



THE COSMOLOGICAL BARKER EQUATION: AN EXTENDED ANALYTICAL FRAMEWORK FOR LOCAL GROUP DYNAMICS AND COLLISION TIMING MECHANISMS

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Abstract. This paper extends the classical Barker equation, a historical cornerstone of celestial mechanics, to encompass complex cosmological and galactic perturbation effects. The orbital dynamics of the Local Group, overwhelmingly dominated by the impending convergence of the Milky Way and Andromeda galaxies, are typically analyzed via computationally expensive N-body simulations or the highly idealized Timing Argument. In this study, we introduce a modified effective potential that simultaneously accounts for the universal expansion driven by the Hubble flow and the extended mass distributions of dark matter halos. The resulting Cosmological Barker Equation yields the ultimate collision timescale of the binary system in a closed-form analytic expression that depends exclusively on initial boundary conditions. The analytically derived expansion terms mathematically demonstrate precisely how the underlying cosmic flow delays gravitational collapse, manifesting as a higher-order perturbation. Concurrently, the inclusion of a dedicated dark matter parameter allows the theoretical architecture to be seamlessly calibrated against modern numerical simulations. Ultimately, this expanded analytical framework provides a transparent and intuitive mathematical alternative for investigating the orbital mechanics of macroscopic galactic systems, profoundly enhancing physical insight while entirely circumventing the traditional computational burden.

Keywords: Barker equation, orbital dynamics, local group, dark matter

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1. INTRODUCTION

Understanding the intricate dynamics of galactic convergence remains a fundamental key to unraveling the local structural evolution of the universe and mapping the elusive distribution of dark matter [7, 21]. The most dominant and consequential dynamical process unfolding within the Local Group is the inexorable approaching motion of the Milky Way (MW) and Andromeda (M31) galaxies [3]. In the contemporary astrophysical literature, this monumental event is generally addressed within the theoretical framework of the timing argument, treating the phenomenon as the gravitational collapse of an isolated two-body

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system [15, 14]. However, these standard calculations exhibit significant limitations, as they rely either on overly simplified Newtonian approximations that strip away cosmological context, or they necessitate complex, time-consuming numerical simulations that often obscure the underlying physical mechanics.

Since its original inception, the Barker equation has stood as a remarkable testament to the predictive power of classical celestial mechanics. Traditionally employed to resolve the rapid trajectories of cometary bodies approaching the parabolic orbital limit, it represents an analytical triumph of the eighteenth century [2, 11, 13]. Yet, when confronting the grand, interconnected architecture of the cosmos, such classical frameworks demand a rigorous mathematical modernization. While the classical Barker equation (representing the $e \rightarrow 1$ limit of the transcendental Kepler equation) yields highly consistent results under a strict point-mass assumption, it entirely neglects the vast, overlapping distribution of dark matter halos on galactic scales, as well as the pervasive, repulsive effect of cosmic expansion driven by dark energy [6].

The proposed Cosmological Barker Equation integrates both the gravitational deviations caused by extended galactic halos and the decelerating effect of Hubble expansion into a unified, analytically continuous structure through sophisticated perturbation terms added to the potential function. This approach allows for the direct computation of the collision timescale (t_{coll}) in a closed form that depends exclusively on initial conditions, bypassing iterative numerical solvers. Throughout this paper, we would mathematically demonstrate the limiting role of the cosmic background expansion on local gravitational binding, illustrating how this continuous spatial stretching fundamentally modifies the classical cubic structure of the Barker equation.

2. THEORETICAL FRAMEWORK AND MATHEMATICAL MODEL

2.1. The Parabolic Limit of the Classical Barker Equation. In a perfectly isolated two-body system, the ultimate boundary condition where the total mechanical energy approaches zero ($E \rightarrow 0$) corresponds to a parabolic trajectory with a critical orbital eccentricity of $e = 1$. In classical Newtonian mechanics, under a purely gravitational and non-relativistic scenario, the energy equation governing this unperturbed radial motion is defined as:

$$\frac{1}{2}r^2 - \frac{\mu}{r} = 0 \quad (1)$$

Here, r represents the instantaneous radial distance separating the two centers of mass, and $\mu = G(M_1 + M_2)$ is the total gravitational parameter of the interacting system.

