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TYPE-1 INTERPOLATING SESQUI-f-HARMONIC MAPS BETWEEN RIEMANNIAN MANIFOLDS

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Abstract. In this paper, first, we introduce and study a new map called a type-1 interpolating sesqui-f-harmonic map. Then, we provide necessary and sufficient conditions for a differentiable curve in a Riemannian space form to be a type-1 interpolating sesqui-f-harmonic. These conditions are presented in a main theorem and investigated in several subcases. Moreover, we analyze type-1 interpolating sesqui-f-harmonic curves on $S^n(1)$ and $H^n(-1)$.

Keywords: f-Harmonic maps, bi-f-harmonic maps, Riemannian manifolds

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1. Introduction

There are some physical applications of harmonic and biharmonic maps in differential geometry. In materials science, harmonic maps are used to study the deformation of materials under different stresses and strains [2]. Moreover, Branding [3] defined a new functional for maps between Riemannian manifolds by interpolating harmonic and biharmonic maps with the motivation of the motion functional for externally curvature bosonic strings used in physics. This functional is called an interpolating sesqui-harmonic map between Riemannian manifolds.

Interpolating sesqui-harmonic maps defined between Riemannian manifolds are similar to harmonic maps, but they also satisfy an additional condition that includes the sesquilinear form. Sesqui-harmonic maps arise in various physical contexts, such as the diffusion of heat and electric currents in conductive materials, the dynamics of fluid flows [6]. For instance, these maps are instrumental in analyzing heat conduction in metallic structures or modeling the stress-strain relationship in elastic media [10]. Furthermore, sesqui-harmonic maps find applications in cosmology, where they contribute to understanding the large-scale distribution

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of matter and the universe's expansion [7]. Finally, this functional, which appears in various places in the physics literature, is called a bosonic string with external curvature in string theory [13, 15].

After defining the interpolating sesqui-harmonic maps between Riemannian manifolds, Branding [4] investigated the analytical results of these maps, especially those with spherical image sets, and focused on the analytical aspects of interpolated sesqui-harmonic maps, which were built on the regularity theory developed for biharmonic maps. He introduced a conservation law for such maps and used it to demonstrate the differentiability of weak solutions.

After these studies, Karaca et al. [12] studied interpolating sesqui-harmonic Legendre curves in Sasakian space forms. Following this, Karaca [11] carried this work to generalized Sasakian space forms. The last study on interpolating sesqui-harmonic maps in the relevant context is by Iqbal et al. [9], where interpolating sesqui-harmonic slant curves are studied in generalized Sasakian space forms, and a definition of an interpolating sesqui-harmonic minimal curve is introduced, which is another interesting new concept.

The f-biharmonic and bi-f-harmonic maps in differential geometry have physical applications similar to those of the harmonic and biharmonic maps. However, adding an f-functional to these maps provides a more precise and accurate representation of surface features, resulting in more accurate results in these applications. An f-functional is used to map local properties of a surface to a target space and to minimize the distortion of angles and lengths during the mapping of the surface to the target space.

In light of all the studies mentioned above, in this paper, we introduce type-1 interpolating sesqui-f-harmonic maps, which can be considered as an intermediate value for f-harmonic and bi-f-harmonic maps between Riemann manifolds. We provide a functional, called a type-1 interpolating sesqui-f-energy integral, which enables the definition of type-1 interpolating sesqui-f-harmonic maps, the corresponding Euler-Lagrange equations, and type-1 interpolating sesqui-f-tension field definitions. Then, in the third section, we investigate the type-1 interpolating sesqui-f-harmonicity conditions for curves on Riemannian manifolds using the type-1 interpolating sesqui-f-harmonicity equations. We analyze these conditions for the curves in terms of the special cases of the elements of the Frenet frame and obtain some absence theorems and results from the data herein.

2. Preliminaries

In this section, we present f-harmonic, bi-f-harmonic, and sesqui-harmonic map equations between Riemannian manifolds.

