CALCULATION

Volume 2, Issue 2, 2026, Pages:88–101

E-ISSN: 3062–2107

www.simadp.com/calculation

SECOND VARIATION OF F-EINSTEIN-HILBERT FUNCTIONAL

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Abstract. This article describes a formula for second variation of generalized Einstein-Hilbert functional on Riemannian manifolds. This work extends the definition of stable Einstein manifolds, and we present some properties.

Keywords: Einstein manifolds, Einstein-Hilbert functional.

2020 Mathematics Subject Classification: Primary 53C25, Secondary 83C05.

1. Introduction

The Einstein-Hilbert functional $\mathcal{E}$ associates to each Riemannian metric g the integral of its scalar curvature S, that is

$$\mathcal{E} : \mathcal{M} \longrightarrow \mathbb{R}, \quad g \longmapsto \mathcal{E}(g) = \int_M S v^g, \tag{1.1}$$

where $\mathcal{M}$ is the set of smooth Riemannian metrics on M, and v^g the volume form with respect to g. It is the action functional that defines the dynamics of gravity in general relativity [3, 4, 5, 6, 8, 16].

One of the simplest modifications to general relativity is the $F(S)$ gravity in which the Lagrangian density F is an arbitrary smooth function of the scalar curvature S of a Riemannian manifold (M, g). When $F(s) = s$, gives the classical Einstein-Hilbert functional, therefore the Einstein gravity, corresponds to $F(S) = S$. The Euler-Lagrange equation of the generalized Einstein-Hilbert functional (it is known by Einstein-Hilbert functional in $f(R)$ gravity, or briefly F-Einstein-Hilbert functional) with respect to g is proved by A. D. Felice, S. Tsujikawa in [7], and T. P. Sotiriou, V. Faraoni in [17].

The second variation of Einstein-Hilbert functional at Einstein metrics was considered in [11]. In [9], K. Kröncke study the second variation of the Einstein-Hilbert functional on Einstein metrics, he find some conditions for stability of Einstein manifolds with respect to the Einstein-Hilbert functional, i.e., that the second variation of the Einstein-Hilbert functional at the metric is nonpositive in the direction of transverse-traceless tensors. Stability properties of compact Riemannian Einstein manifold play a role in mathematical general relativity [1], and in geometric analysis to understand rigidity of Riemannian structures, for example

Received: 2026.03.29 *Accepted: 2026.06.02*

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the dynamical behaviour of the Ricci flow.
In this paper, we extend the definition of the Einstein tensor, where we calculate the first variation of the F-Einstein-Hilbert functional, and we conclude the generalized Einstein tensor. We prove that the generalized Einstein tensor is divergence-free. We study the second variation of the F-Einstein-Hilbert functional on the Riemannian manifold. The second variation formula gives a tool/is a prerequisite for the study the stability of any generalized Einstein manifold, and to see if the F-Einstein-Hilbert functional has extremality properties at some critical points. The smooth function F can be chosen for the existence and the stability of such Riemannian metrics which provide additional information on Riemannian manifolds.

2. F-Einstein-Hilbert functional

First, we give some definitions. Let (M,g) be an n-dimensional Riemannian manifold, and let $X, X_1, ..., X_{q-1}, Y, Z \in \Gamma(TM)$. By R, Ric and S we denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of (M,g). Thus R, Ric and S are defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{2.2}$$

$$\operatorname{Ric}(X,Y) = g(R(X,e_i)e_i,Y), \quad S = \operatorname{Ric}(e_i,e_i), \tag{2.3}$$

where ∇ is the Levi-Civita connection with respect to g, $\{e_1,...,e_n\}$ is an orthonormal frame. Given a smooth function f on M, the gradient of f is defined by

$$g(\operatorname{grad} f, X) = X(f), \tag{2.4}$$

the Hessian of f is defined by

$$(\operatorname{Hess} f)(X,Y) = g(\nabla_X \operatorname{grad} f, Y), \tag{2.5}$$

the Laplacian of f is defined by

$$\Delta f = -\operatorname{Tr}(\operatorname{Hess} f). \tag{2.6}$$

The divergence of $(0,q)$-tensor α on M is defined by

$$(\delta\alpha)(X_1,...,X_{q-1}) = -(\nabla_{e_i}\alpha)(e_i,X_1,...,X_{q-1}). \tag{2.7}$$

The formal adjoint of the divergence $\delta : \Gamma(\otimes^2 T^*M) \longrightarrow \Gamma(T^*M)$ is the map
$\delta^* : \Gamma(T^*M) \longrightarrow \Gamma(\otimes^2 T^*M)$ defined by

$$(\delta^*\alpha)(X,Y) = \frac{1}{2}\big((\nabla_X\alpha)Y + (\nabla_Y\alpha)X\big). \tag{2.8}$$

The formal adjoint of the Levi-civita connection ∇ is given by

$$(\nabla^*\alpha)(X_1,...,X_{q-1}) = -(\nabla_{e_i}\alpha)(e_i,X_1,...,X_{q-1}), \tag{2.9}$$

where $\alpha \in \Gamma(T^*M \otimes T^{(p,q)}M)$, and $\{e_1,...,e_n\}$ is an orthonormal frame.
The composition of $T, Q \in \Gamma(\odot^2 T^*M)$ is defined by

$$(T \circ Q)(X,Y) = T(X,e_i)Q(Y,e_i), \tag{2.10}$$

where $\{e_1, ..., e_n\}$ is an orthonormal frame on M.
For $T \in \Gamma(\otimes^2 T^*M)$, we define $T^\sigma \in \Gamma(\odot^2 T^*M)$ by

$$T^\sigma(X,Y) = \frac{1}{2}\big(T(X,Y) + T(Y,X)\big). \tag{2.11}$$

We define an endomorphism $\overset{\circ}{R} : \Gamma(\odot^2 T^*M) \longrightarrow \Gamma(\odot^2 T^*M)$ by

$$(\overset{\circ}{R}T)(X,Y) = T(R(e_i,X)Y, e_i). \tag{2.12}$$

For $T \in \Gamma(\odot^2 T^*M)$, we define the Lichnerowicz Laplacian by

$$\Delta_L T = \nabla^*\nabla T + \operatorname{Ric} \circ T + T \circ \operatorname{Ric} - 2\overset{\circ}{R}T. \tag{2.13}$$

(For more details, see for example [4], [15]).