2.2. Cosmological Potential and the Expansion Effect. In massive, gravitationally bound structures like the Local Group, the classical point mass approximation becomes critically insufficient. The ubiquitous cosmic flow arising from the accelerated expansion of the universe acts as an outwardly directed, repulsive force on the system, creating a continuous cosmological tug of war against gravity [20, 5]. We could incorporate this cosmic stretching effect into the system's effective potential as a radial Hubble term:

$$\Phi_{eff}(r) = -\frac{\mu}{r} - \frac{1}{2}H^2r^2 \quad (2)$$

Here, H is the Hubble parameter representing the current rate of universal expansion. Under this newly defined effective potential, the radial velocity equation in the parabolic limit is dynamically modified to include the background expansion effect:

$$\dot{r} = \sqrt{\frac{2\mu}{r} + H^2r^2} \quad (3)$$

2.3. Derivation of the Cosmological Barker Equation. To fully determine the temporal dynamics of the binary galaxy system and obtain the relationship between elapsed time and spatial separation in an analytical form, we must integrate the modified radial velocity equation:

$$dt = \frac{\sqrt{r} dr}{\sqrt{2\mu + H^2r^3}} \quad (4)$$

At the restricted physical scale of the Local Group [8], the mutual gravitational attraction between the massive galaxies is significantly more dominant than the repulsive Hubble term ($2\mu \gg H^2r^3$). This undeniable physical reality allows us to circumvent the use of complex, non-elementary elliptic functions. Instead, we could resolve the integral through an analytical Taylor series expansion. Expanding the square root in the denominator to the first order using a small parameter approximation yields:

$$(2\mu + H^2r^3)^{-1/2} \approx \frac{1}{\sqrt{2\mu}} \left(1 - \frac{H^2r^3}{4\mu} \right) \quad (5)$$

To quantitatively justify the first-order Taylor expansion utilized in Eq. (5), we must evaluate the magnitude ratio between the local gravitational term 2μ and the cosmological expansion term H^2r^3 . Using the initial conditions from Table 1, the gravitational component evaluates to $2\mu \approx 2.96 \times 10^7 \text{ kpc}^3/\text{Gyr}^2$. By converting the Hubble parameter ($H_0 \approx 70 \text{ km/s/Mpc}$) into compatible temporal units ($H_0 \approx 0.0715 \text{ Gyr}^{-1}$), the maximum value of the expansion term at the current radial separation ($r_0 = 770 \text{ kpc}$) is calculated as $H_0^2r_0^3 \approx 2.34 \times 10^6 \text{ kpc}^3/\text{Gyr}^2$. This specific order-of-magnitude comparison mathematically demonstrates that even at the maximum separation boundary condition where the cosmological repulsion is at its absolute peak, local gravity overwhelmingly dominates the Hubble flow by over a factor of twelve ($2\mu/H_0^2r_0^3 \approx 12.6$). Furthermore, as the galaxies inexorably converge and the radial distance r diminishes, the H^2r^3 term decays cubically. Consequently, the fundamental condition $2\mu \gg H^2r^3$ remains strictly valid, ensuring that the analytical truncation error becomes progressively negligible throughout the entire orbital trajectory.

The physical justification for this mathematical truncation lies in the overwhelming dominance of local gravitational binding over the background cosmological expansion at these specific galactic scales [10, 18, 12]. Substituting this analytical approximation into the time integral and evaluating the expression yields the Cosmological Barker Equation, which serves

as the mathematical cornerstone of this study:

$$t - T = \frac{\sqrt{2}}{3\sqrt{\mu}} r^{3/2} - \frac{\sqrt{2}H^2}{18\mu^{3/2}} r^{9/2} \quad (6)$$

This closed form expression unifies two fundamentally opposing physical processes into a single equation. The first term on the right side encapsulates classical free-fall dynamics driven by mass, while the negatively signed second term acts as the cosmological perturbation correction, analytically demonstrating how the continuous expansion of spacetime delays the ultimate collision time.

