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INTEGRABILITY FOR THE DERIVATIVE FORMULAS OF THE TYPE-2 BISHOP FRAME AND ITS APPLICATIONS

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Abstract. The main objective of the work is to examine the integrability of the derivative formulae for the type-2 Bishop frame in three-dimensional Euclidean space. We use the coordinate system introduced in [12], which allows for the examination of integration. As an application, we analyze the position vectors of certain curves that are important in mathematics and physical study.

Keywords: Darboux helix, Position vector field, Type-2 Bishop frame, Euclidean 3-space
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1. Introduction

The theory of curves has gone through a long period of development until it reaches a truly modern manner: from the theory of plane curves, with the beginning of calculus, in 1684, the year in which Gottfried Wilhelm Leibniz created it in his Meditaito nova de natura anguli contactus et osculi, to the theory of space curves, reached to the peak point with the infinitesimal calculus. In this development, we have to mention two important things. The first one is the notion of moving frame, as we know it today, created by Gaston Darboux. The second one is the term binormal mentioned in a treatise on space curves by Barre de Saint-Venant. The Frenet frame is a well-known example of a moving frame utilized to describe a space curve in three-dimensional ambient spaces, including Euclidean and Lorentz-Minkowski spaces. The Frenet equations, or Frenet formulae, were first proposed in 1831 by Karl Eduard Senff and Johann Martin Bartels, enhancing the simplicity and utility of the theory of space curves. The scientists were once again discussed in Jean Frederic Frenet's thesis in 1847, published in 1852. Shortly thereafter, those equations were independently discovered by Joseph Alfred Serret in 1851 and are sometimes referred to as the Frenet-Serret equations (for more information at this early stage in history, , see [7]). On the other hand, mathematicians have done a great number of surveys involving the concept of binormal. But it was not until 2010 that the survey of the first moving frame established by the binormal was published by Yılmaz and Turgut. The authors were the first to create the idea of the

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moving frame in a more novel manner than usual in their "A new version of Bishop frame and an application to spherical images." The main principle is that they do this using a common vector field as the binormal vector field of Frenet-Serret frame (for details, see [9]). Later, an analogue of this survey is done in Lorentz-Minkowski 3-space. [8, 10]

The determination of the position vector field of a smooth curve with a certain property—that is, a slant helix, where the principal normal vector field forms a constant angle with a fixed straight line—was investigated in 2010 by Ali and Turgut. They discovered a third-order vector differential equation. By solving the vector differential equation, they obtained the position vector field of a timelike slant helix in Minkowski space, where the straight line is parallel to e_3 [1]. Refer to [2] for slant helices in Euclidean 3-space. In 2011, researchers conducted analogous investigations to ascertain the position vector field of a generic helix using both the Frenet and standard frames in Euclidean three-dimensional space [3]. Refer to [4, 5] for timelike and spacelike generic helices in Minkowski 3-space.

In the past two decades, the problem of determining the position vectors has emerged as an attractive field of study. In recent years, Yerlikaya and his coauthor [12, 13] have approached this problem from a different perspective than those mentioned in the literature. This approach is based on a new coordinate system that will facilitate the integrability of derivative formulas of the Bishop frame. Inspired by these studies, we focus on that of the type-2 Bishop frame and examine the position vector field of several special curves.

2. Preliminaries

When the real vector space $\mathbb{R}^3$ is endowed with the standard flat metric, known as the Euclidean metric, represented by $g = dx_1^2 + dx_2^2 + dx_3^2$, the corresponding space is known as Euclidean space and denoted as $\mathbb{E}^3$, where (x_1, x_2, x_3) constitutes the usual coordinate system of $\mathbb{E}^3$. The norm of an arbitrary vector $w \in \mathbb{E}^3$ is defined as $\|u\| = \sqrt{g(u,u)}$. Furthermore, given two non-zero vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ in $\mathbb{E}^3$, it is important to note that the cross product of u and v is denoted as

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Let $\gamma : J \to \mathbb{E}^3$ be a smooth curve parametrized by the arbitrary parameter t, where J is an open subset of $\mathbb{R}$. The curve γ is referred as a unit speed curve parametrized by the arc lenght s if its velocity vector γ', the first derivative of the curve, fulfills the condition $\|\gamma'\| = 1$. The parameter of γ shall hereafter be denoted as s. In Euclidean 3-space, the Frenet-Serret frame along the curve γ, denoted by $\{t, n, b\}$, has the derivative formula expressed as

$$\begin{pmatrix} t' \\ n' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} t \\ n \\ b \end{pmatrix},$$

where the curvature and the torsion functions of γ are denoted by κ and τ, respectively.