Definition 2.1 ([7], [17]). *We let $\mathcal{M}$ denote the space of Riemannian metrics on a closed orientable manifold M. The generalized Einstein-Hilbert functional (or F-Einstein-Hilbert functional) is defined by*

$$\mathcal{E}_F : \mathcal{M} \longrightarrow \mathbb{R}, \quad g \longmapsto \mathcal{E}_F(g) = \int_M F(S) v^g, \tag{2.14}$$

where S is the scalar curvature of (M,g), and $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-constant smooth function.

The Definition 2.1, is a natural generalization of Einstein-Hilbert functional or the total scalar curvature, when F is the identity map, then $\mathcal{E}_F$ reduces to the usual Einstein-Hilbert functional whose second order infinitesimal behaviour is well understood (see [3, 4, 5, 6, 8, 10, 11, 12, 13, 16]).
Let (M,g) be a closed orientable Riemannian manifold. Consider a smooth one-parameter variation of the metric g, i.e., a smooth family of metrics (g_t) with $-\epsilon < t < \epsilon$, such that $g_0 = g$. Take local coordinates (x^i) on M, and write the metric on M in the usual way as $g_t = g_{i,j}(t,x)dx^i \otimes dx^j$. Write $h = (\partial g_t/\partial t)_{t=0}$, then $h \in \Gamma(\odot^2 T^*M)$ is a symmetric 2-covariant tensor field on M, we get the following.

Theorem 2.1 ([7], [17]). *The first variation of the F-Einstein-Hilbert functional in the direction of h is given by the formula*

$$\frac{d}{dt}\mathcal{E}_F(g_t)\Big|_{t=0} = -\int_M \langle E_F(g), h\rangle v^g, \tag{2.15}$$

where $\langle , \rangle$ is the induced Riemannian metric on $\otimes^2 T^*M$,

$$E_F(g) = F'(S)\operatorname{Ric} - \operatorname{Hess} F'(S) - \big(\Delta F'(S) + \frac{1}{2}F(S)\big)g, \tag{2.16}$$

and F' is the derivative of the function F.

Definition 2.2. *$E_F(g)$ is called the generalized Einstein tensor (or F-Einstein tensor).*

For the proof of Theorem 2.1, we need the following lemma.

Lemma 2.1 ([14], [18]). *Let (M,g) be a Riemannian manifold. Then, the differential at g, in the direction of h, of the volume element and the scalar curvature are given by the following formulas*

$$\frac{\partial v^{g_t}}{\partial t}\Big|_{t=0} = \frac{1}{2}(\operatorname{Tr} h)v^g = \frac{1}{2}\langle g, h\rangle v^g, \tag{2.17}$$

$$\frac{\partial S_t}{\partial t}\Big|_{t=0} = \Delta(\operatorname{Tr} h) + \delta(\delta h) - \langle \operatorname{Ric}, h\rangle. \tag{2.18}$$

Proof of Theorem 2.1. First note that

$$\frac{d}{dt}\mathcal{E}_F(g_t)\Big|_{t=0} = \int_M \Big[\frac{\partial F(S_t)}{\partial t} v^{g_t} + F(S_t)\frac{\partial v^{g_t}}{\partial t}\Big]_{t=0}, \tag{2.19}$$

for all $t \in (-\epsilon, \epsilon)$, we have

$$\frac{\partial F(S_t)}{\partial t} = F'(S_t)\frac{\partial S_t}{\partial t},$$

by the Lemma 2.1, we obtain

$$\begin{aligned}\frac{\partial F(S_t)}{\partial t}\Big|_{t=0} &= F'(S)\Delta(\operatorname{Tr} h) + F'(S)\delta(\delta h)\\ &\quad -F'(S)\langle \operatorname{Ric}, h\rangle.\end{aligned} \tag{2.20}$$

Calculating in a normal frame at $x \in M$ we have

$$\begin{aligned}F'(S)\Delta(\operatorname{Tr} h) &= -F'(S)e_i\big(e_i(\operatorname{Tr} h)\big)\\ &= -e_i\big(F'(S)e_i(\operatorname{Tr} h)\big) + e_i(F'(S))e_i(\operatorname{Tr} h)\\ &= -e_i\big(F'(S)e_i(\operatorname{Tr} h)\big) + e_i\big(e_i(F'(S))\operatorname{Tr} h\big)\\ &\quad -e_i\big(e_i(F'(S))\big)\operatorname{Tr} h,\end{aligned} \tag{2.21}$$

so, the first term in the right-hand side of (2.20), is given by

$$\begin{aligned}F'(S)\Delta(\operatorname{Tr} h) &= \delta\big(F'(S)d(\operatorname{Tr} h)\big) - \delta\big((\operatorname{Tr} h)dF'(S)\big)\\ &\quad +\Delta(F'(S))\langle g, h\rangle.\end{aligned} \tag{2.22}$$

If $f \in C^\infty(M)$ and $\alpha \in \Gamma(T^*M)$, then (see [18], [15])

$$\delta(f\alpha) = -\langle df, \alpha\rangle + f\delta\alpha, \tag{2.23}$$

with $\langle df, \alpha\rangle = \alpha(\operatorname{grad} f)$. Applying this formula, gives

$$F'(S)\delta(\delta h) = \delta\big(F'(S)\delta h\big) + \langle dF'(S), \delta h\rangle, \tag{2.24}$$

by using the following formula (see [18])

$$(\delta T)(Z) = \delta\big(T(\cdot, Z)\big) + \frac{1}{2}\big\langle T, \mathcal{L}_Z g\big\rangle, \tag{2.25}$$

where $\mathcal{L}_Z g$ is the Lie-derivative of g along $Z \in \Gamma(TM)$ (see [15]), and $T \in \Gamma(\odot^2 T^*M)$, we get

$$\begin{aligned}\langle dF'(S), \delta h\rangle &= (\delta h)\big(\operatorname{grad} F'(S)\big)\\ &= \delta\big(h(\cdot, \operatorname{grad} F'(S))\big) + \frac{1}{2}\big\langle h, \mathcal{L}_{\operatorname{grad} F'(S)} g\big\rangle\\ &= \delta\big(h(\cdot, \operatorname{grad} F'(S))\big) + \big\langle h, \operatorname{Hess} F'(S)\big\rangle,\end{aligned} \tag{2.26}$$

by equations (2.24) and (2.26), the second term on the left-hand side of (2.20) is

$$\begin{aligned}F'(S)\delta(\delta h) &= \delta\big(F'(S)\delta h\big) + \delta\big(h(\cdot, \operatorname{grad} F'(S))\big)\\ &\quad +\big\langle h, \operatorname{Hess} F'(S)\big\rangle.\end{aligned} \tag{2.27}$$

