* $\alpha : I \to M$ *with* (3.38), *one has*

$$g(U,U)g(\dot\alpha, \operatorname{grad} f) + \frac{\lambda^2}{2} g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right) g(X,X) + g(\mathcal{T}_U U, X) = 0. \tag{3.44}$$

(3) *Writing* $U = PU + QU$ *as in* (2.23), *the identity*

$$\begin{aligned}&g(\mathcal{T}_{PU}PU, X) + g(\mathcal{T}_{QU}QU, X) + 2g(\mathcal{T}_{PU}QU, X)\\ &\quad + \Big(\sec^2\theta\, g(\phi PU, \phi PU) + g(\omega QU, \omega QU)\Big) g(\dot\alpha, \operatorname{grad} f)\\ &\quad + \frac{\lambda^2}{2} g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), PU + QU\right) g(X,X) = 0\end{aligned} \tag{3.45}$$

holds along every geodesic α.

(4) $\operatorname{grad} f$ *is horizontal, the fibers of* π *are totally umbilical, and*

$$H = -\operatorname{grad} f, \tag{3.46}$$

while λ *is constant along the fibers.*

Proof. We first prove $(i) \iff (ii)$. Let $\alpha : I \to M$ be an arc-length geodesic and write $\dot\alpha$ as in (3.38). Let $\omega(t)$ be the angle between $\dot\alpha$ and $(\ker \pi_*)^{\perp}$. Then

$$g(X,X) = \cos^2\omega, \qquad g(U,U) = \sin^2\omega. \tag{3.47}$$

Using the vertical geodesic equation (3.39), together with (2.15), we get

$$g(U,U)g(\dot{\alpha},\operatorname{grad} f) = g(\mathcal{A}_X X + \mathcal{T}_U X, U) \tag{3.48}$$

if and only if

$$\frac{d}{dt}\big((e^f \circ \alpha)\sin\omega\big) = 0.$$

Indeed, differentiating $g(U,U) = \sin^2\omega$ and using (3.39) gives

$$g(\mathcal{A}_X X + \mathcal{T}_U X, U) = -\sin\omega\cos\omega\,\omega',$$

whereas the Clairaut relation gives

$$g(U,U)g(\dot{\alpha},\operatorname{grad} f) = -\sin\omega\cos\omega\,\omega'.$$

Thus the two identities are equivalent. Now, by (2.12),

$$g(\mathcal{T}_U X, U) = -g(\mathcal{T}_U U, X),$$

and by (2.15) with $X_1 = X_2 = X$,

$$g(\mathcal{A}_X X, U) = -\frac{\lambda^2}{2} g(X,X) g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right).$$

Substituting these two identities into (3.48) gives precisely (3.44). Hence $(i) \iff (ii)$.

Next we prove $(ii) \iff (iii)$. Since the decomposition (2.21) is orthogonal and from (2.23), we have

$$g(U,U) = g(PU,PU) + g(QU,QU).$$

For $PU \in D_\theta$, (2.33) gives

$$g(PU,PU) = \sec^2\theta\, g(\phi PU, \phi PU).$$

For $QU \in D^\perp$, (2.27), (2.24), and (2.2) give

$$g(QU,QU) = g(\omega QU, \omega QU).$$

Moreover, by bilinearity and symmetry of $\mathcal{T}$ on vertical vector fields,

$$\mathcal{T}_U U = \mathcal{T}_{PU} PU + \mathcal{T}_{QU} QU + 2\mathcal{T}_{PU} QU.$$

Substituting these identities into (3.44) yields (3.45). Thus $(ii) \iff (iii)$.

It remains to prove $(ii) \iff (iv)$. Assume first that (3.44) holds. At a point $p \in M$, take a geodesic with initial vector $U + X$, where U is vertical and X is horizontal. Evaluating (3.44) at p gives

$$g(\mathcal{T}_U U, X) = -g(U,U)g(U+X,\operatorname{grad} f) - \frac{\lambda^2}{2} g(X,X) g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right). \tag{3.49}$$

Taking $X = 0$ in (3.49) gives $g(U,\operatorname{grad} f) = 0$ for every vertical U; hence $\operatorname{grad} f$ is horizontal. Therefore (3.49) becomes

$$g(\mathcal{T}_U U, X) = -g(U,U)g(X,\operatorname{grad} f) - \frac{\lambda^2}{2} g(X,X) g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right).$$

Replacing X by cX and comparing the coefficients of c and c^2, we obtain

$$g(\mathcal{T}_U U, X) = -g(U,U)g(X,\operatorname{grad} f)$$

and

$$g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right) = 0.$$

Thus λ is constant along the fibers. Polarizing the first identity gives

$$g(\mathcal{T}_U V, X) = -g(U, V) g(X, \operatorname{grad} f)$$

for all vertical U, V and horizontal X. Comparing this with (2.13), the fibers are totally umbilical and (3.46) hold. Conversely, assume (iv). Then (2.13) and (3.46) give

$$g(\mathcal{T}_U U, X) = -g(U, U) g(X, \operatorname{grad} f).$$

Since $\operatorname{grad} f$ is horizontal, (3.38) implies

$$g(\dot{\alpha}, \operatorname{grad} f) = g(X, \operatorname{grad} f).$$

Since λ is constant along the fibers,

$$\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right) = 0.$$

Substitution into (3.44) gives zero identically. Hence (3.44) holds, and the proof is complete. □