The derivative formulae for the type-2 Bishop frame represented by $\{\zeta_1, \zeta_2, b\}$ along γ are as follows:

$$\begin{pmatrix} \zeta_1' \\ \zeta_2' \\ b' \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\epsilon_1 \\ 0 & 0 & -\epsilon_2 \\ \epsilon_1 & \epsilon_2 & 0 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ b \end{pmatrix}, \tag{2.1}$$

where ϵ_1 and ϵ_2 are the type-2 Bishop curvature functions of γ and ζ_1, ζ_2 are arbitrary unit vector fields in $\mathbb{E}^3$. The geometric apparatus between the type-2 Bishop frame and the Frenet-Serret frame, which we referred to before, is given by

$$\begin{pmatrix} t \\ n \\ b \end{pmatrix} = \begin{pmatrix} \cos\theta(s) & -\sin\theta(s) & 0 \\ \sin\theta(s) & \cos\theta(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ b \end{pmatrix}, \tag{2.2}$$

$$\kappa = -\theta^{'}(s), \;\; \tau = \sqrt{\epsilon_1^2 + \epsilon_2^2}, \tag{2.3}$$

where $\theta(s) = \arctan\left(\frac{\epsilon_1}{\epsilon_2}\right)$. Note that the above apparatus differs from that of the study of Yılmaz and Turgut [9]. By Eq. (2.3) and the angle θ, there exists the following theorem:

Theorem 2.1. [6] *Let $\gamma = \gamma(s)$ be a smooth curve with curvatures $\epsilon_1 \neq 0$ and $\epsilon_2 \neq 0$. γ is a general helix if and only if type-2 Bishop curvatures of the curve satisfy*

$$\frac{\epsilon_1^2}{\left(\epsilon_1^2 + \epsilon_2^2\right)^{\frac{3}{2}}} \left(\frac{\epsilon_2}{\epsilon_1}\right)^{'} = constant.$$

Remark 2.1. *A necessary condition for the type-2 Bishop frame to exist at all points along a curve is that the curvature of the curve should not be zero. If $\kappa = 0$, then the principal normal vector field of the curve denoted by n becomes $(0,0,0)$. This means that the binormal vector field b becomes $(0,0,0)$. This causes a contradiction in the fact that the system $\{\zeta_1, \zeta_2, b\}$ is orthogonal.*

3. Conclusion

Let $\mathbb{E}^3$ endow the Euclidean 3-space and its basis be $B = \{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$. Let the coordinates of a vector relative to the basis B be $\{x_1, x_2, x_3\}$. In [12], the authors established an ordered orthonormal basis $B'' = \{e_1'', e_2'', e_3''\}$ and the corresponding new coordinate system $\{x_1'', x_2'', x_3''\}$ such that

$$e_j'' = \frac{e_3'' \times e_i}{\left\| e_3'' \times e_i \right\|}, \quad j = 1,2 \;\; i = 1,2,3$$

$$\left(e_j'' \times e_3''\right) = \begin{cases} -e_2'' & , \; j = 1 \\ e_1'' & , \; j = 2. \end{cases}$$

Let $\gamma : I \to \mathbb{E}^3$ be a smooth curve parameterized by arc length s, where $s \in I$, and its type-2 Bishop apparatus $\{\zeta_1, \zeta_2, b, \epsilon_1, \epsilon_2\}$ at the point $\gamma(s)$. Let us consider an any curve $\overline{\gamma}$ obtained from γ through a rigid motion, in such a way that the binormal vector field $\overline{b}$ at the point $\overline{\gamma}(s_0)$ of $\overline{\gamma}$ aligns with e_3''. Consequently, due to this motion, $\overline{\zeta}_1$ and $\overline{\zeta}_2$ sit in the plane defined by e_1'' and e_2''. The other vector fields of $\overline{\gamma}$ are designated as $\overline{\zeta}_1$ and $\overline{\zeta}_2$, respectively. Consequently, it is appropriate to discuss the transition matrix between the systems $\left\{\bar{\zeta}_1, \bar{\zeta}_2, \bar{b}\right\}$ and $\{e_1'', e_2'', e_3''\}$, which is structured as follows:

$$\begin{pmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \\ \bar{b} \end{pmatrix} = \begin{pmatrix} \cos\mu(s) & -\sin\mu(s) & 0 \\ \sin\mu(s) & \cos\mu(s) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1'' \\ e_2'' \\ e_3'' \end{pmatrix}, \tag{3.4}$$

where the angle between the vector fields $\bar{b}$ and e_3'' denotes μ.