Substituting (2.22) and (2.27) in (2.20), we obtain

$$\begin{aligned}\frac{\partial F(S_t)}{\partial t}\Big|_{t=0} &= \delta\big(F'(S)d(\operatorname{Tr} h)\big) - \delta\big((\operatorname{Tr} h)dF'(S)\big)\\ &+\Delta(F'(S))\langle g,h\rangle + \delta\big(F'(S)\delta h\big)\\ &+\delta\big(h(\cdot,\operatorname{grad} F'(S))\big) + \big\langle h, \operatorname{Hess} F'(S)\big\rangle\\ &-F'(S)\langle \operatorname{Ric}, h\rangle. \end{aligned}\tag{2.28}$$

From equation (2.28) and the Lemma 2.1, we have

$$\begin{aligned}\Big[\frac{\partial F(S_t)}{\partial t}v^{g_t} + F(S_t)\frac{\partial v^{g_t}}{\partial t}\Big]_{t=0} &= \Big\{\delta\big(F'(S)d(\operatorname{Tr} h)\big) - \delta\big((\operatorname{Tr} h)dF'(S)\big)\\ &+\Delta(F'(S))\langle g,h\rangle + \delta\big(F'(S)\delta h\big)\\ &+\delta\big(h(\cdot,\operatorname{grad} F'(S))\big) + \big\langle h, \operatorname{Hess} F'(S)\big\rangle\\ &-F'(S)\langle \operatorname{Ric}, h\rangle\Big\}v^g + \frac{F(S)}{2}\langle g,h\rangle v^g. \end{aligned}\tag{2.29}$$

Substituting the formula (2.29) in (2.19), and consider the divergence theorem (see [2]), the Theorem 2.1 follows. □

Remark 2.1. *Let $X, Y \in \Gamma(TM)$, we have*

$$\begin{aligned}\operatorname{Hess} F'(S)(X,Y) &= X(Y(F'(S))) - (\nabla_X Y)(F'(S))\\ &= X(F''(S)Y(S)) - F''(S)(\nabla_X Y)(S)\\ &= X(F''(S))Y(S) + F''(S)X(Y(S)) - F''(S)(\nabla_X Y)(S)\\ &= F'''(S)X(S)Y(S) + F''(S)(\operatorname{Hess} S)(X,Y).\end{aligned}$$

According to this formula, the F-Einstein tensor is given by

$$\begin{aligned}E_F(g) &= F'(S)\operatorname{Ric} - F''(S)\operatorname{Hess} S - F'''(S)dS\otimes dS\\ &\quad -\big(F''(S)\Delta S + F'''(S)|\operatorname{grad} S|^2 + \frac{1}{2}F(S)\big)g.\end{aligned}\tag{2.30}$$

Remark 2.2. *Let (M, g) be a Riemannian manifold, we get the following*

- *If $F(s) = s$, for all $s \in \mathbb{R}$, the F-Einstein tensor is given by the formula (see* [4], [14]*)*

$$E_F(g) = E(g) = \operatorname{Ric} - \frac{S}{2}g, \tag{2.31}$$

  is the Einstein tensor.
- *If $F(s) = s^2$, for all $s \in \mathbb{R}$, the F-Einstein tensor is given by (see* [4], [5], [6]*)*

$$E_F(g) = 2S\operatorname{Ric} - 2\operatorname{Hess} S - \big(2\Delta S + \frac{S^2}{2}\big)g. \tag{2.32}$$

From Theorem 2.1, we deduce.

Theorem 2.2 ([7], [17])**.** *A Riemannian metric g is a critical point of the F-Einstein-Hilbert functional if and only if*

$$F'(S)\operatorname{Ric} - \operatorname{Hess} F'(S) - \big(\Delta F'(S) + \frac{1}{2}F(S)\big)g = 0, \tag{2.33}$$

where $F:\mathbb{R}\longrightarrow\mathbb{R}$ is a non-constant smooth function.

By taking traces in (2.33), we obtain

$$SF'(S)+(1-n)\Delta F'(S)-\frac{n}{2}F(S)=0. \tag{2.34}$$

Theorem 2.3. *Let (M,g) be a Riemannian manifold. Then, the divergence of the generalized Einstein tensor is zero (that is, $\delta E_F(g)=0$).*

Proof. Let $F:\mathbb{R}\longrightarrow\mathbb{R}$ be a smooth function, calculating in a normal frame $\{e_i\}$ at $x\in M$, with $X=e_j$, we have

$$\delta E_F(g)(X)=-(\nabla_{e_i}E_F(g))(e_i,X)=-e_i\big(E_F(g)(e_i,X)\big), \tag{2.35}$$

by the definitions of generalized Einstein tensor, and the Hessian tensor, we get

$$\begin{aligned} E_F(g)(e_i,X) &= F'(S)\operatorname{Ric}(e_i,X)-g(\nabla_{e_i}\operatorname{grad}F'(S),X)\\ &\quad -\big(\Delta F'(S)+\frac{1}{2}F(S)\big)g(e_i,X), \end{aligned} \tag{2.36}$$

substituting (2.36) in (2.35), and consider the definition of gradient operator, we obtain

$$\begin{aligned} \delta E_F(g)(X) &= -\operatorname{Ric}(\operatorname{grad}F'(S),X)-F'(S)e_i\big(\operatorname{Ric}(e_i,X)\big)\\ &\quad +g(\nabla_{e_i}\nabla_X\operatorname{grad}F'(S),e_i)+X(\Delta F'(S))+\frac{1}{2}X(F(S)), \end{aligned}$$

by the definitions of the divergence, and the curvature tensor, with $[e_i,X]=0$, we conclude that

$$\begin{aligned} \delta E_F(g)(X) &= -\operatorname{Ric}(\operatorname{grad}F'(S),X)+F'(S)(\delta\operatorname{Ric})(X)\\ &\quad +g(R(e_i,X)\operatorname{grad}F'(S),e_i)+g(\nabla_X\nabla_{e_i}\operatorname{grad}F'(S),e_i)\\ &\quad +X(\Delta F'(S))+\frac{1}{2}X(F(S)), \end{aligned} \tag{2.37}$$

note that

$$\operatorname{Ric}(\operatorname{grad}F'(S),X)=g(R(e_i,X)\operatorname{grad}F'(S),e_i), \tag{2.38}$$

$$F'(S)(\delta\operatorname{Ric})(X)=-\frac{1}{2}X(F(S))=-\frac{1}{2}F'(S)X(S), \tag{2.39}$$

$$g(\nabla_X\nabla_{e_i}\operatorname{grad}F'(S),e_i)=-X(\Delta F'(S)). \tag{2.40}$$

Substituting the formulas (2.38), (2.39) and (2.40) in (2.37), the Theorem 2.3 follows. □

Remark 2.3.