Corollary 3.1. *Let $\pi : (M, g, J) \to (N, g_N)$ be a Clairaut conformal hemi-slant submersion with connected fibers, dilation λ, hemi-slant angle θ, and $r = e^f$. Define $\psi : N \to (0, \infty)$ by*

$$\psi \circ \pi = \frac{1}{\lambda}. \tag{3.50}$$

Then $\pi : (M, g, J) \to (N, \psi^2 g_N)$ is a Clairaut Riemannian hemi-slant submersion with the same Clairaut function $r = e^f$ and the same hemi-slant angle θ.

Proof. By Theorem 3.1, λ is constant along the fibers. Since the fibers are connected, ψ is well defined by (3.50). For $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$, (2.5) gives

$$(\psi^2 g_N)(\pi_* X_1, \pi_* X_2) = \frac{1}{\lambda^2} g_N(\pi_* X_1, \pi_* X_2) = g(X_1, X_2).$$

Thus $\pi : (M, g, J) \to (N, \psi^2 g_N)$ is a Riemannian submersion. Since neither π_* nor the metric g on M is changed, the vertical distribution, the horizontal distribution, and the decomposition (2.21) are unchanged. Hence the anti-invariant condition (2.22) and the slant relation (2.33) remain valid with the same angle θ. Therefore the resulting Riemannian submersion is hemi-slant. Finally, the Clairaut condition (3.37) is unchanged, since the geodesics of (M, g) and the angle $\omega(t)$ with the horizontal distribution are not affected by the conformal change of the target metric. Hence the new Riemannian hemi-slant submersion is Clairaut with $r = e^f$. □

Proposition 3.2. *Let $\pi : (M, g, J) \to (N, g_N)$ be a Clairaut conformal hemi-slant submersion and let $\alpha : I \to M$ be a geodesic with decomposition* (3.38)*. If $U \in D_\theta$, then along α one has*

$$g(\mathcal{T}_U U, X) + \sec^2\theta \, g(\phi U, \phi U) \, g(\dot{\alpha}, \nabla f) + \frac{\lambda^2}{2} g(X, X) \, g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right) = 0 \tag{3.51}$$

if and only if the Clairaut condition holds. Equivalently,

$$g(\mathcal{T}_U U, X) + \csc^2\theta\, g(\omega U, \omega U)\, g(\dot{\alpha}, \nabla f) + \frac{\lambda^2}{2}\, g(X, X)\, g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right) = 0. \tag{3.52}$$

Proof. Since $U \in D_\theta$, the Clairaut condition (3.44) is

$$g(\mathcal{T}_U U, X) + g(U, U) g(\dot{\alpha}, \nabla f) + \frac{\lambda^2}{2} g(X, X) g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right) = 0.$$

Putting $W_1 = W_2 = U$ in (2.33), we obtain

$$g(\phi U, \phi U) = \cos^2\theta\, g(U, U), \qquad g(\omega U, \omega U) = \sin^2\theta\, g(U, U).$$

Hence

$$g(U, U) = \sec^2\theta\, g(\phi U, \phi U), \qquad g(U, U) = \csc^2\theta\, g(\omega U, \omega U).$$

Substitution of these identities into (3.44) gives (3.51) and (3.52), respectively. □

Corollary 3.2. *Let $\alpha : I \to M$ be a geodesic with decomposition* (3.38)*. If $U \in D^\perp$, then the Clairaut condition* (3.44) *is equivalent to*

$$g(\mathcal{T}_U U, X) + g(\omega U, \omega U)\, g(\dot{\alpha}, \nabla f) + \frac{\lambda^2}{2}\, g(X, X)\, g\left(\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right), U\right) = 0. \tag{3.53}$$

Proof. Since $U \in D^\perp$, by (2.27) we have $\phi U = 0$. Thus, from (2.24), we get $JU = \omega U$. Using (2.2), it follows that

$$g(U, U) = g(JU, JU) = g(\omega U, \omega U).$$

Substituting this identity into (3.44), we obtain (3.53). □

Remark 3.1. *Proposition 3.2 and Corollary 3.2 show that the vertical contribution in the Clairaut condition is governed by different geometric quantities on the two summands of* (2.21)*. On the slant distribution D_θ, it is controlled by $g(\phi U, \phi U)$ or equivalently by $g(\omega U, \omega U)$ through the angle θ via* (2.33)*. On the anti-invariant distribution $D^\perp$, the same term is determined solely by $g(\omega U, \omega U)$, since $\phi U = 0$.*

