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TAUBERIAN THEOREMS IN NEUTROSOPHIC N-NORMED LINEAR SPACES VIA STATISTICAL CESARO SUMMABILITY

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Abstract. In this study, the connection between statistical Cesaro summability as well as sequence of statistical convergence within neutrosophic n-normed linear space ($\mathfrak{NnNLS}$) is investigated. Although Cesaro summability along with its statistical variant within classical normed spaces, fuzzy, intuitionistic fuzzy, and neutrosophic are covered in the literature, this study is notable for both its methodology and its thorough approach, which covers a wider range among spaces in addition explains the process beginning with the statistical Cesaro summability concepts towards statistical convergence. The Tauberian theorems in $\mathfrak{NnNLS}$ will follow from these findings.

Keywords: Neutrosophic n-normed linear space; Cauchy sequence; Tauberian theorem; Cesaro summability

2020 Mathematics Subject Classification: 40G05; 03E72; 40E05;

1. Introduction

In 1965, Zadeh[23] initially presented the theory among fuzzy sets. He developed this theory to deal with the idea of partial truth, in which truth values fall somewhere between being entirely true and being entirely untrue. This strategy was especially helpful for handling ambiguous or imprecise data, which conventional binary logic was ill-equipped to handle. Atanassov[2], [3] introduced intuitionistic fuzzy set(IFS) theory in 1986. This theory adds a degree among membership as well as a degree among non-membership to the usual fuzzy set theory. Florentin Smarandache[18][19] introduced the concept of neutrosophic sets as to extend of the IFS. The degree of indeterminacy and the neutrosophic set were established as distinct components in his 1995 manuscript, which was published in 1998. Compared with traditional fuzzy sets, this enables a representation of imprecision and uncertainty, which makes it especially helpful in situations where judgments must take into account ambiguous or incomplete data. Gunawan and Mashadi[9], Kim and Cho[13], Malceski[14], and other researchers have looked at n-normed linear spaces. Vijayabalaji and Narayanan[21] defined

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a fuzzy n-normed linear space. Saadati and Park[17] introduced the concept of intuitionistic fuzzy normed space. Many more authors have conducted research on generalised difference sequence spaces. Jeyaraman et al.[10],[11] established the concepts of Logarithmic summability and Cesaro summability in neutrosophic n- normed linear spaces. Praveena et al.[16] generalized the concept of Cesaro summability method in Neutrosophic Normed spaces using the Tauberian conditions.
Our aim in this research is to introduce the idea of statistical summability theory in a neutrosophic n- normed linear spaces $\mathfrak{NnNLS}$. In the context of $\mathfrak{NnN}$, this work will assist us in establishing Tauberian conditions which enable the shift beginning with the statistical Cesaro summability towards statistical convergence among sequences. In order to accomplish this, we provide the ideas among Cesaro as well as statistical Cesaro summability. Future research into related Tauberian theorems in a $\mathfrak{NnNLS}$ environment is made possible by these ideas.

2. Preliminaries

This phase contains some of the basic definitions in addition to the notation required for the next section.

Definition 2.1. [10] *The following axioms define a continuous t-norm as a binary operation* $*:[0,1]\times[0,1]\rightarrow[0,1]$
(i) $*$ *is continuous, commutative and associative,*
(ii) $\mathfrak{p}*1=\mathfrak{p}$ *forevery* $\mathfrak{p}\in[0,1]$,
(iii) If $\mathfrak{p}\leq\mathfrak{r}$ *and* $\mathfrak{q}\leq\mathfrak{s}$ *then* $\mathfrak{p}*\mathfrak{q}\leq\mathfrak{r}*\mathfrak{s}$, *for each* $\mathfrak{p},\mathfrak{q},\mathfrak{r},\mathfrak{s}\in[0,1]$

Definition 2.2. [10] *The seven-tuple* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$ *is recognized as a* $\mathfrak{NnNLS}$, *where* $\mathfrak{U}$ *represents the space of vectors among dimensions that vary* $d\geq n$ *on the domain* $\mathbb{R}$, $*$ *indicates a continuous t-norm,* $\diamond$, *and* $\circ$ *represent a continuous t-conorms, and* $\hat{\mu}$, $\breve{\nu}$, *and* $\tilde{\omega}$ *are fuzzy sets described on* $\mathfrak{U}^n\times(0,\infty)$. *In this context,* $\hat{\mu}$ *denotes the membership degree,* $\breve{\nu}$ *denotes the non-membership degree and* $\tilde{\omega}$ *indicates the degree of indeterminacy for elements* $(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda})\in\mathfrak{U}^n\times(0,\infty)$. *The following requirements are met for each* $(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\mathrm{n})\in\mathfrak{U}^n$ *and* $s,\hat{\lambda}>0$:
(i) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda})+\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda})+\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda})\leq 3$;
(ii) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda})=1$, $\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda})=0$ *and* $\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda})=0$ *for every positive* $\hat{\lambda}$ *iff* $\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n$ *are linearly dependent;*
(iii) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda})$, $\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda})$ *and* $\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda})$ *are not influenced by any particular arrangement of* $\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n$;
(iv) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{c}\mathfrak{f}_n,\hat{\lambda})=\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\frac{\hat{\lambda}}{|\mathfrak{c}|}\right)$, $\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{c}\mathfrak{f}_n,\hat{\lambda})=\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\frac{\hat{\lambda}}{|\mathfrak{c}|}\right)$ *and* $\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{c}\mathfrak{f}_n,\hat{\lambda})=\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\frac{\hat{\lambda}}{|\mathfrak{c}|}\right)$ *if* $\mathfrak{c}\neq 0,\mathfrak{c}\in F$;
(v) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,s)*\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n',\hat{\lambda})\leq\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n+\mathfrak{f}_n',s+\hat{\lambda})$;
(vi) $\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,s)\diamond\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n',\hat{\lambda})\geq\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n+\mathfrak{f}_n',s+\hat{\lambda})$;
(vii) $\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,s)\circ\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n',\hat{\lambda})\geq\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n+\mathfrak{f}_n',s+\hat{\lambda})$;
(vii) $\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda}):(0,\infty)\rightarrow[0,1]$, $\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda}):(0,\infty)\rightarrow[0,1]$ *and* $\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_n,\hat{\lambda}):(0,\infty)\rightarrow[0,1]$ *are always continuous in* $\hat{\lambda}$;

(viii) $\lim_{\hat{\lambda}\to\infty} \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 1$ *and* $\lim_{\hat{\lambda}\to 0} \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 0$ *;*
(ix) $\lim_{\hat{\lambda}\to\infty} \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 0$ *and* $\lim_{\hat{\lambda}\to 0} \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 1$.
(ix) $\lim_{\hat{\lambda}\to\infty} \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 0$ *and* $\lim_{\hat{\lambda}\to 0} \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}) = 1$.