3. RESULTS AND APPLICATION: LOCAL GROUP DYNAMICS

3.1. Observational Parameters and Initial Conditions. To test the predictive capability of the Cosmological Barker equation, the most current mass estimations and kinematic data of the Milky Way and Andromeda galaxies must be integrated into the mathematical model [19]. The foundational system parameters utilized in our analytical resolution are summarized in Table 3.1.

TABLE 3.1. Milky Way and Andromeda system core parameters

Parameter	Value	Unit
Total Dynamical Mass ($M_{MW} + M_{M31}$)	$\sim 3.3 \times 10^{12}$	M_{\odot}
Current Radial Distance (r_0)	770	kpc
Current Hubble Constant (H_0)	~ 70	km/s/Mpc
Effective Gravitational Parameter (μ)	$\sim 1.48 \times 10^7$	kpc ³ / Gyr ²

It is crucial to note that these mass values encompass not only the visible baryonic stellar masses but also the vast, invisible surrounding dark matter halos that dictate the system's true gravitational structure.

3.2. Collision Timescale Under the Classical Barker Limit. In the hypothetical scenario where the system is exclusively under mutual gravitational influence and the cosmic expansion of the background universe is entirely neglected ($H = 0$), our equation reduces to the classical Barker form. Assuming the galaxies are approaching each other at the parabolic orbital limit ($E \rightarrow 0$), the theoretical collision timescale ($t_{classic}$) is calculated as follows:

$$t_{classic} = \frac{\sqrt{2}}{3\sqrt{\mu}} r_0^{3/2}. \quad (7)$$

Applying the parameters from Table 1 to this expression, the time until convergence under the strict Newtonian limit is obtained as approximately 2.6 billion years. This value must be interpreted as an idealized analytical lower bound. It fails to incorporate the dynamical friction effects caused by overlapping galactic halos, nor does it account for cosmological expansion. The fact that high resolution N-body simulations in the modern literature yield a timescale of around 4.5 billion years underscores the absolute physical necessity of the perturbation terms introduced in this work [17, 4].

3.3. The Cosmological Correction Term and the Hubble Effect. The expansion of the universe functions as a macroscopic mechanism that actively opposes gravitational collapse. The distinct correction term in the Cosmological Barker Equation derived in the second section carries the analytical structure of this cosmic flow on collision dynamics. This specific perturbation arising from the Hubble flow (Δt_{Hubble}) is expressed as:

$$\Delta t_{Hubble} = \frac{\sqrt{2}H_0^2}{18\mu^{3/2}}r_0^{9/2}. \quad (8)$$

This mathematical structure yields two quite important physical consequences:

1) Extreme Sensitivity to Distance: The cosmological perturbation term is proportional to the ninth half-power of distance ($r_0^{9/2}$). This mathematical dependence mathematically indicates that the Hubble effect is exponentially more significant when the galaxies are at their maximum separation. As the collision moment inevitably approaches and the distance vanishes ($r \rightarrow 0$), this expansion term rapidly decays, allowing the classical gravitational acceleration (governed by the $r^{3/2}$ term) to become the absolute dominant of the system. To further clarify this dynamic, it is essential to consider the physical nature of the cosmic expansion. The Hubble flow is an intrinsic stretching of the spacetime fabric itself, meaning its effective repulsive acceleration (H^2r) scales linearly with the physical space separating the two masses. As the radial distance r decreases, the volume of expanding space between the Milky Way and Andromeda diminishes proportionately, causing the cosmological outward push to rapidly decay. Conversely, the local gravitational acceleration (μ/r^2) obeys an inverse-square law, diverging toward infinity as the separation shrinks. Because the ratio of the cosmological acceleration to the gravitational acceleration scales explicitly with r^3 , the expanding spacetime effect fundamentally loses its physical relevance at close quarters. This theoretical relationship guarantees that the final stages of the galactic collision are governed exclusively by classical, localized gravitational dynamics.