- *If $E_F(g)=fg$ for some function f on M, then f is constant function on M (because $\delta E_F(g)=0$).*
- *The condition $E_F(g)=\lambda g$ is equivalent to*

$$F'(S)\operatorname{Ric}-\operatorname{Hess}F'(S)=\mu g, \tag{2.41}$$

for some function μ on M, it is also equivalent to

$$F'(S)\operatorname{Ric}-F''(S)\operatorname{Hess}S-F'''(S)dS\otimes dS=\mu g, \tag{2.42}$$

(see equation (2.30)).

- *If $F(s) = s$, for all $s \in \mathbb{R}$, then $E_F(g) = \lambda g$ if (M,g) is Einstein manifold, that is* $\mathrm{Ric} = \mu g$ *for some constant μ (see [4]).*

Example 2.1. *Let $M = (0,\infty)\times\mathbb{R}^3$ equipped with the Riemannian metric $g = dt^2 + t^2(dx^2 + dy^2 + dz^2)$. Let $F(s) = s^\alpha$ for some constant α. Then, $E_F(g) = 0$ if and only if $\alpha = \frac{1\pm\sqrt{3}}{2}$.*

Example 2.2. *Let $M = \mathbb{S}^n \subset \mathbb{R}^{n+1}$ and $F : \mathbb{R} \longrightarrow \mathbb{R}$ a non-constant smooth function. Then, the induced Riemannian metric $g^{\mathbb{S}^n}$ is a critical point of the F-Einstein-Hilbert functional if and only if $F(s_0) = 0$ and $F'(s_0) = 0$ where $s_0 = n(n-1)$ is the scalar curvature of $(\mathbb{S}^n, g^{\mathbb{S}^n})$.*

Remark 2.4. *The previous examples prove the following results; There is no equivalence between $E_F(g) = 0$ and $E(g) = 0$ where F is a non-constant smooth function. There exist Riemannian Einstein metrics which are critical points of the F-Einstein-Hilbert functional where $F(s) \neq s$.*

3. The second variation of $\mathcal{E}_F$

Let M be a closed orientable manifold. We denote by

$$\mathcal{M}_c = \{g \in \mathcal{M} \mid \mathrm{Vol}(M,g) = \int_M v^g = c\},$$

for some constant $c > 0$. This is a submanifold of $\mathcal{M}$ of codimension 1, and its tangent space at $g \in \mathcal{M}_c$ is given by

$$T_g\mathcal{M}_c = \{T \in \Gamma(\odot^2 T^*M) \mid \int_M \langle g, T\rangle v^g = 0\}.$$

A Riemannian metric g is a critical point of $\mathcal{E}_F|_{\mathcal{M}_c}$ if and only if $E_F(g)$ is orthogonal to $T_g\mathcal{M}_c$, that is $E_F(g) = \lambda g$ for some constant λ. In the following Theorem, we calculate the second derivative of $\mathcal{E}_F(g_t)$ at $t = 0$ where (g_t) $(-\epsilon < t < \epsilon)$ is a smooth one-parameter variation of such Riemannian metric g which enables us to know the extremality properties of $\mathcal{E}_F$. Write

$$h = \frac{\partial g_t}{\partial t}\Big|_{t=0}, \quad k = \frac{\partial^2 g_t}{\partial t^2}\Big|_{t=0}, \tag{3.43}$$

then $h, k \in \Gamma(\odot^2 T^*M)$. Under the notation above we have the following.

Theorem 3.1. *Let (M,g) be a closed orientable Riemannian manifold with volume c. Suppose that $E_F(g) = \lambda g$, for some constant λ, then the second variation of $\mathcal{E}_F|_{\mathcal{M}_c}$ at g in the direction of h is given by*

$$\frac{d^2}{dt^2}\mathcal{E}_F(g_t)\Big|_{t=0} = \int_M \langle T_0(h) + T_1(h), h\rangle v^g,$$

where $T_0(h)$, $T_1(h)$ are defined by

$$\begin{aligned} T_0(h) \quad = \quad & -\frac{F'(S)}{2}\nabla^*\nabla h + F'(S)\overset{\circ}{R}h + F'(S)\delta^*(\delta h) + \frac{1}{2}F'(S)\,\mathrm{Hess}(\mathrm{Tr}\,h) \\ & + \frac{F'(S)}{2}\big[\Delta(\mathrm{Tr}\,h) + \delta(\delta h)\big]g - \frac{1}{2}\big[\lambda + \frac{1}{2}F(S)\big](\mathrm{Tr}\,h)g, \end{aligned}$$

$$\begin{aligned} T_1(h) &= -f\,\mathrm{Ric} + \mathrm{Hess}\, f + (\Delta f)g - h(\nabla.\,\mathrm{grad}\, F'(S),\cdot)^\sigma - (\nabla.h)(\cdot,\mathrm{grad}\, F'(S))^\sigma \\ &+\frac{1}{2}\nabla_{\mathrm{grad}\, F'(S)}h - \langle \delta h + \frac{1}{2}d(\mathrm{Tr}\, h), dF'(S)\rangle g - \frac{1}{2}(\Delta F'(S))(\mathrm{Tr}\, h)g \\ &+\frac{1}{2}\langle \mathrm{Hess}\, F'(S), h\rangle g, \end{aligned}$$

and $f = F''(S)\big[\Delta(\mathrm{Tr}\, h) + \delta(\delta h) - \langle \mathrm{Ric}, h\rangle\big]$.

For the proof of Theorem 3.1, we need the following lemmas.