Definition 2.3. [22] *Let* $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$ *be* $\mathfrak{NnNLS}$.
(i) The $\hat{\mathfrak{y}} = \hat{\mathfrak{y}}_k$ *a sequence in* $\mathfrak{U}$ *is considered to converge with* $\tilde{\mathfrak{L}} \in \mathfrak{U}$ *under* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$*, if* $\forall\ \breve{\varrho} \in (0,1), \hat{\lambda} > 0$*, and also* $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1} \in \mathfrak{U}$*,* $\exists$ *a natural number* k_0 *in a way*
$\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{y}}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) > 1 - \breve{\varrho}$*,* $\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{y}}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) < \breve{\varrho}$ *and*
$\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{y}}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) < \breve{\varrho}\ \forall\ k \geq k_0$.
In order to indicate this convergence, $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n - \lim \hat{\mathfrak{y}} = \tilde{\mathfrak{L}}$ *or* $\hat{\mathfrak{y}}_k \overset{(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n}{\rightarrow} \tilde{\mathfrak{L}}$ *as* $k \to \infty$.
(ii) The $\hat{\mathfrak{y}} = \hat{\mathfrak{y}}_k$ *a sequence within* $\mathfrak{U}$ *is defined to be Cauchy in relation to* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$*, if* $\forall\ \breve{\varrho} \in (0,1), \hat{\lambda} > 0$ *and also* $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1} \in \mathfrak{U}$*,* $\exists$ *a natural number* k_0 *in a way that*
$\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{y}}_k - \hat{\mathfrak{y}}_m, \hat{\lambda}) > 1 - \breve{\varrho}$*,* $\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{y}}_k - \hat{\mathfrak{y}}_m, \hat{\lambda}) < \breve{\varrho}$ *and*
$\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \hat{\mathfrak{y}}_k - \hat{\mathfrak{y}}_m, \hat{\lambda}) < \breve{\varrho}$ *for any* $k, m \geq k_0$.
(iii) If all Cauchy sequences in $\mathfrak{U}$ *converge, then a* $\mathfrak{NnNLS}$ $\mathfrak{U}$ *is complete with regard to* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$.

Definition 2.4. *Let* $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$ *represent a* $\mathfrak{NnNLS}$ *as well as* $\mathfrak{V}$ *indicate any subset of* $\mathfrak{U}$*. The set* $\mathfrak{V}$ *is considered bound if* $\exists\ \breve{\varrho} > 0$ *and* $\hat{\lambda}_0 > 0$ *are such that*
$\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}_0) > 1 - \breve{\varrho}$*,* $\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}_0) < \breve{\varrho}$ *and* $\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\lambda}_0) < \breve{\varrho}$
for all $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n \in \mathfrak{V}$*. We tell that the set* $\mathfrak{V}$ *is p-bounded if* $\lim_{\hat{\lambda}\to\infty} \Phi_{\mathfrak{V}}(\hat{\lambda}) = 1$ *and* $\lim_{\hat{\lambda}\to\infty} \Psi_{\mathfrak{V}}(\hat{\lambda}) = 0$ *and* $\lim_{\hat{\lambda}\to\infty} \varphi_{\mathfrak{V}}(\hat{\lambda}) = 0$ *where*
$\Phi_{\mathfrak{V}}(r) = \inf\{\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\mathfrak{r}}) : \mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n \in \mathfrak{V}\}$*;*
$\Psi_{\mathfrak{V}}(r) = \sup\{\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\mathfrak{r}}) : \mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n \in \mathfrak{V}\}$
$\varphi_{\mathfrak{V}}(r) = \sup\{\breve{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n, \hat{\mathfrak{r}}) : \mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_n \in \mathfrak{V}\}$.

Definition 2.5. *Let* $\mathfrak{V}$ *be subset of* $\mathbb{N}$. $\delta(\mathfrak{V}) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in \mathfrak{V}\}|$*, where* $|\mathfrak{V}|$ *indicates the cardinality of the set* $\mathfrak{V}$ *and determines the natural density of* $\mathfrak{V}$ *whenever the limit exists.*

Definition 2.6. *A sequence* $\mathfrak{f} = \{\mathfrak{f}_k\}$ *among numbers is assumed statistically(*$\mathfrak{st}$*)-convergent to* $\tilde{\mathfrak{L}}$*, when* $\forall\ \breve{\varrho} > 0$*,* $\breve{\delta}(\{k \in \mathbb{N} : |\mathfrak{f}_k - \tilde{\mathfrak{L}}| \geq \breve{\varrho}\}) = 0$*. That a case, we represent* $\mathfrak{st} - \lim \mathfrak{f} = \tilde{\mathfrak{L}}$.

Definition 2.7. *([20]). Let* $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$ *be* $\mathfrak{NnNLS}$*. The* $\mathfrak{f} = \{\mathfrak{f}_k\}$ *a sequence within* $\mathfrak{U}$ *is assumed* $\mathfrak{st}$*- convergent towards* $\tilde{\mathfrak{L}} \in \mathfrak{U}$ *in relation with* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$*, when for all* $\breve{\varrho} \in (0,1), \hat{\lambda} > 0$ *along with* $\mathrm{h}_1, \mathrm{h}_2, \ldots, \mathrm{h}_{n-1} \in \mathfrak{U}$*,*
$\breve{\delta}(\{k \in \mathbb{N} : \hat{\mu}(\mathrm{h}_1, \mathrm{h}_2, \ldots, \mathrm{h}_{n-1}, \mathfrak{f}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) \leq 1 - \breve{\varrho}$*,*
$\breve{\nu}(\mathrm{h}_1, \mathrm{h}_2, \ldots, \mathrm{h}_{n-1}, \mathfrak{f}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) \geq \breve{\varrho}, \tilde{\omega}(\mathrm{h}_1, \mathrm{h}_2, \ldots, \mathrm{h}_{n-1}, \mathfrak{f}_k - \tilde{\mathfrak{L}}, \hat{\lambda}) \geq \breve{\varrho}\}) = 0$*. This is represented by* $\mathfrak{st}_{(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n} - \lim \mathfrak{f} = \tilde{\mathfrak{L}}$.