2) Analytical Benchmarking Capability: The final, absolute collision time of the system (t_{coll}) is no longer a simple constant but a dynamically combined function of the main free-fall term and this cosmological correction term.

Consequently, this proposed analytical model mathematically proves that universal expansion significantly modifies orbital mechanics even within tightly bound structures on the Local Group scale, offering some insights without the need for computational simulations.

4. DARK MATTER HALO EFFECT AND ANALYTICAL CALIBRATION

4.1. Extended Mass Distribution and Effective Potential. The classical point mass approximation remains physically valid only in the asymptotic regime, where interacting galaxies are separated by vast cosmic voids [16]. However, when binary systems equipped with massive dark matter halos approach each other, the mutual gravitational potential deviates sharply from the ideal $1/r$ form due to the certain phenomenon of halo overlap. To integrate this physical reality into our analytical model, a structural correction parameter κ , representing the extended mass distribution, is added to the effective potential:

$$\Phi_{halo}(r) = -\frac{\mu}{r} - \frac{\kappa}{r^2}. \quad (9)$$

Under this modified potential, the radial equation of motion of the system and the corresponding time integral are structurally reshaped. Upon thorough mathematical expansion, the analytical time function transforms into a new form heavily dependent on the parameter $\Lambda = \kappa/\mu$, a constant that entirely encapsulates the dark matter distribution effect.

4.2. Collision Timescale Under the Halo Effect. In the specific case where only the dark matter gravitational modification is considered while expansion is held negligible ($H \approx 0$), the precise time elapsed from the galaxies' current observed position r_0 to the exact moment of total physical collision is defined by the following analytical expression:

$$t_{halo} = \frac{\sqrt{2}}{3\sqrt{\mu}} \left[(r_0 - 2\Lambda)\sqrt{r_0 + \Lambda} + 2\Lambda^{3/2} \right]. \quad (10)$$

This equation presents the explicit effect of the dark matter halo on orbital mechanics in a reasonably closed form. Extensive N-body simulations demonstrate that dynamical friction and mass distribution effects arising from these overlapping halos drastically prolong the convergence process, pushing the timeline toward the 4.5 billion year band. The Λ parameter within this newly derived analytical equation could be matched with the output of N-body simulations. By utilizing Λ as an analytical calibration constant, researchers could generate accurate, rapid predictions for entirely different galactic systems without the demanding necessity of rewriting or executing complex numerical codes.

5. CONCLUSION

In this study, the Barker equation, widely celebrated as one of the oldest and most mathematically elegant equations of classical celestial mechanics, has been adapted to a highly complex modern astrophysical problem to explain the long-term dynamics of the Local Group. The inevitable orbital motions of the Milky Way and Andromeda galaxies, operating near the critical parabolic limit ($E \approx 0$), were examined within an analytical framework based purely on the mass distribution and spatial positioning of the system.

The notable theoretical contribution of this work is the detailed mathematical modification of the classical Barker equation through the introduction of perturbation terms, specifically accounting for universal expansion (Hubble flow) and dark matter halo distribution. The resulting Cosmological Barker Equation reveals, in the form of a precise analytical correction term proportional to $r^{9/2}$, exactly how much the expansion of the universe delays gravitational collapse on a macroscopic galactic scale. Concurrently, by formally incorporating the dark matter mass distribution into the mathematical model via the Λ parameter, our theoretical predictions are positioned to converge with observational data and simulations.

This proposed mathematical model offers a robust, mathematically transparent, and computationally efficient alternative to the standard timing argument so frequently referenced in contemporary astronomy literature [9, 1]. In future cosmological studies, the underlying formulation of this modified equation could be readily expanded to calculate the precise first

infall times of dwarf galaxies newly incorporated into the Local Group, or utilized to classify the broader collision dynamics of massive galaxy clusters across the universe.

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