Theorem 3.2. *Let $\pi : (M, g, J) \to (N, g_N)$ be a Clairaut conformal hemi-slant submersion with connected fibers and $r = e^f$. Let $\alpha : I \to M$ be a geodesic with decomposition* (3.38) *and* (2.23)*. Then the Clairaut condition* (3.44) *is equivalent to*

$$\begin{aligned} &g(\mathcal{T}_{PU} PU, X) + g(\mathcal{T}_{QU} QU, X) + 2g(\mathcal{T}_{PU} QU, X) \\ &\quad + \Big(\sec^2\theta\, g(\phi PU, \phi PU) + g(\omega QU, \omega QU)\Big) g(\dot{\alpha}, \nabla f) = 0. \end{aligned} \tag{3.54}$$

Equivalently,

$$\begin{aligned} &g(\mathcal{T}_{PU} PU, X) + g(\mathcal{T}_{QU} QU, X) + 2g(\mathcal{T}_{PU} QU, X) \\ &\quad + \Big(\csc^2\theta\, g(\omega PU, \omega PU) + g(\omega QU, \omega QU)\Big) g(\dot{\alpha}, \nabla f) = 0. \end{aligned} \tag{3.55}$$

Proof. Since π is a Clairaut conformal hemi-slant submersion with connected fibers, Theorem 3.1 implies that λ is constant along the fibers. Hence

$$\nabla^{\mathcal{V}}\left(\frac{1}{\lambda^2}\right) = 0. \tag{3.56}$$

From (3.44) and (3.56), we obtain

$$g(\mathcal{T}_U U, X) + g(U, U)g(\dot{\alpha}, \nabla f) = 0. \tag{3.57}$$

Now, bilinearity of $\mathcal{T}$, and the symmetry of $\mathcal{T}$ on vertical vectors, we get

$$\begin{aligned}\mathcal{T}_U U &= \mathcal{T}_{PU+QU}(PU + QU)\\ &= \mathcal{T}_{PU}PU + \mathcal{T}_{PU}QU + \mathcal{T}_{QU}PU + \mathcal{T}_{QU}QU\\ &= \mathcal{T}_{PU}PU + 2\mathcal{T}_{PU}QU + \mathcal{T}_{QU}QU.\end{aligned} \tag{3.58}$$

Taking the inner product with X, we obtain

$$g(\mathcal{T}_U U, X) = g(\mathcal{T}_{PU}PU, X) + g(\mathcal{T}_{QU}QU, X) + 2g(\mathcal{T}_{PU}QU, X). \tag{3.59}$$

On the other hand, since the decomposition (2.21) is orthogonal,

$$g(U, U) = g(PU, PU) + g(QU, QU). \tag{3.60}$$

Because $PU \in D_\theta$, putting $W_1 = W_2 = PU$ in (2.33), we have

$$g(\phi PU, \phi PU) = \cos^2\theta\, g(PU, PU) \tag{3.61}$$

and

$$g(\omega PU, \omega PU) = \sin^2\theta\, g(PU, PU). \tag{3.62}$$

Therefore

$$g(PU, PU) = \sec^2\theta\, g(\phi PU, \phi PU) \tag{3.63}$$

and

$$g(PU, PU) = \csc^2\theta\, g(\omega PU, \omega PU). \tag{3.64}$$

Since $QU \in D^\perp$, (2.27) gives

$$\phi QU = 0.$$

Hence, by (2.24),

$$JQU = \omega QU.$$

Using (2.2), we get

$$g(QU, QU) = g(\omega QU, \omega QU). \tag{3.65}$$

Substituting (3.63) and (3.65) into (3.60), we obtain

$$g(U, U) = \sec^2\theta\, g(\phi PU, \phi PU) + g(\omega QU, \omega QU). \tag{3.66}$$

Similarly, from (3.64) and (3.65), we obtain

$$g(U, U) = \csc^2\theta\, g(\omega PU, \omega PU) + g(\omega QU, \omega QU). \tag{3.67}$$

Finally, substituting (3.59) and (3.66) into (3.57), we obtain (3.54). Likewise, substituting (3.59) and (3.67) into (3.57), we obtain (3.55). This completes the proof. □

Lemma 3.1. *Let $\pi : (M, g, J) \to (N, g_N)$ be a Clairaut conformal hemi-slant submersion with connected fibers, dilation λ, and $r = e^f$. Let φ be a positive smooth function on M such that*

$$\mathcal{V}(\operatorname{grad}\varphi) = 0. \tag{3.68}$$

Then $\widetilde{\pi} : (M, \varphi^2 g, J) \to (N, g_N)$, $\widetilde{\pi}(x) = \pi(x)$, *is a Clairaut conformal hemi-slant submersion with dilation*

$$\widetilde{\lambda} = \frac{\lambda}{\varphi} \tag{3.69}$$

and Clairaut function

$$\widetilde{r} = \varphi e^f. \tag{3.70}$$

Proof. Let $\widetilde{g} = \varphi^2 g$. Since $\widetilde{\pi} = \pi$ as a smooth map, one has $\ker \widetilde{\pi}_* = \ker \pi_*$. Because $\widetilde{g}$ is conformal to g, orthogonality is preserved, and hence the horizontal distribution $(\ker \pi_*)^\perp$ is unchanged. Therefore the orthogonal decomposition (2.21) remains valid with respect to $\widetilde{g}$. Moreover, since neither J nor the vertical distribution changes, the anti-invariant condition (2.22) is preserved. For $W_1, W_2 \in \Gamma(D_\theta)$, (2.33) yields

$$\widetilde{g}(\phi W_1, \phi W_2) = \varphi^2 g(\phi W_1, \phi W_2) = \varphi^2 \cos^2\theta\, g(W_1, W_2) = \cos^2\theta\, \widetilde{g}(W_1, W_2).$$

Hence the slant relation (2.33) is preserved with the same angle θ. Thus $\widetilde{\pi}$ is again a conformal hemi-slant submersion.