Lemma 3.1. *Let* $T_t, Q_t \in \Gamma(\odot^2 T^*M)$ *all dependent of time* $t \in (-\epsilon,\epsilon)$ *with* $T_0 = T$ *and* $Q_0 = Q$. *Then*

$$\frac{\partial}{\partial t}\Big|_{t=0}\big\langle T_t, Q_t\big\rangle_t = \big\langle \frac{\partial T_t}{\partial t}\Big|_{t=0}, Q\big\rangle + \big\langle T, \frac{\partial Q_t}{\partial t}\Big|_{t=0}\big\rangle - 2\big\langle T, h\circ Q\big\rangle,$$

where $\langle,\rangle_t$ *is the induced Riemannian metric (with respect to* g_t*) on* $\otimes^2 T^*M$.

Proof. We have

$$\big\langle T_t, Q_t\big\rangle_t = T_t^{ij}Q_t^{ab}g_t^{ia}g_t^{jb},$$

so that

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0}\big\langle T_t, Q_t\big\rangle_t &= \frac{\partial T_t^{ij}}{\partial t}\Big|_{t=0}Q^{ab}g^{ia}g^{jb} + T^{ij}\frac{\partial Q_t^{ab}}{\partial t}\Big|_{t=0}g^{ia}g^{jb} \\ &+T^{ij}Q^{ab}\frac{\partial g_t^{ia}}{\partial t}\Big|_{t=0}g^{jb} + T^{ij}Q^{ab}g^{ia}\frac{\partial g_t^{jb}}{\partial t}\Big|_{t=0}, \end{aligned}$$

since $\frac{\partial g_t^{ia}}{\partial t}\big|_{t=0} = -g^{iu}g^{av}h_{uv}$ and $\frac{\partial g_t^{jb}}{\partial t}\big|_{t=0} = -g^{ju}g^{bv}h_{uv}$ (see [14]), we get

$$\begin{aligned} \frac{\partial}{\partial t}\Big|_{t=0}\big\langle T_t, Q_t\big\rangle_t &= \big\langle \frac{\partial T_t}{\partial t}\Big|_{t=0}, Q\big\rangle + \big\langle T, \frac{\partial Q_t}{\partial t}\Big|_{t=0}\big\rangle \\ &-T^{ij}Q^{ab}g^{iu}g^{av}h_{uv}g^{jb} - T^{ij}Q^{ab}g^{ia}g^{ju}g^{bv}h_{uv}, \end{aligned}$$

note that

$$-T^{ij}Q^{ab}g^{iu}g^{av}h_{uv}g^{jb} - T^{ij}Q^{ab}g^{ia}g^{ju}g^{bv}h_{uv} = -2T^{ij}Q^{ab}g^{iu}g^{av}h_{uv}g^{jb},$$

on the other hand

$$\begin{aligned} -2\langle T, h\circ Q\rangle &= -2T^{ij}(h\circ Q)^{ub}g^{iu}g^{jb} \\ &= -2T^{ij}g^{av}h_{uv}Q^{ab}g^{iu}g^{jb}. \end{aligned}$$

□

Lemma 3.2. *Let* (f_t) $(-\epsilon < t < \epsilon)$ *be a time dependent family of smooth functions on* M *with* $f_0 = f$. *Then, the first variation of the Hessian and the Laplacian are given by*

$$\frac{\partial\, \mathrm{Hess}_t\, f_t}{\partial t}\Big|_{t=0} = \mathrm{Hess}\Big(\frac{\partial f_t}{\partial t}\Big|_{t=0}\Big) - (\nabla.h)(\cdot,\mathrm{grad}\, f)^\sigma + \frac{1}{2}\nabla_{\mathrm{grad}\, f}h,$$

$$\frac{\partial \Delta_t f_t}{\partial t}\Big|_{t=0} = \Delta\Big(\frac{\partial f_t}{\partial t}\Big|_{t=0}\Big) - \big\langle \delta h + \frac{1}{2}d(\mathrm{Tr}\, h), df\big\rangle + \big\langle\, \mathrm{Hess}\, f, h\big\rangle,$$

where $\mathrm{Hess}_t\, f_t$ *(resp.* $\Delta_t f_t$*) is the Hessian (resp. Laplacian) of* f_t *with respect to the metric* g_t.

Proof. By the definition of Hessian (2.5), we obtain

$$\begin{aligned}\frac{\partial \operatorname{Hess}_t f_t}{\partial t}(X,Y) &= \frac{\partial}{\partial t}\Big[X(Y(f_t)) - (\nabla^t_X Y)(f_t)\Big]\\ &= \frac{\partial}{\partial t}\Big[X(Y(f_t)) - g(\nabla^t_X Y, \operatorname{grad} f_t)\Big]\\ &= X(Y(\frac{\partial f_t}{\partial t})) - g(\frac{\partial}{\partial t}\nabla^t_X Y, \operatorname{grad} f_t) - g(\nabla^t_X Y, \operatorname{grad}\frac{\partial f_t}{\partial t}),\end{aligned} \tag{3.44}$$

where ∇^t is the Levi-Civita connection with respect to g_t. The first variation of the Levi-Civita connection in the direction of h is given by the formula

$$g(\frac{\partial}{\partial t}\nabla^t_X Y\Big|_{t=0}, Z) = \frac{1}{2}\Big[(\nabla_X h)(Y,Z) + (\nabla_Y h)(X,Z) - (\nabla_Z h)(X,Y)\Big], \tag{3.45}$$

(see [4]). Here $X, Y, Z \in \Gamma(TM)$ (all independent of time t). We conclude that

$$\frac{\partial \operatorname{Hess}_t f_t}{\partial t}\Big|_{t=0} = \operatorname{Hess}\Big(\frac{\partial f_t}{\partial t}\Big|_{t=0}\Big) - (\nabla . h)(\cdot, \operatorname{grad} f)^\sigma + \frac{1}{2}\nabla_{\operatorname{grad} f} h. \tag{3.46}$$