Definition 2.1. [8] *Let (M, g) and (N, h) be Riemannian manifolds. Then, a harmonic map $\varpi : (M, g) \to (N, h)$ is defined as the critical point of the energy functional*

$$E(\varpi) = \frac{1}{2}\int_M |d\varpi|^2 dv_g$$

where v_g is the volume element of (M, g). Then, by using Euler-Lagrange equation $\tau(\varpi)$ of the energy functional $E(\varpi)$, where it is the tension field of the map ϖ, the map is called as *harmonic if*

$$\tau(\varpi) = tr\nabla d\varpi = 0$$

Definition 2.2. [8] *A map $\varpi : (M, g) \to (N, h)$ is defined as a biharmonic map if it is a critical point of the bienergy functional*

$$E_2(\varpi) = \frac{1}{2}\int_M |\tau(\varpi)|^2 dv_g$$

for all variations. Then, for the bienergy functional $E_2(\varpi)$, the Euler-Lagrange equation $\tau_2(\varpi)$ equals to

$$\tau_2(\varpi) = tr(\nabla^{\varpi}\nabla^{\varpi} - \nabla^{\varpi}_{\nabla})\tau(\varpi) - tr(R^N(d\varpi, \tau(\varpi))d\varpi) = 0$$

where $\tau_2(\varpi)$ is the bitension field of the map ϖ, if ϖ is a biharmonic map where R^N is the curvature tensor field of N defined as

$$R^N(X, Y)Z = \nabla^N_X\nabla^N_Y Z - \nabla^N_Y\nabla^N_X Z - \nabla^N_{[X,Y]}Z$$

for all $X, Y, Z \in \Gamma(TN)$. Here, ∇^{ϖ} is the pull-back connection.

Definition 2.3. [1, 5] *A map $\varpi : (M, g) \to (N, h)$ is said to be an f-harmonic if it is a critical point of the f-energy functional*

$$E_f(\varpi) = \frac{1}{2}\int_M f|d\varpi|^2 dv_g$$

where $f \in C^{\infty}(M, \mathbb{R})$ is a positive smooth function. Then, the f-harmonic map equation obtained by using Euler-Lagrange equation as follows:

$$\tau_f(\varpi) = f\tau(\varpi) + d\varpi(gradf) = 0$$

where $\tau_f(\varpi)$ is the f-tension field of the map ϖ.

Definition 2.4. [16] *A map $\varpi : (M, g) \to (N, h)$ is said to be a bi-f-harmonic if it is a critical point of the bi-f-energy functional*

$$E_{2,f}(\varpi) = \frac{1}{2}\int_M |\tau_f(\varpi)|^2 dv_g$$

The Euler-Lagrange equation for the bi-f-harmonic map is defined by

$$\tau_{2,f}(\varpi) = tr\big(\nabla^{\varpi} f(\nabla^{\varpi}\tau_f(\varpi)) - f\nabla^{\varpi}_{\nabla^N}\tau_f(\varpi) + fR^N(\tau_f(\varpi), d\varpi)d\varpi\big) = 0$$

where $\tau_{2,f}(\varpi)$ is the bi-f-tension field of the map ϖ.

Furthermore, Branding [3] defined an interpolating sesqui-harmonic map between Riemannian manifolds. He introduced an action functional that interpolates between the actions for harmonic and biharmonic maps.

Definition 2.5. [3, 12] *A map ϖ is called as interpolating sesqui-harmonic if it is a critical point of the following action functional that interpolates between the actions for harmonic and biharmonic maps:*

$$E_{\delta_1,\delta_2}(\varpi) = \delta_1\int_M |d\varpi|^2 dv_g + \delta_2\int_M |\tau(\varpi)|^2 dv_g$$

where $\delta_1, \delta_2 \in \mathbb{R}$. *Then, the interpolating sesqui-harmonic equation is provided as follows: For* $\delta_1, \delta_2 \in \mathbb{R}$,

$$\tau_{\delta_1,\delta_2}(\varpi) = \delta_2\tau_2(\varpi) - \delta_1\tau(\varpi) = 0.$$

For more information about interpolating sesqui-harmonic maps, see [3, 4, 12].

3. Interpolating Sesqui-f-Harmonic Curves on Riemannian Manifolds

Branding [3] has obtained a functional by associating the critical points of the energy integral, which is crucial for harmonic maps, with the bi-energy integral that provides rise to biharmonic maps, enabling the definition of interpolated sesqui-harmonic maps between Riemann manifolds. Biharmonic maps are maps that roughly solve the Laplace equation in a space. Moreover, f-biharmonic maps emerge as maps satisfying the multivariable f-Laplace equation. The difference between these two maps arises from how these equations are solved and which properties are emphasized. In this study, we extend Branding's work [3] to a broader class of maps, including f-harmonic and bi-f-harmonic maps. We introduce and thoroughly study the type-1 interpolating sesqui-f-harmonic map, an intermediate concept between f-harmonic and bi-f-harmonic maps on Riemannian manifolds.