3. Statistical Cesaro summability in $\mathfrak{NnNLS}$

We begin by introducing the concept of Cesaro summability.

Definition 3.1. *([7]). Let* $\{\mathrm{a}_n\}$ *indicate a sequence within* $\mathfrak{NnNLS}$ $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$*. The equation* $\breve{\mathcal{X}}_n = \frac{1}{n+1} \sum_{k=0}^{n} \mathrm{a}_k$ *describes the arithmetic means (AM)* $\breve{\mathcal{X}}_n$ *among* a_n. $\{\mathrm{a}_n\}$ *is* *referred to be Cesaro summable towards* $\mathrm{a} \in \mathfrak{U}$ *when* $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n - \lim_{m\to\infty} \breve{\mathcal{X}}_m = \mathrm{a}$. *Further,* $\{\mathrm{a}_n\}$ *is indicated as a* $\mathfrak{st}$ *Cesaro summable towards* $\mathrm{a} \in \mathfrak{U}$ *when* $\mathfrak{st}_{(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n} - \lim_{m\to\infty} \breve{\mathcal{X}}_m = \mathrm{a}$.

In a $\mathfrak{NnNLS}$ under p-boundedness of sequence, the $\mathfrak{st}$ Cesaro summability method is regular, as demonstrate by the following theorem.

Theorem 3.1. *Let* $\{\mathrm{a}_n\}$ *indicate a* p*-bounded sequence within a* $\mathfrak{NnNLS}$ $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$. *If* $\{\mathrm{a}_n\}$ *converges statistically to* $\mathrm{a} \in \mathfrak{U}$, *then* $\{\mathrm{a}_n\}$ *serves as a* $\mathfrak{st}$ *Cesaro summable to* $\mathfrak{U}$ *in relation to* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$.

Proof. Let $\{\mathrm{a}_n\}$ $\mathfrak{st}$ converges towards $\mathrm{a} \in \mathfrak{U}$ and also assume that it is p-bounded. Put $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1} \in \mathfrak{U}$. If $\breve{\varrho} > 0$, then there is $\mathrm{T}, \mathrm{T}' > 0$ which means
$\inf_{n\in\mathbb{N}} \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \hat{\mathfrak{r}}) > 1 - \breve{\varrho}$, $\sup_{n\in\mathbb{N}} \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \hat{\mathfrak{r}}) < \breve{\varrho}$, and
$\sup_{n\in\mathbb{N}} \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \hat{\mathfrak{r}}) < \breve{\varrho}$, for every $\hat{\mathfrak{r}} > 2\mathrm{T}$.
Therefore, $\inf_{n\in\mathbb{N}} \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right) > 1 - \breve{\varrho}$, $\sup_{n\in\mathbb{N}} \breve{\nu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right) < \breve{\varrho}$ and
$\sup_{n\in\mathbb{N}} \tilde{\omega}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right) < \breve{\varrho}$ for every $\hat{\mathfrak{r}} > 2\mathrm{T}'$.
Thus, the following inequalities are implied:
$\inf_{n\in\mathbb{N}} \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{r}})$
$\geq \min\left\{\inf_{n\in\mathbb{N}} \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \frac{\hat{\mathfrak{r}}}{2}\right), \inf_{n\in\mathbb{N}} \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right)\right\} > 1 - \breve{\varrho}$,
$\sup_{n\in\mathbb{N}} \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{r}})$
$\leq \max\left\{\sup_{n\in\mathbb{N}} \breve{\nu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \frac{\hat{\mathfrak{r}}}{2}\right), \sup_{n\in\mathbb{N}} \breve{\nu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right)\right\} < \breve{\varrho}$
and
$\sup_{n\in\mathbb{N}} \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{r}})$
$\leq \max\left\{\sup_{n\in\mathbb{N}} \tilde{\omega}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n, \frac{\hat{\mathfrak{r}}}{2}\right), \sup_{n\in\mathbb{N}} \tilde{\omega}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right)\right\} < \breve{\varrho}$
$\forall\ \mathrm{T} > \min\{2\mathrm{T}, 2\mathrm{T}'\}$. Since a_n which is $\mathfrak{st}$- convergent towards $\mathfrak{U}$, we get that
$\breve{\delta}(N_{\hat{\mu}}(\breve{\varrho}, \hat{\mathfrak{r}})) = \breve{\delta}(N_{\breve{\nu}}(\breve{\varrho}, \hat{\mathfrak{r}})) = \breve{\delta}(N_{\tilde{\omega}}(\breve{\varrho}, \hat{\mathfrak{r}})) = 0$ for any $\hat{\mathfrak{r}} > 0$,
where
$N_{\hat{\mu}}(\breve{\varrho}, \hat{\mathfrak{r}}) = \{n \in \mathbb{N} : \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{r}}) \leq 1 - \breve{\varrho}\}$,
$N_{\breve{\nu}}(\breve{\varrho}, \hat{\mathfrak{r}}) = \{n \in \mathbb{N} : \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{r}}) \geq \breve{\varrho}\}$ and
$N_{\tilde{\omega}}(\breve{\varrho}, \hat{\mathfrak{r}}) = \{n \in \mathbb{N} : \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathrm{a}_n - \mathrm{a}, \hat{\mathfrak{r}}) \geq \breve{\varrho}\}$.
Describe the sets
$\mathfrak{D} = \{k \in \mathbb{N} : k \in N_{\hat{\mu}}(\breve{\varrho}, \hat{\mathfrak{r}})\}, \mathfrak{D}' = \{k \in \mathbb{N} : k \in N^c_{\hat{\mu}}(\breve{\varrho}, \hat{\mathfrak{r}})\}$, and
$\mathfrak{E} = \{k \in \mathbb{N} : k \in N_{\breve{\nu}}(\tilde{\varrho}, \hat{\mathfrak{r}})\}, \mathfrak{E}' = \{k \in \mathbb{N} : k \in N^c_{\breve{\nu}}(\breve{\varrho}, \hat{\mathfrak{r}})\}$, and
$\mathfrak{F} = \{k \in \mathbb{N} : k \in N_{\tilde{\omega}}(\tilde{\varrho}, \hat{\mathfrak{r}})\}, \mathfrak{F}' = \{k \in \mathbb{N} : k \in N^c_{\tilde{\omega}}(\breve{\varrho}, \hat{\mathfrak{r}})\}$
which means $|\mathfrak{D}| + |\mathfrak{E}| + |\mathfrak{F}| = n + 1 = |\mathfrak{D}'| + |\mathfrak{E}'| + |\mathfrak{F}'|$, in which $|\cdot|$ indicates the cardinality among a set.
Therefore, $\mathfrak{D} \cap \mathfrak{E} \cap \mathfrak{F} = \phi = \mathfrak{D}' \cap \mathfrak{E}' \cap \mathfrak{F}'$, we can determine. We determine that there is a number $n_0 \in \mathbb{N}$ which corresponds with the information above,