Furthermore, it is noteworthy that the rigid motion transforming $\bar{\gamma}(s_0)$ into $\gamma(s_0)$ and $\bar{\zeta}_1, \bar{\zeta}_2, \bar{b}$ into ζ_1, ζ_2, b is, in fact, identical to the aforementioned rigid motion. Hence, we write

$$\zeta_1 = \bar{\zeta}_1,\ \ \zeta_2 = \bar{\zeta}_2,\ \ b = \bar{b}$$

for any $s = s_0$.

By establishing $i = 2$ and $j = 2$ by the argument that $e_3'' = b = (b_1, b_2, b_3)$, we derive

$$e_1'' = \frac{1}{\sqrt{1-b_2^2}}\left(-b_1 b_2, 1-b_2^2, -b_2 b_3\right) \tag{3.5}$$

and

$$e_2'' = \frac{1}{\sqrt{1-b_2^2}}\left(-b_3, 0, b_1\right). \tag{3.6}$$

Theorem 3.1. *Let $\{e_1'', e_2'', e_3''\}$ be the new ordered orthonormal basis obtained from the natural ordered orthonormal basis of $\mathbb{E}^3$ and $\epsilon_1(s)$, $\epsilon_2(s)$ be differentiable functions, where s belongs to an open interval in $\mathbb{R}$. According to the new coordinate system, the binormal vector field $b = (b_1, b_2, b_3)$ in an indirect solution triplet of Eq. (2.1), which is determined by Eqs. (3.7) and (3.8) is given by*

$$\begin{cases} b_1(s) &= \cos f_1(s)\cos f_2(s) \\ \\ b_2(s) &= \sin f_1(s) \\ \\ b_3(s) &= \cos f_1(s)\sin f_2(s) \end{cases}$$

where

$$f_1(s) = c_1 + \int \left(\cos\mu(s)\ \epsilon_1(s) + \sin\mu(s)\ \epsilon_2(s)\right) ds, \tag{3.7}$$

$$f_2(s) = c_2 + \int \frac{-\sin\mu(s)\ \epsilon_1(s) + \cos\mu(s)\ \epsilon_2(s))}{\cos f_1(s)}\, ds \tag{3.8}$$

and c_1, c_2 are constants.

Proof. Let $\{e_1'', e_2'', e_3''\}$ be the new ordered orthonormal basis derived from the natural ordered orthonormal basis in the Euclidean 3-space. Thus, Eqs. (3.5) and (3.6) are valid. Using Eq. (3.4), a relationship between type-2 Bishop vector fields and the vector fields of the new system is appeared as

$$\begin{aligned} \zeta_1(s) &= \cos\mu(s)\ e_1''(s) - \sin\mu(s)\ e_2''(s) \\ \zeta_2(s) &= \sin\mu(s)\ e_1''(s) + \cos\mu(s)\ e_2''(s). \end{aligned} \tag{3.9}$$

We will now compute the elements of the binormal vector field $b(s)$. Substituting Eqs. (3.5) and (3.6) into Eq. (3.9) and putting it into Eq. (2.1), we get

$$\frac{db_1}{ds} = \frac{-1}{\sqrt{1-b_2^2}}\left[\{\ \cos\mu(s)\ \epsilon_2 - \sin\mu(s)\ \epsilon_1\}\, b_3 + \{\ \cos\mu(s)\ \epsilon_1 + \sin\mu(s)\ \epsilon_2\}\, b_1 b_2\right] \tag{3.10}$$

$$\frac{db_2}{ds} = \{\cos\mu(s)\ \epsilon_1 + \sin\mu(s)\ \epsilon_2\}\sqrt{1-b_2^2}, \tag{3.11}$$

$$\frac{db_3}{ds} = \frac{-1}{\sqrt{1-b_2^2}}\left[\{\ \sin\mu(s)\ \epsilon_1 - \cos\mu(s)\ \epsilon_2\}\, b_1 + \{\ \cos\mu(s)\ \epsilon_1 + \sin\mu(s)\ \epsilon_2\}\, b_2 b_3\right] \tag{3.12}$$