3.1. **Type-1 Interpolating Sesqui-f-Harmonic Maps on Riemannian Manifolds.**

In this section, first, we define the following functional

$$E_{\rho_1,\rho_2,f}(\varpi) = \rho_1 \int\limits_M f|d\varpi|^2 dv_g + \rho_2 \int\limits_M |\tau_f(\varpi)|^2 dv_g$$

where $\rho_1, \rho_2 \in \mathbb{R} \setminus \{0\}$. Let

$$\begin{aligned} \pi : M \times (-\epsilon, \epsilon) &\to N \\ (x,t) &\to \pi(x,t) = \varpi_t(x) \end{aligned}$$

be a smooth variation of ϖ with variation vector fields v, where (M,g) and (N,h) are Riemannian manifolds. Denote the pull-back connection on the vector bundle $\pi^{-1} : TN \to M \times (-\epsilon, \epsilon)$ by ∇^π. Since any vector field $X \in \Gamma(TM)$ can be considered as a vector field on $M \times (-\epsilon, \epsilon)$, then $\left[\frac{\partial}{\partial t}, X\right] = 0$. As a first step, we calculate the following equality:

$$\frac{d}{dt}E_{\rho_1,\rho_2,f}(\varpi_t; M)|_{t=0} = \rho_1 \left(\frac{1}{2}\frac{d}{dt}\int\limits_M f|d\varpi_t|^2 dv_g|_{t=0} \right) + \rho_2 \left(\frac{1}{2}\frac{d}{dt}\int\limits_M |\tau_f(\varpi_t)|^2 dv_g|_{t=0} \right)$$

It is well known from [14] that

$$\frac{1}{2}\frac{d}{dt}\int\limits_M f|d\varpi_t|^2 dv_g|_{t=0} = -\int\limits_M h(\tau_f(\varpi), v) dv_g \tag{3.1}$$

and

$$\frac{1}{2}\frac{d}{dt}\int\limits_M |\tau_f(\varpi_t)|^2 dv_g|_{t=0} = -\int\limits_M h(\tau_{2,f}(\varpi), v) dv_g \tag{3.2}$$

where

$$\tau_f(\varpi) = f\tau(\varpi) + d\varpi(\text{grad}f)$$

and

$$\tau_{2,f}(\varpi) = \text{tr}\big(\nabla^\varpi f(\nabla^\varpi \tau_f(\varpi)) - f\nabla^\varpi_{\nabla^N}\tau_f(\varpi) + fR^N(\tau_f(\varpi), d\varpi)d\varpi\big).$$

By using (3.1) and (3.2),

$$\frac{d}{dt}E_{\rho_1,\rho_2,f}(\varpi_t;M)|_{t=0} = -\rho_1\int\limits_M h(\tau_f(\varpi),v)dv_g - \rho_2\int\limits_M h(\tau_{2,f}(\varpi),v)dv_g$$

which implies

$$\frac{d}{dt}E_{\rho_1,\rho_2,f}(\varpi_t;M)|_{t=0} = -\int\limits_M <\rho_1\tau_f(\varpi)+\rho_2\tau_{2,f}(\varpi),v)> dv_g. \tag{3.3}$$

Definition 3.1. *A map $\varpi:(M,g)\to(N,h)$ between Riemannian manifolds is called a type-1 interpolating sesqui-f-harmonic map if it is a critical point of the functional*

$$E_{\rho_1,\rho_2,f} = \rho_1\int\limits_M f|d\varpi|^2dv_g + \rho_2\int\limits_M |\tau_f(\varpi)|^2dv_g$$

The Euler-Lagrange equation associated with (3.3) is

$$\tau_{\rho_1,\rho_2,f}(\varpi) = \rho_1\tau_f(\varpi)+\rho_2\tau_{2,f}(\varpi)$$

where $\tau_f(\varpi)$ and $\tau_{2,f}(\varpi)$ are the f-tension and bi-f-tension field of ϖ, respectively, and $\rho_1,\rho_2\in\mathbb{R}\setminus\{0\}$.

Theorem 3.1. *A map $\varpi:(M,g)\to(N,h)$ is a type-1 interpolating sesqui-f-harmonic map if and only if $\rho_1\tau_f(\varpi)+\rho_2\tau_{2,f}(\varpi)=0$.*

Remark 3.1. *From the relevant definitions, the following hold:*
If f is a constant, then f-harmonicity induces to harmonicity. In this case, interpolating sesqui-f-harmonic maps overlap with interpolating sesqui-harmonic maps defined by Branding [3] *such that $\delta_1=-\rho_1 f$ and $\delta_2=-\rho_2 f$.*
It is well known from [14] *that any f-harmonic map is bi-f-harmonic. Thus, f-harmonic maps are always interpolating sesqui-f-harmonic maps, for any $\rho_1,\rho_2\in\mathbb{R}$.*