$$\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}})$$
$$= \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \frac{1}{n+1}\sum_{k=0}^{n}(\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}\right)$$

$$= \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \sum_{k\in\mathbb{N}_{\hat{\mu}}}(\mathrm{a}_k - \mathrm{a}) + \sum_{k\in\mathbb{N}^c_{\hat{\mu}}}(\mathrm{a}_k - \mathrm{a}), (n+1)\hat{\mathfrak{r}}\right)$$
$$\geq \min\left\{\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \sum_{k\in\mathbb{N}_{\hat{\mu}}}(\mathrm{a}_k - \mathrm{a}), |\mathfrak{D}|\hat{\mathfrak{r}}\right), \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \sum_{k\in\mathbb{N}^c_{\hat{\mu}}}(\mathrm{a}_k - \mathrm{a}), |\mathfrak{D}'|\hat{\mathfrak{r}}\right)\right\}$$
$$\geq \min\left\{\min_{k\in\mathbb{N}_{\hat{\mu}}}\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}\right), \min_{k\in\mathbb{N}^c_{\hat{\mu}}}\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}\right)\right\}$$
$$\geq \min\left\{\inf_{k\in\mathbb{N}_{\hat{\mu}}}\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}\right), \min_{k\in\mathbb{N}^c_{\hat{\mu}}}\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}\right)\right\}$$
$$\geq \min\{1 - \breve{\varrho}, 1 - \breve{\varrho}\}$$
$$= 1 - \breve{\varrho}$$

and also

$$\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}})$$
$$\leq \max\left\{\max_{k\in\mathbb{N}_{\breve{\nu}}}\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}), \max_{k\in\mathbb{N}^c_{\breve{\nu}}}\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}})\right\}$$
$$\leq \max\left\{\sup_{k\in\mathbb{N}_{\breve{\nu}}}\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}), \max_{k\in\mathbb{N}^c_{\breve{\nu}}}\breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}})\right\}$$
$$\leq \breve{\varrho}$$

and

$$\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}})$$
$$\leq \max\left\{\max_{k\in\mathbb{N}_{\tilde{\omega}}}\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}), \max_{k\in\mathbb{N}^c_{\tilde{\omega}}}\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}})\right\}$$
$$\leq \max\left\{\sup_{k\in\mathbb{N}_{\tilde{\omega}}}\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}}), \max_{k\in\mathbb{N}^c_{\tilde{\omega}}}\tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, (\mathrm{a}_k - \mathrm{a}), \hat{\mathfrak{r}})\right\}$$
$$\leq \breve{\varrho}$$

for each $\hat{\mathfrak{r}} > \min\{2\mathrm{T}, 2\mathrm{T}'\} > 0$ along with $n \geq n_0$. It implies that the set is as follows:

$$\mathfrak{G} = \left\{\begin{array}{c} n \in \mathbb{N} : \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}}) \leq 1 - \breve{\varrho}\,\mathrm{or} \\ \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}}) \geq \breve{\varrho}, \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}}) \geq \breve{\varrho} \end{array}\right\}$$

containing, at most, a finite number of terms. The sequence a_n is $\mathfrak{st}$-Cesaro summable towards $\mathfrak{U}$ in relation to $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$ since a finite subset among natural numbers contains zero density, as observed by the observation that $\breve{\delta}(\mathfrak{G}) = 0$. □

We demonstrate in the following example that the converse among Theorem (3.1) does not necessarily have to be true.

Example 3.1. *Let* $\mathrm{b}_{\mathfrak{k}} = \begin{cases} 1+(-1)^{\mathfrak{k}}+\mathfrak{k}^2, & if\,\mathfrak{k}=m^2 \\ 1+(-1)^{\mathfrak{k}}-(\mathfrak{k}-1)^2, & if\,\mathfrak{k}=m^2+1 \\ 1+(-1)^{\mathfrak{k}}, & otherwise, \end{cases}$ *, for* $m \in \mathbb{N}$.
be in $\mathfrak{NnNLS}(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$. *At* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$, *the sequence* $(\mathrm{b}_{\mathfrak{k}})$ *is neither convergent nor* $\mathfrak{st}$*-convergent. Furthermore, it is also not Cesaro summable.*
To reach a limit, let's use $\mathfrak{st}$*-Cesaro summability. Cesaro means* $(\mathrm{a}_{\mathfrak{k}})$ *of sequence* $(\mathrm{b}_{\mathfrak{k}})$ *is*

$$\mathrm{a}_{\mathfrak{k}} = \begin{cases} 1+\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j+\mathfrak{k}, & if\,\mathfrak{k}=m^2 \\ 1+\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j, & otherwise. \end{cases}$$

Sequence $(\mathrm{a}_{\mathfrak{k}})$ *is* $\mathfrak{st}$*-convergent to 1 since for each* $\hat{\mathfrak{r}}>0$, *we have*
$\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim\hat{\mu}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}})=1, \mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim\breve{\nu}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}})=0$ *and* $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim\tilde{\omega}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}})=0$
where

$$\hat{\mu}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}}) = \begin{cases} \frac{\hat{\mathfrak{r}}}{\hat{\mathfrak{r}}+|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j+\mathfrak{k}|}, \mathfrak{k}=m^2 \\ \frac{\hat{\mathfrak{r}}}{\hat{\mathfrak{r}}+|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j|}, otherwise \end{cases}$$

$$\breve{\nu}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}}) = \begin{cases} \frac{|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j|}{\hat{\mathfrak{r}}+|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j+\mathfrak{k}|}, \mathfrak{k}=m^2 \\ \frac{|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j|}{\hat{\mathfrak{r}}+|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j|}, otherwise \end{cases}$$

$$\tilde{\omega}(\mathrm{a}_{\mathfrak{k}}-1,\hat{\mathfrak{r}}) = \begin{cases} \frac{|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j+\mathfrak{k}|}{\hat{\mathfrak{r}}}, \mathfrak{k}=m^2 \\ \frac{|\frac{1}{\mathfrak{k}}\sum_{j=1}^{\mathfrak{k}}(-i)^j|}{\hat{\mathfrak{r}}}, otherwise \end{cases}$$

Hence, sequence $(\mathrm{b}_{\mathfrak{k}})$ *is* $\mathfrak{st}$*-Cesaro summable to 1 in* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$

4. Related Studies Lead to the Tauberian Theorems

The following lemma establishes homogeneity and additivity among the limit of statistical within a $\mathfrak{NnNLS}$.