Now let $X_1, X_2 \in \Gamma((\ker \pi_*)^\perp)$. By (2.5),

$$g_N(\widetilde{\pi}_* X_1, \widetilde{\pi}_* X_2) = g_N(\pi_* X_1, \pi_* X_2) = \lambda^2 g(X_1, X_2) = \frac{\lambda^2}{\varphi^2}\widetilde{g}(X_1, X_2).$$

Thus $\widetilde{\pi}$ is horizontally conformal with dilation (3.69).

By Theorem 3.1, the fibers are totally umbilical and satisfy (3.46). Under the conformal change $\widetilde{g} = \varphi^2 g$, condition (3.68) implies

$$\widetilde{H} = -\widetilde{\mathrm{grad}}(f + \log\varphi).$$

Moreover, (3.68) and Theorem 3.1 imply that both φ and λ are constant along the fibers. Hence $\widetilde{\lambda} = \lambda/\varphi$ is constant along the fibers. Applying Theorem 3.1 once more, we conclude that $\widetilde{\pi}$ is Clairaut with

$$\widetilde{r} = e^{f + \log\varphi} = \varphi e^f.$$

□

The following result refines the vertical curvature formula for Clairaut conformal submersions by taking into account the hemi-slant decomposition of the vertical distribution. While the global scalar curvature identity is the vertical trace of the usual Clairaut conformal submersion formula, the identities below split this trace into its slant, anti-invariant and mixed hemi-slant components.

Theorem 3.3. *Let* $\pi : (M, g, J) \to (N, g_N)$ *be a Clairaut conformal hemi-slant submersion with connected fibers, dilation* λ*, and* $r = e^f$*. Put* $q = \dim(\ker \pi_*)$, $p = \dim D_\theta$, $s = \dim D^\perp$, *so that* $q = p + s$*. For any orthonormal vertical vector fields* $U, V \in \Gamma(\ker \pi_*)$*, one has*

$$K(U, V) = \widehat{K}(U, V) - \|\nabla f\|^2, \tag{3.71}$$

where K *denotes the sectional curvature of* M *and* $\widehat{K}$ *denotes the sectional curvature of the corresponding fiber. Consequently, with respect to the hemi-slant decomposition* (2.21)*, the vertical scalar curvature splits as*

$$K^{\mathcal{V}} = \widehat{K} - q(q-1)\|\nabla f\|^2, \tag{3.72}$$

and more precisely,

$$K^{D_\theta} = \widehat{K}^{D_\theta} - p(p-1)\|\nabla f\|^2, \tag{3.73}$$

$$K^{D^\perp} = \widehat{K}^{D^\perp} - s(s-1)\|\nabla f\|^2, \tag{3.74}$$

$$K^{D_\theta, D^\perp} = \widehat{K}^{D_\theta, D^\perp} - 2ps\|\nabla f\|^2. \tag{3.75}$$

Hence

$$K^{\mathcal{V}} = K^{D_\theta} + K^{D^\perp} + K^{D_\theta, D^\perp}. \tag{3.76}$$

In particular, for every unit vector field $W \in \Gamma(D_\theta)$ *with* $\cos\theta \neq 0$,

$$K\left(W, \frac{1}{\cos\theta}\phi W\right) = \widehat{K}\left(W, \frac{1}{\cos\theta}\phi W\right) - \|\nabla f\|^2. \tag{3.77}$$

Proof. Since π is a Clairaut conformal hemi-slant submersion, Theorem 3.1 gives that ∇f is horizontal, the fibers are totally umbilical, (3.46) holds, and λ is constant along the fibers. In particular, the conformal correction term in (2.15) vanishes for $X_1 = X_2$ along the Clairaut fibers. By (2.13) and (3.46), for all vertical vector fields $U, V \in \Gamma(\ker \pi_*)$ we have

$$\mathcal{T}_U V = -g(U, V)\nabla f. \tag{3.78}$$

Let U, V be orthonormal vertical vector fields. The Gauss equation for the fibers gives

$$K(U, V) = \widehat{K}(U, V) + g(\mathcal{T}_U V, \mathcal{T}_V U) - g(\mathcal{T}_U U, \mathcal{T}_V V). \tag{3.79}$$

Using (3.78), we obtain

$$\mathcal{T}_U V = 0, \qquad \mathcal{T}_U U = -\nabla f, \qquad \mathcal{T}_V V = -\nabla f.$$