Let $\alpha: I\to(N,h)$ be a differentiable curve on an n-dimensional Riemannian manifold (N,h) parametrized by the arclength s, where I is an open interval and $\alpha^{'}=T$. Considering the f-tension field and bi-f-tension field equations in [16],

$$\tau_f(\varpi) = f\nabla_T T + f^{'}T$$

and

$$\tau_{2,f}(\varpi) = (ff^{'''}+f^{'}f^{''})T + (3ff^{''}+2(f^{'})^2)\nabla_T T + 4ff^{'}\nabla_T^2 T + f^2\nabla_T^3 T + f^2R^N(\nabla_T T,T)T$$

which imply

$$\begin{aligned}\tau_{\rho_1,\rho_2,f}(\varpi) = {} & (\rho_1 f^{'}+\rho_2(ff^{'''}+f^{'}f^{''}))T + (\rho_1 f+\rho_2(3(ff^{''}+2(f^{'})^2))\nabla_T T + 4\rho_2 ff^{'}\nabla_T^2 T \\ & +\rho_2 f^2\nabla_T^3 T + \rho_2 f^2R^N(\nabla_T T,T)T.\end{aligned}$$

Thus, the following proposition can be obtained:

Proposition 3.1. *Let $\alpha : I \to (N, h)$ be a differentiable curve parametrized by its arclength. Then, α is a type-1 interpolating sesqui-f-harmonic curve if and only if*

$$\begin{aligned} 0 = &(\rho_1 f' + \rho_2(ff''' + f'f''))T + (\rho_1 f + \rho_2(3(ff'' + 2(f')^2))\nabla_T T + 4\rho_2 ff'\nabla_T^2 T \\ &+\rho_2 f^2\nabla_T^3 T + \rho_2 f^2 R^N(\nabla_T T, T)T. \end{aligned} \tag{3.4}$$

Let $\{E_1 = T, E_2, ..., E_n\}$ be a Frenet frame along α defined an n-dimensional Riemannian manifold (N, h), where E_2 is the unit normal vector field along α and for every $j \in \{3, 4, 5, \dots, n\}$, E_j is a unit vector field such that

$$\begin{cases} \nabla_T T = k_1 E_2 \\ \nabla_T E_2 = -k_1 T + k_2 E_3 \\ \nabla_T E_r = -k_{r-1}E_{r-1} + k_r E_{r+1}, \quad r \in \{3, 4, ..., n-1\} \\ \nabla_T E_n = -k_{n-1}E_{n-1} \end{cases}.$$

Here, $k_1 = ||\nabla_T T||$ and $k_2, ..., k_{n-1}$ are nonnegative real-valued functions. Following the equations calculated in [16],

$$\nabla_T^2 T = -k_1^2 T + k_1' E_2 + k_1 k_2 E_3 \tag{3.5}$$

$$\nabla_T^3 T = -3k_1 k_1' T + (k_1'' - k_1^3 - k_1 k_2^2)E_2 + (2k_1' k_2 + k_1 k_2')E_3 + k_1 k_2 k_3 E_4 \tag{3.6}$$

and

$$R^N(\nabla_T T, T)T = k_1 R^N(E_2, T)T. \tag{3.7}$$

In view of (3.5)-(3.7) in (3.4), the following theorem can be obtained:

Theorem 3.2. *A differentiable curve $\alpha : I \to (N, h)$ parametrized by its arclength is a type-1 interpolating sesqui-f-harmonic map if and only if*

$$\begin{aligned} 0 = &(\rho_1 f' + \rho_2(ff''' + f'f'') - 4\rho_2 ff'k_1^2 - 3\rho_2 f^2 k_1 k_1')T + (\rho_1 f k_1 + \rho_2 k_1(3ff'' + 2(f')^2) \\ &+4\rho_2 ff'k_1' + \rho_2 f^2(k_1'' - k_1^3 - k_1 k_2^2))E_2 + (4\rho_2 ff'k_1 k_2 + \rho_2 f^2(2k_1' k_2 + k_1 k_2'))E_3 \\ &+(\rho_2 f^2 k_1 k_2 k_3)E_4 + \rho_2 f^2 k_1 R^N(E_2, T)T. \end{aligned}$$

If (N, h) is a Riemannian manifold of constant sectional curvature, then the following theorem can be obtained:

Theorem 3.3. *Let $\alpha : I \to (N(c), h)$ be a differentiable curve on a Riemannian space form $N(c)$, parametrized by its arclength. Then, α is a type-1 interpolating sesqui-f-harmonic curve if and only if*

$$\begin{cases} \rho_1 f' + \rho_2(ff''' + f'f'') - 4\rho_2 ff'k_1^2 - 3\rho_2 f^2 k_1 k_1' = 0 \\ \rho_1 f k_1 + \rho_2 k_1(3ff'' + 2(f')^2) + 4\rho_2 ff'k_1' + \rho_2 f^2(k_1'' - k_1^3 - k_1 k_2^2) + c\rho_2 f^2 k_1 = 0 \\ 4\rho_2 ff'k_1 k_2 + \rho_2 f^2(2k_1' k_2 + k_1 k_2') = 0 \\ \rho_2 f^2 k_1 k_2 k_3 = 0 \end{cases}. \tag{3.8}$$

From the first equation in (3.8), we consider the case in which α is a geodesic as follows:

Theorem 3.4. *A geodesic curve on a Riemannian manifold is a type-1 interpolating sesqui-f-harmonic curve if and only if*

$$\frac{\rho_1}{\rho_2} = -\frac{(ff^{''})^{'}}{f^{'}}.$$

Assume that $\alpha : I \to E^n$ is a differentiable curve on the n-dimensional Euclidean space parametrized by its arclength, where $I \subseteq \mathbb{R}$ is an open interval. We investigate the following special cases:

Case I. Let $k_1 \in \mathbb{R} \setminus \{0\}$ and $k_2 = 0$. Then, (3.8) reduces to

$$\begin{cases} \rho_1 f^{'} + \rho_2 (ff^{'''} + f^{'} f^{''}) - 4\rho_2 ff^{'} k_1^2 = 0 \\ \rho_1 f k_1 + \rho_2 k_1 (3ff^{''} + 2(f^{'})^2) - \rho_2 f^2 k_1^3 = 0 \end{cases}$$

which implies

$$\begin{cases} \rho_1 f^{'} = \rho_2 (4ff^{'} k_1^2 - (ff^{''})^{'}) \\ \rho_1 f = \rho_2 (-3ff^{''} - 2(f^{'})^2 + f^2 k_1^2) \end{cases}. \tag{3.9}$$

From the second equation in (3.9),

$$ff^{''} = \frac{\rho_2 f^2 k_1^2 - \rho_1 f - 2\rho_2 (f^{'})^2}{3\rho_2}.$$

By putting the last equation into the first equation in (3.9),

$$f^{'}\left(5\rho_2 f k_1^2 + 2\rho_2 f^{''} - \rho_1\right) = 0.$$

Thus, the following theorem can be obtained:

Theorem 3.5. *Let $\alpha : I \to E^n$ be a differentiable circle in the n-dimensional Euclidean space parametrized by its arclength with $k_1 \in \mathbb{R} \setminus \{0\}$ and $k_2 = 0$. Then, α is a type-1 interpolating sesqui-f-harmonic curve if and only if*

$$f(s) = c_1 \cos\left(\sqrt{\frac{5}{2}} k_1 s\right) + c_2 \sin\left(\sqrt{\frac{5}{2}} k_1 s\right) + \frac{\rho_1}{5k_1^2 \rho_2}.$$

Case II. Let $k_1, k_2 \in \mathbb{R} \setminus \{0\}$. In this case, (3.8) reduces to

$$\begin{cases} \rho_1 f' + \rho_2 (ff''' + f'f'') - 4\rho_2 ff' k_1^2 = 0 \\ \rho_1 f k_1 + \rho_2 k_1 (3ff'' + 2(f')^2) + \rho_2 f^2 (-k_1^3 - k_1 k_2^2) = 0 \\ 4\rho_2 ff' k_1 k_2 = 0 \\ \rho_2 f^2 k_1 k_2 k_3 = 0 \end{cases}$$

which implies

$$\begin{cases} \rho_1 + \rho_2 f(-k_1^2 - k_2^2) = 0 \\ f^{'} = 0 \\ k_3 = 0 \end{cases}.$$

Since f is a positive real-valued and nonconstant function, we obtain the following theorem:

Theorem 3.6. *There does not exist a type-1 interpolating sesqui-f-harmonic helix in the n-dimensional Euclidean space.*