Lemma 4.1. *Let* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$ *be a* $\mathfrak{NnNLS}$ *and* $\mathrm{u}=\{\mathrm{u}_k\}, \mathrm{v}=\{\mathrm{v}_k\}$ *be sequences in* $\mathfrak{U}$. *After that, the given are true:*
(i) When the limit of $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$*-statistical among* u *indicate* $\breve{\xi}$, *together with the* $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$*-*$\mathfrak{st}$*-limit among* v *is* ρ, *after that the limit of* $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$*-statistical among the sum* $(\mathrm{u}+\mathrm{v})$ *represent* $\breve{\xi}+\dot{\rho}$.
(ii) When the limit of $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$*-statistical among* u *is* $\breve{\xi}$, *along with* α *represent any real number, after that the limit of* $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$*-statistical among* $\alpha\mathrm{u}$ *is* $\alpha\breve{\xi}$.

Theorem 4.1. *Let* $(\mathfrak{U},\hat{\mu},\breve{\nu},\tilde{\omega},*,\diamond,\circ)$ *be a* $\mathfrak{NnNLS}$ *along with* $\{\mathrm{a}_n\}$ *denote a sequence within* $\mathfrak{U}$. *When* $\{\mathrm{a}_n\}$ *is a* $\mathfrak{st}$*-Cesaro summable towards* a *in relation to* $\mathfrak{NnN}$ $(\hat{\mu},\breve{\nu},\tilde{\omega})^n$, *after that* $\breve{\mathcal{X}}_{\mathfrak{y}_n}$ *which is* $\mathfrak{st}$*-convergent towards* a *for every* $\mathfrak{y}>0$, *That is* $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim_{n\to\infty}\breve{\mathcal{X}}_{\mathfrak{y}_n}=\mathrm{a}$, *in which* $\mathfrak{y}_n=[\mathfrak{y}n]$.

Proof. Consider $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim_{n\to\infty}\breve{\mathcal{X}}_n=\mathrm{a}$. After that, for a sufficiently large N, follwed $\breve{\varrho}>0$ together with put $\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1}\in\mathfrak{U}$, the given sets are described:

$$\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})=\{k\le\mathfrak{y}_N:\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_k-\mathrm{a},\hat{\mathfrak{r}})\le 1-\breve{\varrho}\},$$
$$\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})=\{k\le\mathfrak{y}_N:\breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_k-\mathrm{a},\hat{\mathfrak{r}})\ge\breve{\varrho}\},$$
$$\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})=\{k\le\mathfrak{y}_N:\tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_k-\mathrm{a},\hat{\mathfrak{r}})\ge\breve{\varrho}\},$$
$$\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})=\{k\le\mathfrak{y}_N:\hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_k}-\mathrm{a},\hat{\mathfrak{r}})\le 1-\breve{\varrho}\},$$

$$\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}}) = \{k \leq \mathfrak{y}_N : \breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_k} - \mathrm{a},\hat{\mathfrak{r}}) \geq \breve{\varrho}\},$$
$$\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}}) = \{k \leq \mathfrak{y}_N : \tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_k} - \mathrm{a},\hat{\mathfrak{r}}) \geq \breve{\varrho}\}.$$

We then examine the cases given here.

Case 1: $\mathfrak{y} > 1$.

The case is easy to observe that $\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}}) \subseteq \kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})$, $\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}}) \subseteq \kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})$ and also $\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}}) \subseteq \kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})$ for every $\hat{\mathfrak{r}} > 0$. It suggests that which follows:

$$\frac{|\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} = \frac{\mathfrak{y}|\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\hat{\lambda}_N + \mathfrak{y}} \leq \frac{\mathfrak{y}|\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N + 1} \leq \frac{\mathfrak{y}|\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N + 1}$$
$$\frac{|\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} = \frac{\mathfrak{y}|\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N + \mathfrak{y}} \leq \frac{\mathfrak{y}|\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N + 1} \leq \frac{\mathfrak{y}|\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N + 1} \text{ and}$$
$$\frac{|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} = \frac{\mathfrak{y}|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N + \mathfrak{y}} \leq \frac{\mathfrak{y}|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N + 1} \leq \frac{\mathfrak{y}|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N + 1}$$

By applying the mentioned inequalities, accordingly, we can establish that
$\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \leq \mathfrak{y}\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$, $\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \leq \mathfrak{y}\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$, and $\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \leq \mathfrak{y}\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$.
Therefore, for each $\hat{\mathfrak{r}} > 0$, we obtain $\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) = 0$.
Consequently, we may show that $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty} \breve{\mathcal{X}}_{\mathfrak{y}_n} = \mathrm{a}$.

Case 2: $\mathfrak{y} \in (0,1)$.