Due to the fact that Eq. (3.11) is a type of separable equations, it is simpler to solve compared to other equations, and as a result, the answer ends up being

$$b_2 = \sin\underbrace{\left[c_1 + \int (\ \cos\mu(s)\ \epsilon_1 + \sin\mu(s)\ \epsilon_2)\ ds\right]}_{=f_1(s)}. \tag{3.13}$$

On the other hand, especially since Eqs. (3.10) and (3.12) are non-linear differential equations, it is beneficial to introduce a new variable $g(s)$ rather than solving them directly, which adheres to the following situation:

$$b_1^2 + b_2^2 + b_3^2 = 1,$$

from which

$$b_1 = \cos f_1(s)\ \cos f_2(s), \quad b_3 = \cos f_1(s)\ \sin f_2(s). \tag{3.14}$$

Substituting Eqs. (3.13) and (3.14) into Eq. (3.10), we obtain

$$f_2(s) = c_2 + \int \frac{(-\sin\mu(s)\ \epsilon_1 + \cos\mu(s)\ \epsilon_2)}{\cos f_1(s)}\ ds, \tag{3.15}$$

which completes the proof.

□

The other important that this work will attain can be understood by finding its tangent vector field for the position vector field of a curve. By the proposition that we have just achieved, it may easily be calculated:

For this, we begin by getting the type-2 Bishop vector fileds ζ_1 and ζ_2. Substituting Eqs. (3.5) and (3.6) into Eq. (3.9), for $\zeta_1 = (\zeta_{1_1}, \zeta_{1_2}, \zeta_{1_3})$ and $\zeta_2 = (\zeta_{2_1}, \zeta_{2_2}, \zeta_{2_3})$, we get

$$\zeta_{1_1} = \sin\mu(s)\sin f_2(s) - \cos\mu(s)\sin f_1(s)\cos f_2(s) \tag{3.16}$$

$$\zeta_{1_2} = \cos\mu(s)\cos f_1(s) \tag{3.17}$$

$$\zeta_{1_3} = -\cos\mu(s)\sin f_1(s)\sin f_2(s) - \sin\mu(s)\cos f_2(s) \tag{3.18}$$

and

$$\zeta_{2_1} = -\sin\mu(s)\sin f_1(s)\cos f_2(s) - \cos\mu(s)\sin f_2(s) \tag{3.19}$$

$$\zeta_{2_2} = \sin\mu(s)\cos f_1(s) \tag{3.20}$$

$$\zeta_{2_3} = \cos\mu(s)\cos f_2(s) - \sin\mu(s)\sin f_1(s)\sin f_2(s). \tag{3.21}$$

From Eq. (2.2) taking into account ζ_1 and ζ_2, we have the following remark.

Remark 3.1. *When referring to the position vector field, represented as γ, it is important to remember the following equation:*

$$\frac{d\gamma}{ds} = t. \tag{3.22}$$

With this relation, it is more convenient to perform the operation with the tangent vector field than the binormal vector field. Referring to the proposition 3.1, we have the following relations:

$$t_1 = \frac{-1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\sin f_1\cos f_2\left\{\epsilon_2\cos\mu - \epsilon_1\sin\mu\right\} - \sin f_2\left\{\epsilon_1\cos\mu + \epsilon_2\sin\mu\right\}\right)$$

$$t_2 = \frac{\cos f_2}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\left\{\epsilon_2\cos\mu - \epsilon_1\sin\mu\right\}\right) \tag{3.23}$$

$$t_3 = \frac{-1}{\sqrt{\epsilon_1^2+\epsilon_2^2}}\left(\sin f_1\sin f_2\left\{\epsilon_2\cos\mu - \epsilon_1\sin\mu\right\} + \cos f_2\left\{\epsilon_1\cos\mu + \epsilon_2\sin\mu\right\}\right)$$

4. Applications

Some remarkable curves share the characteristic that a vector field makes a constant angle with a fixed line in space. In the type-2 Bishop frame, two curves exhibit the specified property:

Inclined Curve: A smooth curve is classified as an inclined curve if the vector field ζ_1 (or ζ_2) within its osculating plane forms a constant angle with a fixed line in space. It is analytically defined by the constancy of the ratio of Bishop curvatures ϵ_1 and ϵ_2, as presented by Özyılmaz in the Euclidean 3-space $\mathbb{E}^3$. [6].