Case III. Let $k_1 \in \mathbb{R} \setminus \{0\}$ and k_2 be a nonconstant and positive real-valued function. Then, by using (3.8),

$$\begin{cases} \rho_1 f^{'} + \rho_2(ff^{'''} + f^{'}f^{''}) - 4\rho_2 ff^{'}k_1^2 = 0 \\ \rho_1 fk_1 + \rho_2 k_1(3ff^{''} + 2(f^{'})^2) + \rho_2 f^2(-k_1^3 - k_1k_2^2) = 0 \\ 4\rho_2 ff^{'}k_1k_2 + \rho_2 f^2 k_1 k_2^{'} = 0 \\ \rho_2 f^2 k_1k_2k_3 = 0 \end{cases}$$

which implies

$$\begin{cases} \rho_1 f^{'} + \rho_2(ff^{'''} + f^{'}f^{''} - 4ff^{'}k_1^2) = 0 \\ \rho_1 f + \rho_2(3ff^{''} + 2(f^{'})^2 - f^2(k_1^2 + k_2^2)) = 0 \\ \rho_2(4f^{'}k_1k_2 + fk_1k_2^{'}) = 0 \\ k_2k_3 = 0 \end{cases}$$

Therefore, the following theorem can be obtained:

Theorem 3.7. *Let $\alpha : I \to E^n$ be a differentiable curve in the n-dimensional Euclidean space parametrized by its arclength with $k_1 \in \mathbb{R} \setminus \{0\}$ and the nonconstant and positive real-valued function k_2. Then, α is a type-1 interpolating sesqui-f-harmonic curve if and only if $f(s) = c_1 k_2(s)^{-\frac{1}{4}}$ such that c_1 is a positive integral constant, $k_3 = 0$, and*

$$33(k_2^{'})^3 - 52k_2k_2^{'}k_2^{''} + 16k_2^2k_2^{'''} - 48k_1^2k_2^2k_2^{'} + 16k_2^{'}k_2^4 = 0.$$

Case IV. Let k_1 be a nonconstant and nonnegative real-valued function and $k_2 = 0$. Then, the following theorem can be obtained:

Theorem 3.8. *Let $\alpha : I \to E^n$ be a differentiable curve in the n-dimensional Euclidean space parametrized by its arclength with the nonconstant and nonnegative real-valued function k_1 and $k_2 = 0$. Then, α is a type-1 interpolating sesqui-f-harmonic curve if and only if*

$$\rho_1 f^{'} + \rho_2(ff^{'''} + f^{'}f^{''}) - 4\rho_2 ff^{'}k_1^2 - 3\rho_2 f^2 k_1 k_1^{'} = 0$$

and

$$\rho_1 fk_1 + \rho_2 k_1(3ff^{''} + 2(f^{'})^2) + 4\rho_2 ff^{'}k_1^{'} + \rho_2 f^2(k_1^{''} - k_1^3) = 0$$

Case V. Let k_1 be a nonconstant and nonnegative real-valued function and $k_2 \in \mathbb{R} \setminus \{0\}$. By using (3.8),

$$\begin{cases} \rho_1 f^{'} + \rho_2(ff^{'''} + f^{'}f^{''}) - 4\rho_2 ff^{'}k_1^2 - 3\rho_2 f^2 k_1 k_1^{'} = 0 \\ \rho_1 fk_1 + \rho_2 k_1(3ff^{''} + 2(f^{'})^2) + 4\rho_2 ff^{'}k_1^{'} + \rho_2 f^2(k_1^{''} - k_1^3 - k_1k_2^2) = 0 \\ 4\rho_2 ff^{'}k_1k_2 + 2\rho_2 f^2 k_1^{'}k_2 = 0 \\ k_1k_3 = 0 \end{cases}$$

which implies

$$\begin{cases} \rho_1 f^{'} + \rho_2(ff^{'''} + f^{'}f^{''}) - 4\rho_2 ff^{'}k_1^2 - 3\rho_2 f^2 k_1 k_1^{'} = 0 \\ \rho_1 fk_1 + \rho_2 k_1(3ff^{''} + 2(f^{'})^2) + 4\rho_2 ff^{'}k_1^{'} + \rho_2 f^2(k_1^{''} - k_1^3 - k_1k_2^2) = 0 \\ 2f^{'}k_1 + fk_1^{'} = 0 \\ k_3 = 0 \end{cases}$$

Thus, the following theorem can be obtained:

Theorem 3.9. *Let $\alpha : I \to E^n$ be a differentiable curve in the n-dimensional Euclidean space parametrized by its arclength with the nonconstant and nonnegative real-valued function k_1 and $k_2 \in \mathbb{R} \setminus \{0\}$. Then, α is a type-1 interpolating sesqui-f-harmonic curve if and only if $f(s) = c_2 k_1(s)^{-\frac{1}{2}}$ such that c_2 is a positive integral constant, $k_3 = 0$, and*

$$15(k_1^{'})^3 - 18k_1k_1^{'}k_1^{''} + 4k_1^2k_1^{'''} + 12k_1^4k_1^{'} + 4k_1^2k_2^2k_1^{'} = 0$$