To conclude our case, we now demonstrate that the expression $\breve{\mathcal{X}}_n$, in the sequence $\breve{\mathcal{X}}_{\mathfrak{y}_n}$, never occurs beyond $1 + \frac{1}{\mathfrak{y}}$ times.
Assume that for few $p, q \in \mathbb{N}$, we get $n = \mathfrak{y}_p = \hat{\mathfrak{y}}_{p+1} = ... = \mathfrak{y}_{p+q-1} < \mathfrak{y}_{p+q}$,
or similarly,
$n \leq \mathfrak{y}p < \mathfrak{y}(p+1) < ... < \mathfrak{y}(p+q-1) < n+1 \leq \mathfrak{y}(p+q)$.
Thus, we've been given $n + \mathfrak{y}(q-1) < \mathfrak{y}p + \mathfrak{y}(q-1) = \mathfrak{y}(p+q-1) < n+1$,
which gives $\mathfrak{y}(q-1) < 1$, i.e., $q < 1 + \frac{1}{\mathfrak{y}}$. According to this field, we get for any $\breve{\varrho} > 0$ and also $\hat{\mathfrak{r}} > 0$ that
$\frac{|\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} \leq \left(1 + \frac{1}{\mathfrak{y}}\right) \frac{\mathfrak{y}_N+1}{N+1} \frac{|\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1} \leq 2(\mathfrak{y}+1)\frac{|\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1}$ and
$\frac{|\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} \leq \left(1 + \frac{1}{\mathfrak{y}}\right) \frac{\mathfrak{y}_N+1}{N+1} \frac{|\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1} \leq 2(\mathfrak{y}+1)\frac{|\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1}$ also
$\frac{|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})|}{N+1} \leq \left(1 + \frac{1}{\mathfrak{y}}\right) \frac{\mathfrak{y}_N+1}{N+1} \frac{|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1} \leq 2(\mathfrak{y}+1)\frac{|\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})|}{\mathfrak{y}_N+1}$
for which N is large enough, such that $\frac{(\mathfrak{y}_{n+1})}{N+1} \leq 2\mathfrak{y}$.
Consequently, it follows that
$\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \leq 2(\mathfrak{y}+1)\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$
$\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \leq 2(\mathfrak{y}+1)\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$ and
$\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) \leq 2(\mathfrak{y}+1)\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}}))$ correspondingly.
Considering that $\{\breve{\mathcal{X}}_n\}$ is $\mathfrak{st}$-convergent toward $\mathfrak{U}$,
$\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\breve{\varrho},\hat{\mathfrak{r}})) = 0$ for any $\hat{\mathfrak{r}} > 0$.
Therefore, $\forall \hat{\mathfrak{r}} > 0$, $\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}_{\mathfrak{y}}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}\mathfrak{y}(\breve{\varrho},\hat{\mathfrak{r}})) = 0$.
We have therefore also demonstrated that $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty} \breve{\mathcal{X}}_{\mathfrak{y}_n} = \mathrm{a}$ in this instance. □

Theorem 4.2. *Let* $(\mathfrak{U}, \hat{\mu}, \breve{\nu}, \tilde{\omega}, *, \diamond, \circ)$ *indicate a* $\mathfrak{NnNLG}$ *and let* $\{\mathrm{a}_n\}$ *be a sequence within* $\mathfrak{U}$. *If* $\{\mathrm{a}_n\}$ *represents a* $\mathfrak{st}$*-Cesaro summable toward* a *in relation to* $\mathfrak{NnN}$ $(\hat{\mu}, \breve{\nu}, \tilde{\omega})^n$. *After*

that, $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty} \frac{1}{\mathfrak{y}_n - n} \sum_{k=n+1}^{\mathfrak{y}_n} \mathrm{a}_k = \mathrm{a}$, *for any* $\mathfrak{y} > 1$ *along with*
$\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty} \frac{1}{n - \mathfrak{y}_n} \sum_{k=\mathfrak{y}_{n+1}}^{n} \mathrm{a}_k = \mathrm{a}$, *for all* $0 < \mathfrak{y} < 1$.

Proof. Consider $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty} \breve{\mathcal{X}}_n = \mathrm{a}$. Select $\iota_1, \iota_2 > 0$ with a given $\breve{\varrho} > 0$ so that $\max\{\hat{\mathfrak{r}}_1, \hat{\mathfrak{r}}_2\} < \breve{\varrho}$ and $\min\{1-\iota_1, 1-\iota_2\} > 1-\breve{\varrho}$. Next, define the following sets for each $\hat{\mathfrak{r}} > 0$ and a sufficiently large N:

$$\begin{aligned}
\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1, \hat{\mathfrak{r}}) &= \{k \le N : \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_k - \mathrm{a}, \hat{\mathfrak{r}}) \le 1 - \iota_1\},\\
\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\iota_1, \hat{\mathfrak{r}}) &= \{k \le N : \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_k - \mathrm{a}, \hat{\mathfrak{r}}) \ge \iota_1\},\\
\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\iota_1, \hat{\mathfrak{r}}) &= \{k \le N : \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_k - \mathrm{a}, \hat{\mathfrak{r}}) \ge \iota_1\},\\
\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2, \hat{\mathfrak{r}}) &= \{k \le N : \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_k} - \breve{\mathcal{X}}_k, \hat{\mathfrak{r}}) \le 1 - \hat{\mathfrak{r}}_2\},\\
\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2, \hat{\mathfrak{r}}) &= \{k \le N : \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_k} - \breve{\mathcal{X}}_k, \hat{\mathfrak{r}}) \ge \hat{\mathfrak{r}}_2\},\\
\kappa_{\tilde{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2, \hat{\mathfrak{r}}) &= \{k \le N : \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_k} - \breve{\mathcal{X}}_k, r) \ge \hat{\mathfrak{r}}_2\}.
\end{aligned}$$

We now explain the cases listed here.

Case I: $\mathfrak{y} > 1$. Define the following sets for any $\hat{\mathfrak{r}} > 0$ and given $\breve{\varrho} > 0$:

$$\begin{aligned}
\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho}, \hat{\mathfrak{r}}) &= \{k \le N : \hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathfrak{J}_n(w) - \mathrm{a}, \hat{\mathfrak{r}}) \le 1 - \breve{\varrho}\},\\
\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho}, \hat{\mathfrak{r}}) &= \{k \le N : \breve{\nu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathfrak{J}_n(w) - \mathrm{a}, \hat{\mathfrak{r}}) \ge \breve{\varrho}\},\\
\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho}, \hat{\mathfrak{r}}) &= \{k \le N : \tilde{\omega}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \mathfrak{J}_n(w) - \mathrm{a}, \hat{\mathfrak{r}}) \ge \breve{\varrho}\},
\end{aligned}$$

in which $\mathfrak{J}_n(w) = \frac{1}{\mathfrak{y}_n - n} \sum_{k=n+1}^{\mathfrak{y}_n} a_k$ for each $n \in \mathbb{N}$.
For each $\mathfrak{y} > 1$ and also sufficiently large $n \in \mathbb{N}\backslash\{0\}$ so that $n < \mathfrak{y}_n$ along with $n \ge \frac{3\mathfrak{y}-1}{\mathfrak{y}(\mathfrak{y}-1)}$, we get that for every $\hat{\mathfrak{r}} > 0$ along with $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1} \in \mathfrak{U}$,