By the Lemma 3.1, the first variation of $\Delta_t f_t$ is given by

$$\begin{aligned}\frac{\partial \Delta_t f_t}{\partial t}\Big|_{t=0} &= -\frac{\partial}{\partial t}\Big|_{t=0}\big\langle \operatorname{Hess}_t f_t, g_t\big\rangle_t\\ &= -\big\langle \frac{\partial}{\partial t}\operatorname{Hess}_t f_t\Big|_{t=0}, g\big\rangle - \big\langle \operatorname{Hess} f, h\big\rangle + 2\big\langle \operatorname{Hess} f, h\circ g\big\rangle,\end{aligned} \tag{3.47}$$

by equations (3.46), (3.47), with $h \circ g = h$, we have

$$\begin{aligned}\frac{\partial \Delta_t f_t}{\partial t}\Big|_{t=0} &= \Delta\Big(\frac{\partial f_t}{\partial t}\Big|_{t=0}\Big) + \operatorname{Tr}(\nabla . h)(\cdot, \operatorname{grad} f)^\sigma\\ &\quad -\frac{1}{2}\operatorname{Tr}\nabla_{\operatorname{grad} f} h + \big\langle \operatorname{Hess} f, h\big\rangle,\end{aligned} \tag{3.48}$$

and note that

$$\operatorname{Tr}(\nabla . h)(\cdot, \operatorname{grad} f)^\sigma = -\big\langle \delta h, df\big\rangle, \tag{3.49}$$

$$-\frac{1}{2}\operatorname{Tr}\nabla_{\operatorname{grad} f} h = -\frac{1}{2}\big\langle d(\operatorname{Tr} h), df\big\rangle. \tag{3.50}$$

The proof is completed. □

Proof of Theorem 3.1. First note that

$$\frac{d^2}{dt^2}\mathcal{E}_F(g_t)\Big|_{t=0} = -\frac{d}{dt}\Big|_{t=0}\int_M \langle E_F(g_t), \frac{\partial g_t}{\partial t}\rangle_t v^{g_t}, \tag{3.51}$$

by the variational formulas in Lemma 3.1, we have

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{E}_F(g_t)\Big|_{t=0} &= -\int_M \langle \frac{\partial}{\partial t}E_F(g_t)\Big|_{t=0}, h\rangle v^g\\ &\quad -\int_M \langle E_F(g), k\rangle v^g\\ &\quad +2\int_M \langle E_F(g), h\circ h\rangle v^g\\ &\quad -\frac{1}{2}\int_M \langle E_F(g), h\rangle(\operatorname{Tr} h) v^g.\end{aligned} \tag{3.52}$$

Since $E_F(g) = \lambda g$, we obtain

$$2\int_M \langle E_F(g), h\circ h\rangle v^g = 2\lambda\int_M |h|^2 v^g, \tag{3.53}$$

$$-\frac{1}{2}\int_M \langle E_F(g), h\rangle(\operatorname{Tr} h)v^g = -\frac{\lambda}{2}\int_M (\operatorname{Tr} h)^2 v^g. \tag{3.54}$$

Since $\operatorname{Vol}(M, g_t) = c$, we have

$$\frac{d^2}{dt^2}\Big|_{t=0}\operatorname{Vol}(M, g_t) = \int_M \frac{\partial^2 v^{g_t}}{\partial t^2}\Big|_{t=0} = 0, \tag{3.55}$$

by equation (3.55), and Lemma 2.1, we get

$$\frac{1}{2}\int_M \frac{\partial}{\partial t}\Big|_{t=0}\Big[(\operatorname{Tr}_t \frac{\partial g_t}{\partial t})v^{g_t}\Big] = 0, \tag{3.56}$$

where $\operatorname{Tr}_t \frac{\partial g_t}{\partial t}$ is the trace of $\frac{\partial g_t}{\partial t}$ with respect to g_t, from equation (3.56), and the Lemmas 2.1 and 3.1, we obtain

$$\begin{aligned} 0 &= \int_M \Big[\frac{\partial}{\partial t}(\operatorname{Tr}_t \frac{\partial g_t}{\partial t})\Big|_{t=0} v^g + (\operatorname{Tr} h)\frac{\partial v^{g_t}}{\partial t}\Big|_{t=0}\Big] \\ &= \int_M \Big[\frac{\partial}{\partial t}\langle g_t, \frac{\partial g_t}{\partial t}\rangle_t\Big|_{t=0} v^g + \frac{1}{2}(\operatorname{Tr} h)^2 v^g\Big] \\ &= \int_M \big[|h|^2 + (\operatorname{Tr} k) - 2\langle g, h\circ h\rangle + \frac{1}{2}(\operatorname{Tr} h)^2\big]v^g \\ &= \int_M \big[-|h|^2 + (\operatorname{Tr} k) + \frac{1}{2}(\operatorname{Tr} h)^2\big]v^g, \end{aligned} \tag{3.57}$$

by equation (3.57), the second term on the left-hand side of (3.52) is

$$\begin{aligned} -\int_M \langle E_F(g), k\rangle v^g &= -\lambda\int_M (\operatorname{Tr} k)v^g \\ &= \int_M \big[-\lambda|h|^2 + \frac{\lambda}{2}(\operatorname{Tr} h)^2\big]v^g. \end{aligned} \tag{3.58}$$

We compute

$$\begin{aligned} \frac{\partial}{\partial t}E_F(g_t)\Big|_{t=0} &= \frac{\partial F'(S_t)}{\partial t}\Big|_{t=0}\operatorname{Ric} + F'(S)\frac{\partial \operatorname{Ric}_t}{\partial t}\Big|_{t=0} \\ &\quad -\frac{\partial \operatorname{Hess}_t F'(S_t)}{\partial t}\Big|_{t=0} - \frac{\partial \Delta_t F'(S_t)}{\partial t}\Big|_{t=0} g \\ &\quad -\Delta F'(S)h - \frac{1}{2}\frac{\partial F(S_t)}{\partial t}\Big|_{t=0} g - \frac{1}{2}F(S)h, \end{aligned} \tag{3.59}$$

note that, from the Lemma 2.1, we have

$$\frac{\partial F(S_t)}{\partial t}\Big|_{t=0} = F'(S)\big[\Delta(\operatorname{Tr} h) + \delta(\delta h) - \langle \operatorname{Ric}, h\rangle\big], \tag{3.60}$$

$$\frac{\partial F'(S_t)}{\partial t}\Big|_{t=0} = F''(S)\big[\Delta(\operatorname{Tr} h) + \delta(\delta h) - \langle \operatorname{Ric}, h\rangle\big], \tag{3.61}$$

by Lemma 3.1, and the definition of Lichnerowicz Laplacian, we get

$$\begin{aligned}\frac{\partial \operatorname{Ric}_t}{\partial t}\Big|_{t=0} &= \frac{1}{2}\Delta_L h - \delta^*(\delta h) - \frac{1}{2}\operatorname{Hess}(\operatorname{Tr} h)\\ &= \frac{1}{2}\nabla^*\nabla h - \overset{\circ}{R}h + \frac{1}{2}\operatorname{Ric}\circ h + \frac{1}{2}h\circ\operatorname{Ric}\\ &\quad -\delta^*(\delta h) - \frac{1}{2}\operatorname{Hess}(\operatorname{Tr} h), \end{aligned}\tag{3.62}$$