Darboux Helix: A smooth curve is classified as a Darboux helix if the Darboux vector $w = -\epsilon_2\zeta_1 + \epsilon_1\zeta_2$ makes a constant angle with a fixed line in space. The curvatures ϵ_1 and ϵ_2 of a Darboux helix adhere to the subsequent equation:

$$\frac{\left(\epsilon_1^2+\epsilon_2^2\right)^{\frac{3}{2}}}{\epsilon_1^2}\frac{1}{\left(\frac{\epsilon_2}{\epsilon_1}\right)'} = \text{constant} \tag{4.24}$$

[11]. In Eq. (4.24), remark that the ratio $\frac{\epsilon_2}{\epsilon_1}$ must not be constant. According to the theorem (2.1), an inclined curve is a general helix, but not a Darboux helix.

A new coordinate system is presented in the preceding section, facilitating the integrability of the derivative formulas for the type-2 Bishop frame. This results in a theorem that demonstrates only one of the triplets of the indirect solutions of Eq. (2.1). In this section, we examine the necessary conditions for the indirect solution to achieve stability. Alternatively, we assess the nature of the integration measure.

It is widely recognized that the curvatures of a curve remain constant until a rigid motion is encountered. Consequently, the type-2 Bishop curvatures $\bar{\epsilon_1}$ and $\bar{\epsilon_2}$ of $\bar{r}$ must satisfy the subsequent conditions:

$$\epsilon_1 = \bar{\epsilon_1}, \quad \epsilon_2 = \bar{\epsilon_2},$$

where ϵ_1 and ϵ_2 are the type-2 Bishop curvature functions of r.

From the theorem (3.1), we have Eqs. (3.16) and (3.19) mentioned the previous section. Consequently, we seek to determine the curvatures $\bar{\epsilon_1}$ and $\bar{\epsilon_2}$, respectively. By differentiating Equation (3.16) with regard to s, we obtain

$$\bar{\epsilon_1} = \sqrt{\left(\frac{d\mu}{ds} - \sin f_1 \, \frac{df_2}{ds}\right)^2 + \epsilon_1^2}. \tag{4.25}$$

Similarly, it is evident that another curvature is represented by

$$\bar{\epsilon_2} = \sqrt{\left(\frac{d\mu}{ds} - \sin f_1 \, \frac{df_2}{ds}\right)^2 + \epsilon_2^2}. \tag{4.26}$$

Lemma 4.1. *Let $\gamma(s)$ be a curve in the Euclidean 3-space and let s be its arc length parameter. Assume that the differentiable functions $\epsilon_1(s)$ and $\epsilon_2(s)$ be the type-2 Bishop curvatures of γ. If the following relation holds*

$$\frac{d\mu}{ds} - \sin f_1(s) \, \frac{df_2}{ds} = 0, \tag{4.27}$$

then there exist "steady" solutions satisfying Eq. (2.1), where $f_1(s)$ and $f_2(s)$ are given by Eqs. (3.7) and (3.8), respectively.

Based on Lemma 4.1, we can examine two possible situations.

Case 1: Assuming $\mu =$ constant, Eq. (4.27) is simplified to

$$\sin f_1(s)\frac{df_2}{ds} = 0. \tag{4.28}$$

Suppose that $\sin f_1(s)$ equals zero. Thus, we have $f_1 = 0$ or $f_1 = 2\pi k$, $k \in \mathbb{Z}$. Consequently, the integrand in Eq. (3.7) may be considered as

$$\cos\mu \; \epsilon_1(s) + \sin\mu \; \epsilon_2(s) = 0. \tag{4.29}$$

The following statements are deduced from the last equality.

- when $\sin\mu = 0$ (or $\cos\mu = 0$), we have $\epsilon_1 = 0$ (or $\epsilon_2 = 0$). This never occurs.
- If $\sin\mu \neq 0$ and $\cos\mu \neq 0$, the resulting position vector field is an inclined curve with

$$\frac{\epsilon_2}{\epsilon_1} = -\cot\mu.$$

Note that the function $f_2(s)$ can be determined from the aforementioned relation using Eq.(4.29), specifically $f_2(s) = c_2 + \frac{-1}{\sin\mu}\int \epsilon_1(s)\, ds$. Furthermore, when the position vector field refers to an inclined curve, its straight line can determine $d = (a, b, c)$ with the help of Eq. (3.16):

$$\langle \zeta_1, d\rangle = -\cos\mu\ b + \sin\mu\left\{a\sin f_2(s) - c\cos f_2(s)\right\}$$

The concept of inclined curves indicates that the necessary and sufficient condition for the previous inner product to remain constant is the achievement of the following relations.

$$a\sin f_2(s) - c\cos f_2(s) = 0,$$

$$b = \pm 1.$$

Hence, we obtain $d = (0, \pm 1, 0)$. This provides knowledge about the plane where the straight line is spanned.