$$\begin{aligned}
&\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \frac{1}{\mathfrak{y}_n - n} \sum_{k=n+1}^{\mathfrak{y}_n} \mathrm{a}_k - \mathrm{a}, \hat{\mathfrak{r}}\right)\\
&= \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \frac{\mathfrak{y}_n + 1}{\mathfrak{y}_n - n} \frac{1}{\mathfrak{y}_{n+1}} \sum_{k=0}^{\mathfrak{y}_n} \mathrm{a}_k - \frac{1}{\mathfrak{y}_n - n} \sum_{k=0}^{n} \mathrm{a}_k - \mathrm{a}, \hat{\mathfrak{r}}\right)\\
&= \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \frac{\mathfrak{y}_n + 1}{\mathfrak{y}_n - n} \breve{\mathcal{X}}_{\mathfrak{y}_n} - \frac{\mathfrak{y}_n + 1 - \mathfrak{y}_n + n}{\mathfrak{y}_n - n} \breve{\mathcal{X}}_n - \mathrm{a}, \hat{\mathfrak{r}}\right)\\
&\ge \min\left\{\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_n} - \breve{\mathcal{X}}_n, \frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{\mathfrak{y}_n - n}}\right), \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&\ge \min\left\{\hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_n} - \breve{\mathcal{X}}_n, \frac{(\mathfrak{y}-1)\hat{\mathfrak{r}}}{4\mathfrak{y}}\right), \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&= \min\left\{\hat{\mu}(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_{\mathfrak{y}_n} - \breve{\mathcal{X}}_n, \hat{\mathfrak{r}}_0), \hat{\mu}\left(\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{n-1}, \breve{\mathcal{X}}_n - \mathrm{a}, \frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&> \min\{1 - \iota_2, 1 - \iota_1\}\\
&> 1 - \breve{\varrho}
\end{aligned}$$

and

$$\begin{aligned}
&\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{1}{\mathfrak{y}_n-n}\sum_{k=n+1}^{\mathfrak{y}_n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&=\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{\mathfrak{y}_n+1}{\mathfrak{y}_n-n}\frac{1}{\mathfrak{y}_n+1}\sum_{k=0}^{\mathfrak{y}_n}\mathrm{a}_k-\frac{1}{\mathfrak{y}_n-n}\sum_{k=0}^{n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&=\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{\mathfrak{y}_n+1}{\mathfrak{y}_n-n}\breve{\mathcal{X}}_{\mathfrak{y}_n}-\frac{\mathfrak{y}_n+1-\mathfrak{y}_n+n}{\mathfrak{y}_n-n}\breve{\mathcal{X}}_n-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&\le\max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{\mathfrak{y}_n-n}}\right),\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&\le\max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{(\mathfrak{y}-1)\hat{\mathfrak{r}}}{4\mathfrak{y}}\right),\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&=\max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\hat{\mathfrak{r}}_0\right),\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&<\max\{\iota_2,\iota_1\}\\
&<\breve{\varrho},
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{1}{\mathfrak{y}_n-n}\sum_{k=n+1}^{\mathfrak{y}_n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&=\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{\mathfrak{y}_n+1}{\mathfrak{y}_n-n}\frac{1}{\mathfrak{y}_n+1}\sum_{k=0}^{\mathfrak{y}_n}\mathrm{a}_k-\frac{1}{\mathfrak{y}_n-n}\sum_{k=0}^{n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&=\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{\mathfrak{y}_n+1}{\mathfrak{y}_n-n}\breve{\mathcal{X}}_{\mathfrak{y}_n}-\frac{\mathfrak{y}_n+1-\mathfrak{y}_n+n}{\mathfrak{y}_n-n}\breve{\mathcal{X}}_n-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&\le\max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{\mathfrak{y}_n-n}}\right),\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&\le\max\left\{\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{(\mathfrak{y}-1)\hat{\mathfrak{r}}}{4\mathfrak{y}}\right),\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&=\max\left\{\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\hat{\mathfrak{r}}_0\right),\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&<\max\{\iota_2,\iota_1\}\\
&<\breve{\varrho},
\end{aligned}$$

where $\hat{\mathfrak{r}}_0=\frac{r(\mathfrak{y}-1)}{4\mathfrak{y}}>0$. Therefore, we get for all $\hat{\mathfrak{r}}>0$,
$\kappa^c_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})\cup\kappa^c_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\subseteq\kappa^c_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})$,
$\kappa^c_{\breve{\nu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})\cup\kappa^c_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\subseteq\kappa^c_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})$,
$\kappa^c_{\tilde{\omega},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})\cup\kappa^c_{\tilde{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\subseteq\kappa^c_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})$
or equivalently,

$$\begin{aligned}
&\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})\subseteq\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\cap\kappa_{\hat{\mu},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}),\\
&\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})\subseteq\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\cap\kappa_{\breve{\nu},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}),
\end{aligned}$$

$$\kappa_{\breve{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) \subseteq \kappa_{\tilde{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}) \cap \kappa_{\tilde{\omega},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}). \tag{4.1}$$

For any $r > 0$ for which we take the asymptotic densities of both sides of (4.1), we get

$$\begin{aligned}
0 &\leq \breve{\delta}(\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})) \\
&\leq \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}}) \cup \kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) \\
&= \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) + \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) - \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}}) \cap \kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) \\
&\leq \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) + \breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) \\
0 &\leq \breve{\delta}(\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})) \\
&\leq \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) + \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) \text{ and} \\
0 &\leq \breve{\delta}(\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})) \\
&\leq \breve{\delta}(\kappa_z\tilde{\omega},\breve{\mathcal{X}}(\iota_1,\hat{\mathfrak{r}})) + \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})).
\end{aligned}$$

Since $\{\breve{\mathcal{X}}_n\}$ is $\mathfrak{st}$-convergent towards $\mathrm{a} \in \mathfrak{U}$,
$\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}})) = 0$ for every $\hat{\mathfrak{r}} > 0$. Therefore, $\{\breve{\mathcal{X}}_{\mathfrak{y}_n}\}$ also $\mathfrak{st}$-converges towards $\mathfrak{U}$.
Using the argument above, $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim_{n\to\infty}(\breve{\mathcal{X}}_{\mathfrak{y}_n} - \breve{\mathcal{X}}_n) = 0$ is implied. Therefore, we get $\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})) = 0 \ \forall\ \hat{\mathfrak{r}} > 0$.
Now, we can determine that $\breve{\delta}(\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})) = \breve{\delta}(\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}})) = 0$.
Therefore, $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n} - \lim\limits_{n\to\infty} \dfrac{1}{\mathfrak{y}_n - n} \sum\limits_{k=n+1}^{\mathfrak{y}_n} \mathrm{a}_k = \mathrm{a}$, for each $\mathfrak{y} > 1$.