Substitution into (3.79) gives (3.71). Now choose a local orthonormal frame adapted to (2.21),

$$\{E_1, \dots, E_p, Z_1, \dots, Z_s\},$$

where $E_a \in \Gamma(D_\theta)$ and $Z_r \in \Gamma(D^\perp)$. Taking the trace of (3.71) over all ordered distinct vertical pairs gives (3.72). Restricting the same trace to the ordered pairs inside D_θ gives (3.73), restricting it to the ordered pairs inside $D^\perp$ gives (3.74), and taking the ordered mixed pairs (E_a, Z_r) and (Z_r, E_a) gives (3.75). Summing these three contributions yields (3.76). Finally, let $W \in \Gamma(D_\theta)$ be unit. By (2.33),

$$g(\phi W, \phi W) = \cos^2\theta.$$

Moreover, by (2.2), $g(W, \phi W) = 0$. Hence

$$\left\{W, \frac{1}{\cos\theta}\phi W\right\}$$

is an orthonormal pair in D_θ. Applying (3.71) to this pair gives (3.77). □

Corollary 3.3. *Let* $\pi : (M, g, J) \to (N, g_N)$ *be a Clairaut conformal hemi-slant submersion with connected fibers, dilation* λ, *and* $r = e^f$. *Let* $q = \dim(\ker \pi_*)$, $p = \dim D_\theta$ *and* $s = \dim D^\perp$. *Then, for any unit vector field* $E \in \Gamma(D_\theta)$,

$$\mathrm{Ric}^{\mathcal{V}}(E, E) = \widehat{\mathrm{Ric}}(E, E) - (q-1)\|\nabla f\|^2. \tag{3.80}$$

More precisely,

$$\mathrm{Ric}^{D_\theta}(E,E) = \widehat{\mathrm{Ric}}^{D_\theta}(E,E) - (p-1)\|\nabla f\|^2, \tag{3.81}$$

$$\mathrm{Ric}^{D_\theta,D^\perp}(E,E) = \widehat{\mathrm{Ric}}^{D_\theta,D^\perp}(E,E) - s\|\nabla f\|^2. \tag{3.82}$$

Similarly, for any unit vector field $Z \in \Gamma(D^\perp)$,

$$\mathrm{Ric}^{\mathcal{V}}(Z,Z) = \widehat{\mathrm{Ric}}(Z,Z) - (q-1)\|\nabla f\|^2, \tag{3.83}$$

with the refined decomposition

$$\mathrm{Ric}^{D^\perp}(Z,Z) = \widehat{\mathrm{Ric}}^{D^\perp}(Z,Z) - (s-1)\|\nabla f\|^2, \tag{3.84}$$

$$\mathrm{Ric}^{D^\perp,D_\theta}(Z,Z) = \widehat{\mathrm{Ric}}^{D^\perp,D_\theta}(Z,Z) - p\|\nabla f\|^2. \tag{3.85}$$

Proof. Let

$$\{E_1,\dots,E_p,Z_1,\dots,Z_s\}$$

be a local orthonormal frame adapted to the hemi-slant decomposition (2.21). By Theorem 3.3, for any orthonormal vertical pair U,V, we have

$$K(U,V) = \widehat{K}(U,V) - \|\nabla f\|^2.$$

Fix a unit vector field $E \in \Gamma(D_\theta)$. Taking the trace over all vertical basis vectors orthogonal to E, we obtain

$$\mathrm{Ric}^{\mathcal{V}}(E,E) = \widehat{\mathrm{Ric}}(E,E) - (q-1)\|\nabla f\|^2,$$

which gives (3.80). If the trace is restricted to the D_θ-directions only, there are $p-1$ terms, and hence (3.81) follows. If the trace is taken over the $D^\perp$-directions, there are s terms, which gives (3.82).

The proof for a unit vector field $Z \in \Gamma(D^\perp)$ is identical: tracing first over all vertical directions gives (3.83), tracing over $D^\perp$ gives (3.84), and tracing over D_θ gives (3.85). □

We first record the global harmonicity criterion in the present setting. This criterion is the Clairaut conformal submersion harmonicity condition specialized to the hemi-slant case. The contribution of the hemi-slant structure appears in the refined decomposition of the tension field given in the next theorem.

Theorem 3.4. *Let $\pi : (M^{2m}, g, J) \to (N^n, g_N)$ be a Clairaut conformal hemi-slant submersion with connected fibers, dilation λ, and $r = e^f$. Put*

$$q = \dim(\ker \pi_*) = 2m - n.$$

Then π is harmonic if and only if

$$q \,\mathrm{grad}\, f = (n-2)\,\mathrm{grad}^{\mathcal{H}}(\log \lambda). \tag{3.86}$$

Proof. Let $\{U_1,\dots,U_q\}$ be a local orthonormal frame of $\ker \pi_*$ and let $\{X_1,\dots,X_n\}$ be a local orthonormal frame of $(\ker \pi_*)^\perp$ consisting of basic vector fields. By (2.16), the tension field is

$$\tau(\pi) = \sum_{i=1}^{q} (\nabla \pi_*)(U_i,U_i) + \sum_{a=1}^{n} (\nabla \pi_*)(X_a,X_a).$$

For the vertical part, using (2.19) and (2.14), we obtain

$$\sum_{i=1}^{q}(\nabla\pi_*)(U_i,U_i) = -\pi_*\left(\sum_{i=1}^{q}\mathcal{T}_{U_i}U_i\right) = -q\,\pi_* H. \tag{3.87}$$