In light of this information, the position vector field of an inclined curve having a straight line spanned by e_2 is computed using Eq.(3.23) as the following:

$$\gamma(s) = (d_1, d_2 s, d_3)\,, \tag{4.30}$$

where d_i $(i = 1, 2, 3)$ is a constant of integration. The last equality expresses to us that the above position vector field is a geodesic, which gives rise to a contradiction with the creation of the type-2 Bishop frame according to Remark 2.1.

Let $\frac{df_2}{ds} = 0$. Thus, it is evident that $f_2 =$ constant. Hence, the integrand in Eq. (3.8) is

$$-\sin\mu\ \epsilon_1(s) + \cos\mu\ \epsilon_2(s) = 0. \tag{4.31}$$

The following statements are deduced from the last equality.

- when $\cos\mu = 0$ (or $\sin\mu = 0$), we have $\epsilon_1 = 0$ (or $\epsilon_2 = 0$). This never occurs.
- If $\sin\mu \neq 0$ and $\cos\mu \neq 0$, the resulting position vector field is an inclined curve with

$$\frac{\epsilon_2}{\epsilon_1} = \tan\mu.$$

Using Eq.(4.29), we can get the function $f_1(s)$ as follows: $f_1(s) = c_1 + \frac{1}{\cos\mu}\int \epsilon_1(s)\, ds$. Also, if the position vector field corresponds to an inclined curve, its straight line may compute $d = (a, b, c)$ with the help of Eq. (3.16).

$$\langle \zeta_1, d\rangle = \cos\mu\sin f_1\left\{-a\cos m - c\sin m\right\} - \cos\mu\cos f_1\ b + \sin\mu\left\{a\sin m - c\cos m\right\},$$

where m depends on the constantity of f_2. From the definition of inclined curves, the previously mentioned dot product remains constant if and only if the subsequent relations are satisfied.:

$$a\cos m + c\sin m = 0,$$
$$a\sin m - c\cos m = 1,$$
$$b = 0,$$

from which we get $d = (\sin m, 0, -\cos m)$. This provides details about the plane where the straight line is located.

Similarly, it is easy to see the position vector field of an inclined curve having a straight line spanned by e_1 and e_3 as the following:

$$\gamma(s) = (\sin m\ s, d_1, \cos m\ s)\,.$$

This causes a contradiction for the same reason. Thus, we have the following result:

Corollary 4.1. *There does not exist any inclined curve with the type-2 Bishop curvatures $\epsilon_1(s)$ and $\epsilon_2(s)$ in $\mathbb{E}^3$.*

For the last one, we have

Case 2: When μ is not constant, three subcases emerge as

- $f_1 =$ constant, $f_2 =$ constant
- $f_1 \neq$ constant, $f_2 =$ constant
- $f_1 =$ constant, $f_2 \neq$ constant

Upon analyzing the first two items, we identify a contradiction with the claim that $\mu \neq$ constant. As a result, these subcases do not happen. We will now analyze the last item.

By the constancy of the function f_1, we have the following.

$$\cos\mu(s)\epsilon_1(s) + \sin\mu(s)\epsilon_2(s) = 0. \tag{4.32}$$

Combining the previous equation and Eq. (3.8), we get the function f_1 as

$$c_2 + \frac{-1}{n}\int\sqrt{\epsilon_1^2(s)+\epsilon_2^2(s)}\ ds, \tag{4.33}$$

where $n = \cos c_1$. Hence, Eq. (4.27) becomes

$$\frac{d\mu}{ds} + m\sqrt{\epsilon_1^2(s)+\epsilon_2^2(s)} = 0, \tag{4.34}$$

from which we get

$$\mu(s) = -m\int\sqrt{\epsilon_1^2(s)+\epsilon_2^2(s)}\ ds, \tag{4.35}$$

where $m = \frac{\sqrt{1-n^2}}{n}$. From Eqs. (4.32) and (4.35), we obtain

$$m = \frac{\epsilon_1^2\left(\frac{\epsilon_2}{\epsilon_1}\right)'}{\left(\epsilon_1^2+\epsilon_2^2\right)^{\frac{3}{2}}}. \tag{4.36}$$