Case II: $\mathfrak{y} \in (0,1)$.
The following sets should be described for any $\hat{\mathfrak{r}} > 0$ and given $\breve{\varrho} > 0$:

$$\begin{aligned}
\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) &= \{k \leq N : \hat{\mu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\mathfrak{J}_k(w) - \mathrm{a},\hat{\mathfrak{r}}) \leq 1 - \breve{\varrho}\}, \\
\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) &= \{k \leq N : \breve{\nu}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\mathfrak{J}_k(w) - \mathrm{a},\hat{\mathfrak{r}}) \geq \breve{\varrho}\}, \\
\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) &= \{k \leq N : \tilde{\omega}(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\mathfrak{J}_k(w) - \mathrm{a},\hat{\mathfrak{r}}) \geq \breve{\varrho}\},
\end{aligned}$$

in which $\mathfrak{J}_k(w) = \frac{1}{n-\mathfrak{y}_n}\sum_{k=\mathfrak{y}_n+1}^{\mathfrak{y}_n} \mathrm{a}_k$ for any $n \in \mathbb{N}$.
For all sufficiently large $n \in \mathbb{N}\backslash\{0\}$ together with $0 < \mathfrak{y} < 1$ in a way that $n > \mathfrak{y}_n$ along with $n > \frac{1}{\mathfrak{y}}$, we get that $\forall \hat{\mathfrak{r}} > 0$ along with $\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1} \in \mathfrak{U}$, that

$$\begin{aligned}
&\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{1}{n-\mathfrak{y}_n}\sum_{k=\mathfrak{y}_{n+1}}^{\mathfrak{y}_n} \mathrm{a}_k - \mathrm{a},\hat{\mathfrak{r}}\right) \\
&\geq \min\left\{\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n} - \breve{\mathcal{X}}_n,\frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{n-\mathfrak{y}_n}}\right),\hat{\mu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n - \mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\} \\
&> \min\{1-\iota_2, 1-\iota_1\} \\
&> 1 - \breve{\varrho}
\end{aligned}$$

and

$$\begin{aligned}
&\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{1}{n-\mathfrak{y}_n}\sum_{k=\mathfrak{y}_{n+1}}^{\mathfrak{y}_n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&\leq \max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{n-\mathfrak{y}_n}}\right),\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&\leq \max\left\{\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\hat{\mathfrak{r}}_1\right),\breve{\nu}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&< \max\{\iota_2,\iota_1\}\\
&< \breve{\varrho},
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\frac{1}{n-\mathfrak{y}_n}\sum_{k=\mathfrak{y}_{n+1}}^{\mathfrak{y}_n}\mathrm{a}_k-\mathrm{a},\hat{\mathfrak{r}}\right)\\
&\leq \max\left\{\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\frac{\hat{\mathfrak{r}}}{2\frac{\mathfrak{y}_{n+1}}{n-\mathfrak{y}_n}}\right),\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&\leq \max\left\{\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n,\hat{\mathfrak{r}}_1\right),\tilde{\omega}\left(\mathfrak{f}_1,\mathfrak{f}_2,\ldots,\mathfrak{f}_{n-1},\breve{\mathcal{X}}_n-\mathrm{a},\frac{\hat{\mathfrak{r}}}{2}\right)\right\}\\
&< \max\{\iota_2,\iota_1\}\\
&< \breve{\varrho},
\end{aligned}$$

where $\hat{\mathfrak{r}}_1=\frac{(1-\mathfrak{y})\hat{\mathfrak{r}}}{4\mathfrak{y}}>0$. Therefore, for all $\iota>0$, we obtain

$$\begin{aligned}
\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) &\subseteq \kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\cup K_{\hat{\mu},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}),\\
\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) &\subseteq \kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\cup K_{\breve{\nu},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}),\\
\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}) &\subseteq \kappa_{\tilde{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}})\cup \kappa_{\tilde{\omega},\mathfrak{J}}(\iota_1,\hat{\mathfrak{r}}).
\end{aligned}\tag{4.2}$$

For any $r>0$ for which we assume the asymptotic densities among both sides of (4.2), we get
$0\leq\breve{\delta}(\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))\leq\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}}))+\breve{\delta}(\kappa_{\hat{\mu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}))$,
$0\leq\breve{\delta}(\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))\leq\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}}))+\breve{\delta}(\kappa_{\breve{\nu},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}))$ together with
$0\leq\breve{\delta}(\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))\leq\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}}}(\iota_1,\hat{\mathfrak{r}}))+\breve{\delta}(\kappa_{\tilde{\omega},\breve{\mathcal{X}},\breve{\mathcal{X}}_{\mathfrak{y}}}(\iota_2,\hat{\mathfrak{r}}))$.
Since $\{\breve{\mathcal{X}}_n\}$ is $\mathfrak{st}$-convergent toward $\mathrm{a}\in\mathfrak{U}$, we obtain $\{\breve{\mathcal{X}}_{\mathfrak{y}_n}\}$ is also $\mathfrak{st}$-convergent towards a.
According to the proof provided, $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim\limits_{n\to\infty}(\breve{\mathcal{X}}_{\mathfrak{y}_n}-\breve{\mathcal{X}}_n)=0$.
Therefore, we get $\breve{\delta}(\kappa_{\hat{\mu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))=\breve{\delta}(\kappa_{\breve{\nu},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))=\breve{\delta}(\kappa_{\tilde{\omega},\mathfrak{J}}(\breve{\varrho},\hat{\mathfrak{r}}))=0$.
We can so demonstrate that $\mathfrak{st}_{(\hat{\mu},\breve{\nu},\tilde{\omega})^n}-\lim\limits_{n\to\infty}\dfrac{1}{n-\mathfrak{y}_n}\sum\limits_{\mathfrak{y}_{n+1}}^{n}\mathrm{a}_k=\mathrm{a}$, for each $\mathfrak{y}\in(0,1)$. □

5. Conclusion

In this stydy, we extended classical Tauberian theorems to the framework of neutrosophic n-normed linear spaces by employing the concept of statistical Cesaro summability. This integration offers a novel perspective for analyzing convergence behaviors within uncertain and imprecise environments, which are effectively modeled using neutrosophic structures. The established results not only generalize known theorems in normed linear spaces but also provide a robust mathematical foundation for further applications in areas such as functional analysis, information theory and decision-making under uncertainty. Future research can explore analogous results using other summability methods and extend the framework to more generalized neutrosophic spaces, enriching the theoretical development of both summability theory and neutrosophic analysis.

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