from equations (3.59), (3.60), (3.61), (3.62), and the Lemma 3.2, we have

$$\begin{aligned}\frac{\partial}{\partial t}E_F(g_t)\Big|_{t=0} &= f\operatorname{Ric} + F'(S)\Big[\frac{1}{2}\nabla^*\nabla h - \overset{\circ}{R}h + \frac{1}{2}\operatorname{Ric}\circ h + \frac{1}{2}h\circ\operatorname{Ric}\\ &\quad -\delta^*(\delta h) - \frac{1}{2}\operatorname{Hess}(\operatorname{Tr} h)\Big] - \operatorname{Hess} f + (\nabla.h)(\cdot, \operatorname{grad} F'(S))^\sigma\\ &\quad -\frac{1}{2}\nabla_{\operatorname{grad} F'(S)}h - (\Delta f)g + \Big\langle \delta h + \frac{1}{2}d(\operatorname{Tr} h), dF'(S)\Big\rangle g\\ &\quad -\Big\langle \operatorname{Hess} F'(S), h\Big\rangle g - (\Delta F'(S))h - \frac{F'(S)}{2}\Big[\Delta(\operatorname{Tr} h)\\ &\quad +\delta(\delta h) - \langle \operatorname{Ric}, h\rangle\Big]g - \frac{1}{2}F(S)h, \end{aligned}\tag{3.63}$$

where $f = F''(S)\big[\Delta(\operatorname{Tr} h) + \delta(\delta h) - \langle \operatorname{Ric}, h\rangle\big]$.

Note that, from the definitions of the composition (2.10) and the F-Einstein tensor (2.16), and the condition $E_F(g) = \lambda g$, we get

$$\begin{aligned}\frac{F'(S)}{2}\big(\operatorname{Ric}\circ h + h\circ\operatorname{Ric}\big) &= \big[\lambda + \Delta F'(S) + \frac{1}{2}F(S)\big]h\\ &\quad + h(\nabla.\operatorname{grad} F'(S), \cdot)^\sigma, \end{aligned}\tag{3.64}$$

$$\begin{aligned}\frac{F'(S)}{2}\langle \operatorname{Ric}, h\rangle g &= \frac{1}{2}\big[\lambda + \Delta F'(S) + \frac{1}{2}F(S)\big](\operatorname{Tr} h)g\\ &\quad + \frac{1}{2}\langle \operatorname{Hess} F'(S), h\rangle g. \end{aligned}\tag{3.65}$$

From equations (3.52), (3.53), (3.54), (3.58), (3.63), (3.64) and (3.65), we have

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{E}_F(g_t)\Big|_{t=0} &= \int_M \Big\langle -f\operatorname{Ric} - \frac{F'(S)}{2}\nabla^*\nabla h + F'(S)\overset{\circ}{R}h\\ &\quad - h(\nabla.\operatorname{grad} F'(S), \cdot)^\sigma + F'(S)\delta^*(\delta h)\\ &\quad + \frac{1}{2}F'(S)\operatorname{Hess}(\operatorname{Tr} h) + \operatorname{Hess} f - (\nabla.h)(\cdot, \operatorname{grad} F'(S))^\sigma\\ &\quad + \frac{1}{2}\nabla_{\operatorname{grad} F'(S)}h + (\Delta f)g - \langle \delta h + \frac{1}{2}d(\operatorname{Tr} h), dF'(S)\rangle g\\ &\quad + \frac{F'(S)}{2}[\Delta(\operatorname{Tr} h) + \delta(\delta h)]g - \frac{1}{2}\big[\lambda + \Delta F'(S)\\ &\quad + \frac{1}{2}F(S)\big](\operatorname{Tr} h)g + \frac{1}{2}\langle \operatorname{Hess} F'(S), h\rangle g, h\Big\rangle v^g, \end{aligned}\tag{3.66}$$

the Theorem follows from equation (3.66). □

Remark 3.1. *If $F(s) = s$, for all $s \in \mathbb{R}$. Note that, the condition $E_F(g) = \lambda g$ is equivalent to* $\operatorname{Ric} = \big[\lambda + \frac{S}{2}\big]g$. *That is, g is Einstein Riemannian metric with constant $\mu = \lambda + \frac{S}{2}$. In* *this case, we have*

$$T_0(h) = -\frac{1}{2}\nabla^*\nabla h + \overset{\circ}{R}h + \delta^*(\delta h) + \frac{1}{2}\,\mathrm{Hess}(\mathrm{Tr}\,h) + \frac{1}{2}\big[\Delta(\mathrm{Tr}\,h) + \delta(\delta h)\big]g - \frac{\mu}{2}(\mathrm{Tr}\,h)g,$$

and $T_1(h) = 0$. *From the formula*

$$(\mathrm{Tr}\,h)\delta(\delta h) = \delta\big((\mathrm{Tr}\,h)\delta h\big) + \delta\big(h(\cdot, \mathrm{grad}(\mathrm{Tr}\,h))\big) + \big\langle\, \mathrm{Hess}(\mathrm{Tr}\,h), h\big\rangle,$$

and the divergence theorem (see [2]*), the second variation of* $\mathcal{E}_F|_{\mathcal{M}_c}$ *at* g *in the direction of* h *is given by (see* [4], [9]*)*

$$\frac{d^2}{dt^2}\mathcal{E}_F(g_t)\Big|_{t=0} = \int_M \big\langle -\frac{1}{2}\nabla^*\nabla h + \overset{\circ}{R}h + \delta^*(\delta h) + \frac{1}{2}\Delta(\mathrm{Tr}\,h)g + \delta(\delta h)g - \frac{\mu}{2}(\mathrm{Tr}\,h)g, h\big\rangle v^g.$$

Definition 3.1. *A Riemannian manifold* (M, g) *is said to be* F*-Einstein if* $E_F(g) = \lambda g$ *for some constant* λ*, where* $F : \mathbb{R} \longrightarrow \mathbb{R}$ *is a non-constant smooth function. We call* λ *the* F*-Einstein constant of* g*. We say that a closed orientable* F*-Einstein manifold is stable (resp. strictly stable) if for any* $h \in TT = \mathrm{Tr}^{-1}(0) \cap \delta^{-1}(0)$ *(such tensors are called transverse traceless or* TT*-tensors)*

$$\mathcal{E}''_F(h) = \int_M \langle \widehat{T}_0(h) + \widehat{T}_1(h), h\rangle v^g \leq 0 \quad (\textit{resp. } < 0),$$

where $\widehat{T}_0$, $\widehat{T}_1$ *are the restrictions of* T_0, T_1 *to* TT *respectively, given by*

$$\widehat{T}_0(h) = -\frac{F'(S)}{2}\big[\nabla^*\nabla h - 2\overset{\circ}{R}h\big],$$

$$\widehat{T}_1(h) = -\widehat{f}\,\mathrm{Ric} + \frac{1}{2}\nabla_{\mathrm{grad}\,F'(S)}h,$$

and $\widehat{f} = f|_{TT} = -F''(S)\langle \mathrm{Ric}, h\rangle$.