Since the ratio of $\epsilon_2(s)$ to $\epsilon_1(s)$ is non-constant due to Eq. (4.32), we have $m \neq 0$. Therefore, Eq. (4.36) is

$$\frac{1}{m} = \frac{\left(\epsilon_1^2 + \epsilon_2^2\right)^{\frac{3}{2}}}{\epsilon_1^2} \frac{1}{\left(\frac{\epsilon_2}{\epsilon_1}\right)'} = \text{constant}.$$

By substituting Eqs. (4.32) and (4.33) into Eq. (3.23) with the help of Eq. (4.24), we get the position vector of a Darboux helix.

Proposition 4.1. *Let γ be a Darboux helix in $\mathbb{E}^3$ and $\epsilon_1(s)$, $\epsilon_2(s)$ be its type-2 Bishop curvatures. Thus, its position vector field is calculated:*

$$\gamma(s) = \Big(-\sqrt{1-n^2}\int \cos\Big(c_2 + \int p(s)\ ds\Big)\ ds + \ d_1,\ ns + \ d_2,$$
$$-\sqrt{1-n^2}\int \sin\Big(c_2 + \int p(s)\ ds\Big)\ ds + \ d_3\Big),$$

where $p(s) = \frac{-1}{n}\int \sqrt{\epsilon_1^2(s) + \epsilon_2^2(s)}ds$ and $n \neq 1$, c_2 and d_i for $i = 1,2,3$ are constant.

Example 4.1. *Substituting $\epsilon_1(s) = \tan\left(arcsin\frac{s}{5}\right)$ and $\epsilon_2(s) = 1$ in Proposition 4.1, we get the position vector of Darboux helix in the sense of type-2 Bishop frame as follows:*

$$r(s) = \left(\frac{\sqrt{26}}{26}\ s + d_1, \frac{5}{\sqrt{26}}\int \cos\left[c_2 - \sqrt{26}\arcsin\frac{s}{5}\right] ds + d_2,\right.$$
$$\left.\frac{5}{\sqrt{26}}\int \sin\left[c_2 - \sqrt{26}\arcsin\frac{s}{5}\right] ds + d_3\right).$$

Plotting for $d_1 = d_2 = d_3 = c_2 = 0$, we have the following figure.

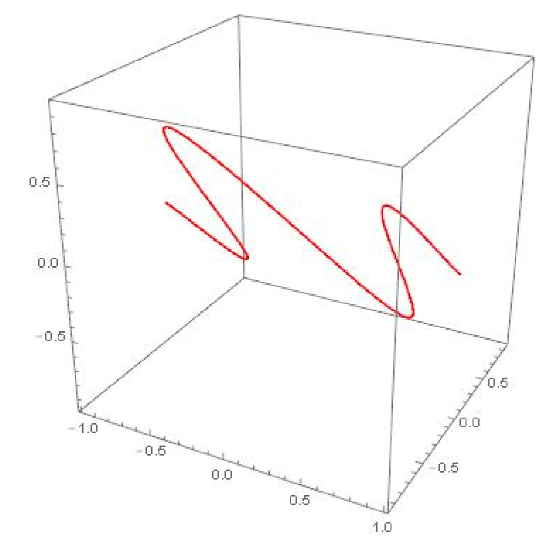

FIGURE 1. The Darboux helix with $k_1(s) = \tan(\arcsin ms)$, $k_2(s) = 1$ for $m = \frac{1}{5}$

Remark 4.1. *Taking $n = 1$ in the aforementioned statement shows that the position vector of the Darboux helix is expressed as Eq. (4.30). In light of theorem 2.1, we derive the following corollary based on result 4.1.*

Corollary 4.2. *A Darboux helix with the type-2 Bishop curvature functions $\epsilon_1(s)$ and $\epsilon_2(s)$ in $\mathbb{E}^3$ is a general helix, vice versa.*

Remark 4.2. *This study examines the outcomes when $i = 2$. The geometric interpretation of the results for $i = 1$ and $i = 3$ refers to the displacement of the components of the curve.*

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