Since π is Clairaut, Theorem 3.1 gives (3.46). Hence

$$\sum_{i=1}^{q}(\nabla\pi_*)(U_i,U_i) = q\,\pi_*(\operatorname{grad} f). \tag{3.88}$$

For the horizontal part, applying (2.17) with $X_1 = X_2 = X_a$, we have

$$\begin{aligned}\pi_*(\mathcal{H}\nabla_{X_a}X_a) = \nabla^N_{\pi_*X_a}\pi_*X_a + \lambda^2 X_a\left(\frac{1}{\lambda^2}\right)\pi_*X_a \\ -\frac{\lambda^2}{2}\pi_*\left(\operatorname{grad}^{\mathcal{H}}\frac{1}{\lambda^2}\right).\end{aligned} \tag{3.89}$$

On the other hand, by the definition of the second fundamental form (2.16),

$$(\nabla\pi_*)(X_a,X_a) = \nabla^N_{\pi_*X_a}\pi_*X_a - \pi_*(\nabla_{X_a}X_a).$$

Since $\pi_*(\nabla_{X_a}X_a) = \pi_*(\mathcal{H}\nabla_{X_a}X_a)$, it follows from (3.89) that

$$(\nabla\pi_*)(X_a,X_a) = -\lambda^2 X_a\left(\frac{1}{\lambda^2}\right)\pi_*X_a + \frac{\lambda^2}{2}\pi_*\left(\operatorname{grad}^{\mathcal{H}}\frac{1}{\lambda^2}\right). \tag{3.90}$$

Equivalently, this is the special case $X_1 = X_2 = X_a$ of (2.18). Summing (3.90) over $a = 1,\dots,n$, we get

$$\begin{aligned}\sum_{a=1}^{n}(\nabla\pi_*)(X_a,X_a) = -\lambda^2\sum_{a=1}^{n}X_a\left(\frac{1}{\lambda^2}\right)\pi_*X_a \\ +\frac{n\lambda^2}{2}\pi_*\left(\operatorname{grad}^{\mathcal{H}}\frac{1}{\lambda^2}\right).\end{aligned} \tag{3.91}$$

By (2.34),

$$\operatorname{grad}^{\mathcal{H}}\left(\frac{1}{\lambda^2}\right) = \sum_{a=1}^{n}X_a\left(\frac{1}{\lambda^2}\right)X_a.$$

Therefore

$$\sum_{a=1}^{n}X_a\left(\frac{1}{\lambda^2}\right)\pi_*X_a = \pi_*\left(\operatorname{grad}^{\mathcal{H}}\frac{1}{\lambda^2}\right).$$

Substituting this into (3.91), we obtain

$$\sum_{a=1}^{n}(\nabla\pi_*)(X_a,X_a) = \frac{(n-2)\lambda^2}{2}\pi_*\left(\operatorname{grad}^{\mathcal{H}}\frac{1}{\lambda^2}\right). \tag{3.92}$$

Since

$$\frac{\lambda^2}{2}\operatorname{grad}^{\mathcal{H}}\left(\frac{1}{\lambda^2}\right) = -\operatorname{grad}^{\mathcal{H}}(\log\lambda),$$

we get from (3.88) and (3.92)

$$\tau(\pi) = \pi_*\left(q\operatorname{grad} f - (n-2)\operatorname{grad}^{\mathcal{H}}(\log\lambda)\right).$$

By Theorem 3.1, $\operatorname{grad} f$ is horizontal. Hence both $q \operatorname{grad} f$ and $(n-2)\operatorname{grad}^{\mathcal{H}}(\log\lambda)$ are horizontal vector fields. Since π_* is injective on $(\ker\pi_*)^\perp$, we conclude that

$$\tau(\pi) = 0$$

if and only if

$$q \operatorname{grad} f = (n-2)\operatorname{grad}^{\mathcal{H}}(\log\lambda).$$

This proves (3.86). ☐

Theorem 3.5. *Let $\pi : (M^{2m}, g, J) \to (N^n, g_N)$ be a Clairaut conformal hemi-slant submersion with connected fibers, dilation λ, and $r = e^f$. Put $p = \dim D_\theta$, $s = \dim D^\perp$, $q = p + s$. Let*

$$H_\theta = \frac{1}{p}\sum_{a=1}^{p} \mathcal{T}_{E_a}E_a, \qquad H_\perp = \frac{1}{s}\sum_{r=1}^{s} \mathcal{T}_{Z_r}Z_r,$$

where $\{E_1, \dots, E_p\}$ and $\{Z_1, \dots, Z_s\}$ are local orthonormal frames of D_θ and $D^\perp$, respectively. Then

$$H_\theta = H_\perp = -\operatorname{grad} f. \tag{3.93}$$

Moreover, the tension field decomposes as

$$\tau(\pi) = p\,\pi_*(\operatorname{grad} f) + s\,\pi_*(\operatorname{grad} f) - (n-2)\pi_*(\operatorname{grad}^{\mathcal{H}}\log\lambda). \tag{3.94}$$