CASE VI: Let k_1 and k_2 be nonconstant and nonnegative real-valued functions. From the third equation in (3.8),

$$\begin{cases} 4f^{'}k_1k_2 + f(2k_1^{'}k_2 + k_1k_2^{'}) = 0 \\ \dfrac{f^{'}}{f} = -\dfrac{1}{2}\dfrac{k_1^{'}}{k_1} - \dfrac{1}{4}\dfrac{k_2^{'}}{k_2} \end{cases}$$

which implies

$$f(s) = c_3 k_1(s)^{-\frac{1}{2}} k_2(s)^{-\frac{1}{4}}$$

where c_3 is a positive integral constant. Therefore, the following theorem can be obtained:

Theorem 3.10. *Let $\alpha : I \to E^n$ be a differentiable curve in the n-dimensional Euclidean space parametrized by its arclength with nonconstant and nonnegative real-valued functions k_1 and k_2. Then, α is a type-1 interpolating sesqui-f-harmonic curve if and only if $f(s) = c_3 k_1(s)^{-\frac{1}{2}} k_2(s)^{-\frac{1}{4}}$ such that c_3 is a positive constant, $k_3 = 0$,*

$$\rho_1 f^{'} + \rho_2(ff^{'''} + f^{'}f^{''}) - 4\rho_2 ff^{'}k_1^2 - 3\rho_2 f^2 k_1 k_1^{'} = 0$$

and

$$\rho_1 fk_1 + \rho_2 k_1(3ff^{''} + 2(f^{'})^2) + 4\rho_2 ff^{'}k_1^{'} + \rho_2 f^2(k_1^{''} - k_1^3 - k_1k_2^2) = 0$$

3.2. **Type-1 Interpolating Sesqui-f-Harmonic Curves in $S^n(1)$ and $H^n(-1)$.** This section presents type-1 interpolating sesqui-f-harmonic curves in $S^n(1)$ and $H^n(-1)$. In view of (3.8), we have the following theorem:

Theorem 3.11. *Let $\alpha : I \to S^n(1)$ be a differentiable curve parametrized by its arclength. Then, α is a type-1 interpolating sesqui-f-harmonic curve if and only if*

$$\rho_1 f^{'} + \rho_2(ff^{'''} + f^{'}f^{''}) - 4\rho_2 ff^{'}k_1^2 - 3\rho_2 f^2 k_1 k_1^{'} = 0$$

$$\rho_1 fk_1 + \rho_2 k_1(3ff^{''} + 2(f^{'})^2) + 4\rho_2 ff^{'}k_1^{'} + \rho_2 f^2(k_1^{''} - k_1^3 - k_1k_2^2) + \rho_2 f^2 k_1 = 0$$

$$4\rho_2 ff^{'}k_1k_2 + \rho_2 f^2(2k_1^{'}k_2 + k_1k_2^{'}) = 0$$

and

$$\rho_2 f^2 k_1 k_2 k_3 = 0$$

Theorem 3.12. *Let $\alpha : I \to S^n(1)$ be a differentiable curve parametrized by its arclength. If $k_1 = 0$, then α is a type-1 interpolating sesqui-f-harmonic curve if and only if $\frac{\rho_1}{\rho_2} = -\frac{(ff^{''})^{'}}{f^{'}}$. If $k_1 \in \mathbb{R} \setminus \{0\}$ and $k_2 = 0$, then α is a type-1 interpolating sesqui-f-harmonic curve if and only if*

$$f(s) = c_1 \cos\left(\sqrt{\frac{5k_1^2+1}{2}}s\right) + c_2 \sin\left(\sqrt{\frac{5k_1^2+1}{2}}s\right) + \frac{\rho_1}{(5k_1^2+1)\rho_2}.$$

If $k_1, k_2 \in \mathbb{R} \setminus \{0\}$, then α is a type-1 interpolating sesqui-f-harmonic curve if and only if

$$f(s) = \frac{\rho_1}{4\rho_2}s^2 + c_1 s + c_2$$

where $-5k_1^2 + k_2^2 = 1$, or

$$f(s) = c_1 \cos\left(\sqrt{\frac{-5k_1^2+k_2^2-1}{2}}s\right) + c_2 \sin\left(\sqrt{\frac{-5k_1^2+k_2^2-1}{2}}s\right) + \frac{\rho_1}{(5k_1^2-k_2^2+1)\rho_2}$$

where $-5k_1^2 + k_2^2 > 1$, or

$$f(s) = c_1 e^{\sqrt{\frac{5k_1^2-k_2^2+1}{2}}s} + c_2 e^{-\sqrt{\frac{5k_1^2-k_2^2+1}{2}}s} + \frac{\rho_1}{(5k_1^2-k_2^2+1)\rho_2}$$

where $-5k_1^2 + k_2^2 < 1$.