Remark 3.2.

- *In the Definition 3.1,* $\mathrm{Tr}^{-1}(0)$ *(resp.* $\delta^{-1}(0)$*) denotes the space of symmetric* $(0, 2)$*-tensor fields, whose trace (resp. divergence) vanishes on* (M, g).
- *By using* $\delta h = 0$ *and symmetry of* h*, we obtain the following formulas*

$$\langle \mathrm{Hess}\,\widehat{f}, h\rangle = -\delta\big[h(\mathrm{grad}\,\widehat{f}, \cdot)\big];$$

$$-\langle h(\nabla.\,\mathrm{grad}\,F'(S), \cdot)^\sigma, h\rangle - \langle (\nabla.h)(\cdot, \mathrm{grad}\,F'(S))^\sigma, h\rangle = \delta\left[(h \circ h)(\mathrm{grad}\,F'(S), \cdot)\right].$$

  This explains the disappearance of these terms in $\langle \widehat{T}_1(h), h\rangle$ *after integration over* M.
- *The Definition 3.1, is a natural generalization of stable Einstein manifold (see* [4, 9, 11, 12]*).*
- *We call the operator* $\Delta^F_E(h) = -2\big(\widehat{T}_0(h) + \widehat{T}_1(h)\big)$ *the* F*-Einstein operator. Thus, an* F*-Einstein manifold* (M, g) *is stable, if the* F*-Einstein operator is nonnegative on* TT*-tensors, and strictly stable if it is positive on* TT*-tensors. If* $F(s) = s$ *for all*

$s \in \mathbb{R}$, then the F-Einstein operator reduces to the usual Einstein operator $\Delta_E(h) = \nabla^*\nabla h - 2\overset{\circ}{R}h$.

Theorem 3.2. *Let $F \in C^\infty(\mathbb{R})$. We assume that $F'(s) \geq 0$ (resp. $F'(s) > 0$) for all $s \in \mathbb{R}$. Then, any Einstein manifold of negative sectional curvature is stable (resp. strictly stable) F-Einstein manifold.*

Proof. Let (M, g) be an Einstein manifold with Einstein constant μ, i.e., $\mathrm{Ric} = \mu g$. Thus, (M, g) is F-Einstein manifold with F-Einstein constant $\lambda = \mu F'(S) - \frac{1}{2}F(S)$. Moreover, $\widehat{f} = 0$ and the F-Einstein operator becomes

$$\Delta_E^F(h) = F'(S)\big[\nabla^*\nabla h - 2\overset{\circ}{R}h\big].$$

Hence, if $F'(S) \geq 0$ (resp. $F'(S) > 0$) and the sectional curvature of (M, g) is negative, then (M, g) is stable (resp. strictly stable) F-Einstein manifold (see [9, 10]). □

Remark 3.3. *Let (M, g) be an F-Einstein manifold with F-Einstein constant λ. We assume that (M, g) has constant scalar curvature. If $F'(S) > 0$, according to (2.16), the Riemannian manifold (M, g) is Einstein with Einstein constant $\mu = F'(S)^{-1}\big(\lambda + \frac{1}{2}F(S)\big)$. Moreover, if the sectional curvature of (M, g) is negative, then (M, g) is strictly stable. Here, if the manifold M is even-dimensional, by using (2.34) with $E_F(g) = \lambda g$, we can consider the smooth function $F(s) = -2\lambda + c\, s^{n/2}$ for some $c \in \mathbb{R}$.*

Theorem 3.3. *Let $F \in C^\infty(\mathbb{R})$ and (M, g) be a closed orientable F-Einstein manifold of constant sectional curvature $c > 0$. We assume that $F'(s) \geq 0$ and $F''(s) \leq 0$ for all $s \in \mathbb{R}$. Then, (M, g) is stable. Moreover, if $F'(s) > 0$ for all $s \in \mathbb{R}$, then (M, g) is strictly stable.*

Proof. A straightforward calculation shows that if $h \in TT$,

$$\begin{aligned} -\frac{F'(S)}{2}\langle \nabla^*\nabla h, h\rangle &= -\frac{1}{2}\delta\big[F'(S)\langle \nabla.h, h\rangle\big] - \frac{1}{2}\langle \nabla_{\operatorname{grad} F'(S)} h, h\rangle \\ &\quad -\frac{F'(S)}{2}\operatorname{Tr}\langle \nabla.h, \nabla.h\rangle. \end{aligned} \tag{3.67}$$

By using $\operatorname{Tr} h = 0$, we find that

$$F'(S)\langle \overset{\circ}{R}h, h\rangle = -c\, F'(S)|h|^2. \tag{3.68}$$

From equations (3.67) and (3.69), we conclude that

$$\begin{aligned} \mathcal{E}_F''(h) &= \int_M \Big[-\frac{1}{2}F'(S)\operatorname{Tr}\langle \nabla.h, \nabla.h\rangle - c\, F'(S)|h|^2 \\ &\quad + F''(S)\langle \mathrm{Ric}, h\rangle^2\Big] v^g. \end{aligned} \tag{3.69}$$

Theorem 3.3 follows from equation (3.69), the assumptions $F' \geq 0$, $F'' \leq 0$, and $c > 0$. □

Corollary 3.1. *The n-dimensional unit sphere $\mathbb{S}^n$ is a strictly stable F-Einstein manifold for all $F \in C^\infty(\mathbb{R})$ such that $F' > 0$ and $F'' \leq 0$.*

Acknowledgments. The author would like to thank the editor and the reviewers for their useful remarks and suggestions.

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