Consequently, π is harmonic if and only if

$$p \operatorname{grad} f + s \operatorname{grad} f = (n-2)\operatorname{grad}^{\mathcal{H}}\log\lambda. \tag{3.95}$$

Equivalently,

$$q \operatorname{grad} f = (n-2)\operatorname{grad}^{\mathcal{H}}\log\lambda. \tag{3.96}$$

Proof. Let $\{E_1, \dots, E_p, Z_1, \dots, Z_s\}$ be a local orthonormal frame adapted to (2.21). Since π is Clairaut, Theorem 3.1 gives (3.46). Hence, by the totally umbilical condition (2.13),

$$\mathcal{T}_{E_a}E_a = g(E_a, E_a)H = H = -\operatorname{grad} f$$

and

$$\mathcal{T}_{Z_r}Z_r = g(Z_r, Z_r)H = H = -\operatorname{grad} f.$$

Taking traces over D_θ and $D^\perp$ separately gives

$$H_\theta = \frac{1}{p}\sum_{a=1}^{p} \mathcal{T}_{E_a}E_a = -\operatorname{grad} f, \qquad H_\perp = \frac{1}{s}\sum_{r=1}^{s} \mathcal{T}_{Z_r}Z_r = -\operatorname{grad} f,$$

which proves (3.93).

Now, using (2.19), we split the vertical part of the tension field as

$$\begin{aligned}\sum_{i=1}^{q}(\nabla\pi_*)(U_i, U_i) &= \sum_{a=1}^{p}(\nabla\pi_*)(E_a, E_a) + \sum_{r=1}^{s}(\nabla\pi_*)(Z_r, Z_r)\\ &= -\pi_*\left(\sum_{a=1}^{p}\mathcal{T}_{E_a}E_a\right) - \pi_*\left(\sum_{r=1}^{s}\mathcal{T}_{Z_r}Z_r\right)\\ &= p\,\pi_*(\operatorname{grad} f) + s\,\pi_*(\operatorname{grad} f). \end{aligned}\tag{3.97}$$

On the other hand, the horizontal trace computed in Theorem 3.4 gives

$$\sum_{\alpha=1}^{n}(\nabla\pi_*)(X_\alpha, X_\alpha) = -(n-2)\pi_*(\mathrm{grad}^{\mathcal{H}}\log\lambda). \tag{3.98}$$

Combining (3.97) and (3.98) yields (3.94). Since $\mathrm{grad}\, f$ is horizontal by Theorem 3.1, and $\mathrm{grad}^{\mathcal{H}}\log\lambda$ is horizontal, the injectivity of π_* on $(\ker\pi_*)^\perp$ gives (3.95). □

As immediate consequences of the global harmonicity criterion, we obtain the following special cases. They are included here for completeness and to clarify the behavior of the Clairaut function in the hemi-slant setting.

Corollary 3.4. *Let $\pi : (M^{2m}, g, J) \to (N^2, g_N)$ be a Clairaut conformal hemi-slant submersion with connected fibers and $r = e^f$. Then π is harmonic if and only if $\mathrm{grad}\, f = 0$. In particular, π is harmonic if and only if the Clairaut function r is constant.*

Corollary 3.5. *Let $\pi : (M^{2m}, g, J) \to (N^n, g_N)$ be a Clairaut conformal hemi-slant submersion with connected fibers and constant dilation λ. Then π is harmonic if and only if $\mathrm{grad}\, f = 0$. Equivalently, a homothetic Clairaut conformal hemi-slant submersion is harmonic if and only if its fibers are minimal.*

Corollary 3.6. *Let $\pi : (M^{2m}, g, J) \to (N^n, g_N)$ be a harmonic Clairaut conformal hemi-slant submersion with connected fibers, dilation λ, and $r = e^f$. Then*

$$q\,\Delta f = (n-2)\,\mathrm{div}\big(\mathrm{grad}^{\mathcal{H}}\log\lambda\big), \tag{3.99}$$

where $q = \dim(\ker\pi_*) = 2m - n$.

Proof. Since π is harmonic, Theorem 3.4 gives $q\;\mathrm{grad}\, f = (n-2)\,\mathrm{grad}^{\mathcal{H}}\log\lambda$. Taking divergence on both sides and using (2.35), we obtain

$$q\;\mathrm{div}(\mathrm{grad}\, f) = (n-2)\,\mathrm{div}\big(\mathrm{grad}^{\mathcal{H}}\log\lambda\big).$$

By (2.36), this gives (3.99). □

Example 3.1. *Let $M = \mathbb{R}^6 \setminus \{\rho = 0\}$ with standard coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$ and standard Euclidean metric g. Let J be the standard Kähler structure defined by*

$$J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \qquad J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}, \qquad i = 1, 2, 3.$$