In view of (3.8), the following theorem can be obtained:

Theorem 3.13. *Let $\alpha : I \to H^n(-1)$ be a differentiable curve parametrized by its arclength. Then, α is a type-1 interpolating sesqui-f-harmonic curve if and only if*

$$\rho_1 f^{'} + \rho_2(ff^{'''} + f^{'}f^{''}) - 4\rho_2 ff^{'}k_1^2 - 3\rho_2 f^2 k_1 k_1^{'} = 0$$

$$\rho_1 f k_1 + \rho_2 k_1(3ff^{''} + 2(f^{'})^2) + 4\rho_2 ff^{'}k_1^{'} + \rho_2 f^2(k_1^{''} - k_1^3 - k_1 k_2^2) - \rho_2 f^2 k_1 = 0$$

$$4\rho_2 ff^{'}k_1 k_2 + \rho_2 f^2(2k_1^{'}k_2 + k_1 k_2^{'}) = 0$$

and

$$\rho_2 f^2 k_1 k_2 k_3 = 0$$

Theorem 3.14. *Let $\alpha : I \to H^n(-1)$ be a differentiable curve parametrized by its arclength. If $k_1 = 0$, then α is a type-1 interpolating sesqui-f-harmonic curve if and only if $\frac{\rho_1}{\rho_2} = -\frac{(ff^{''})^{'}}{f^{'}}$. If $k_1 \in \mathbb{R} \setminus \{0\}$ and $k_2 = 0$, then α is a type-1 interpolating sesqui-f-harmonic curve if and only if*

$$f(s) = \frac{\rho_1}{4\rho_2}s^2 + c_1 s + c_2$$

where $k_1 = \pm\frac{1}{\sqrt{5}}$, or

$$f(s) = c_1 \cos\left(\sqrt{\frac{5k_1^2-1}{2}}s\right) + c_2 \sin\left(\sqrt{\frac{5k_1^2-1}{2}}s\right) + \frac{\rho_1}{(5k_1^2-1)\rho_2}$$

where $k_1 \in \left(-\infty, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$, *or*

$$f(s) = c_1 e^{\sqrt{\frac{1-5{k_1}^2}{2}}s} + c_2 e^{-\sqrt{\frac{1-5{k_1}^2}{2}}s} + \frac{\rho_1}{(1-5k_1^2)\rho_2}$$

where $k_1 \in \left(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

If $k_1, k_2 \in \mathbb{R} \setminus \{0\}$, *then* α *is a type-1 interpolating sesqui-*f*-harmonic curve if and only if*

$$f(s) = \frac{\rho_1}{4\rho_2}s^2 + c_1 s + c_2$$

where $5k_1^2 - k_2^2 = 1$, *or*

$$f(s) = c_1 \cos\left(\sqrt{\frac{5k_1^2 - k_2^2 - 1}{2}}s\right) + c_2 \sin\left(\sqrt{\frac{5k_1^2 - k_2^2 - 1}{2}}s\right) + \frac{\rho_1}{(5k_1^2 - k_2^2 - 1)\rho_2}$$

where $5k_1^2 - k_2^2 > 1$, *or*

$$f(s) = c_1 e^{\sqrt{\frac{-5k_1^2 + k_2^2 + 1}{2}}s} + c_2 e^{-\sqrt{\frac{-5k_1^2 + k_2^2 + 1}{2}}s} + \frac{\rho_1}{(5k_1^2 - k_2^2 - 1)\rho_2}$$

where $5k_1^2 - k_2^2 < 1$.

4. Conclusion

In this paper, we focus on type-1 interpolating sesqui-f-harmonic maps. Moreover, we obtain necessary and sufficient conditions for a differentiable curve in a Riemannian space form to be type-1 interpolating sesqui-f-harmonic. While the results establish a consistent theoretical framework for sesqui-f-harmonic maps, further investigations may focus on different space forms and possible applications in mathematical physics and mechanics, following the geometric approaches discussed in this paper. Additionally, future studies can investigate similar intermediate concepts between f-harmonic and f-biharmonic maps on Riemannian manifolds.

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