Fix $0 < \theta < \frac{\pi}{2}$ and define

$$\xi_1 = -\sin\theta\, y_1 + \cos\theta\, x_2, \qquad \xi_2 = y_2, \qquad \xi_3 = y_3,$$

with $\rho^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$. Let $(N, g_N) = (\mathbb{R}^3, g_{\mathbb{R}^3})$ and define

$$\pi : M \to N, \qquad \pi = \frac{1}{\rho^2}(\xi_1, \xi_2, \xi_3).$$

Put

$$E_1 = \frac{\partial}{\partial x_1}, \quad E_2 = \frac{\partial}{\partial y_1}, \quad E_3 = \frac{\partial}{\partial x_2}, \quad E_4 = \frac{\partial}{\partial y_2}, \quad E_5 = \frac{\partial}{\partial x_3}, \quad E_6 = \frac{\partial}{\partial y_3}.$$

Since the Euclidean inversion

$$F(\xi) = \frac{\xi}{\|\xi\|^2}$$

is a local diffeomorphism on $\mathbb{R}^3 \setminus \{0\}$*, the vertical distribution of* $\pi = F \circ (\xi_1, \xi_2, \xi_3)$ *coincides with the kernel of* $d(\xi_1, \xi_2, \xi_3)$*. Therefore*

$$\ker \pi_* = \operatorname{span}\{U_1, U_2, U_3\},$$

where

$$U_1 = E_1, \qquad U_2 = \cos\theta\, E_2 + \sin\theta\, E_3, \qquad U_3 = E_5.$$

An orthonormal frame of the horizontal distribution is

$$X_1 = -\sin\theta\, E_2 + \cos\theta\, E_3, \qquad X_2 = E_4, \qquad X_3 = E_6.$$

Indeed,

$$X_i(\xi_j) = \delta_{ij}, \qquad i, j = 1, 2, 3.$$

Let

$$D_\theta = \operatorname{span}\{U_1, U_2\}, \qquad D^\perp = \operatorname{span}\{U_3\}.$$

Then

$$\ker \pi_* = D_\theta \oplus D^\perp.$$

Moreover,

$$JU_1 = E_2 = \cos\theta\, U_2 - \sin\theta\, X_1,$$

and

$$JU_2 = J(\cos\theta\, E_2 + \sin\theta\, E_3) = -\cos\theta\, U_1 + \sin\theta\, X_2.$$

Thus, for every nonzero $U \in D_\theta$*, the angle between* JU *and* D_θ *is constant and equal to* θ*. Hence* D_θ *is a slant distribution with slant angle* θ*. Also,*

$$JU_3 = JE_5 = E_6 = X_3 \in (\ker \pi_*)^\perp,$$

so $D^\perp$ *is anti-invariant. Since* $0 < \theta < \frac{\pi}{2}$ *and* $D^\perp \neq \{0\}$*,* π *is proper hemi-slant. We now verify the conformality. The inversion* $F(\xi) = \xi/\|\xi\|^2$ *satisfies*

$$\langle dF_\xi(a), dF_\xi(b)\rangle = \frac{1}{\rho^4}\langle a, b\rangle.$$

Since $X_i(\xi_j) = \delta_{ij}$*, for horizontal vector fields* X, Y *we obtain*

$$g_N(\pi_* X, \pi_* Y) = \frac{1}{\rho^4} g(X, Y).$$

Therefore π *is a conformal hemi-slant submersion with dilation* $\lambda = \frac{1}{\rho^2}$*. Since* λ *is not constant on* M*,* π *is not a Riemannian submersion. Next, the fibers of* π *coincide with the affine fibers of the linear map*

$$(x_1, y_1, x_2, y_2, x_3, y_3) \mapsto (\xi_1, \xi_2, \xi_3).$$

Hence the fibers are totally geodesic in the Euclidean space, and so $\mathcal{T}_U V = 0$ *for all vertical vector fields* U, V*. Thus* $H = 0$*. Taking* $f = 0$*,* $r = e^f = 1$*, we have* $H = 0 = -\operatorname{grad} f$*. Moreover,*

$$\frac{1}{\lambda^2} = \rho^4$$

depends only on the horizontal variables ξ_1, ξ_2, ξ_3*. Since*

$$U_i(\xi_j) = 0, \qquad i = 1, 2, 3, \quad j = 1, 2, 3,$$

we get

$$U_i(\rho^4) = 0.$$

Therefore (3.56) *holds. By Theorem 3.1,* π *is a Clairaut conformal hemi-slant submersion with Clairaut function* $r = 1$*. Finally, we check the harmonicity. Since* $q = \dim(\ker \pi_*) = 3$ *and* $n = 3$*, Theorem 3.4 gives*

$$3 \,\mathrm{grad}\, f = \mathrm{grad}^{\mathcal{H}}(\log \lambda).$$

Here $\mathrm{grad}\, f = 0$*, while*

$$\log \lambda = -2 \log \rho.$$

Hence $\mathrm{grad}^{\mathcal{H}}(\log \lambda)$ *is not identically zero. Therefore* π *is not harmonic. Thus this example provides a non-harmonic, non-Riemannian, proper Clairaut conformal hemi-slant submersion. Since* $f = 0$*, the curvature formula in Theorem 3.3 reduces to* $K^{\mathcal{V}} = \widehat{K}$*. Indeed, the fibers are affine Euclidean subspaces, so both* $K^{\mathcal{V}}$ *and* $\widehat{K}$ *vanish.*

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