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GEOMETRIC STUDY OF RICCI SOLITONS IN PERFECT FLUID SPACETIMES WITH LORENTZIAN CONCIRCULAR STRUCTURE MANIFOLDS

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Abstract. In this article, we examine the interaction between Ricci solitons and the properties of the geometric structure in a perfect fluid spacetimes that admits a Lorentzian Concircular structure manifold with a Concircular curvature tensor. We investigate the conditions under which a Ricci soliton exists within such a framework and analyze its implications on the curvature properties of the spacetime. The study focuses on the influence of the soliton potential on the energy-momentum tensor of the perfect fluid and examines the interplay between the Ricci curvature and the Concircular structure. Further, we establish key geometric conditions that characterize the nature of the Ricci soliton in this setting and derive significant constraints on the manifolds topology. Our findings contribute to the broader understanding of the role of Ricci solitons in relativistic fluids and their impact on spacetime geometry.

Keywords: Ricci soliton, Einstein, perfect fluid, Lorentz space.

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1. Introduction

Geometric flows have emerged as a crucial tool in the study of Riemannian and semi-Riemannian manifolds, as well as in the theory of general relativity. In his foundational work, Hamilton [12] identified the Ricci flow as an effective method for refining the structure of a manifold. This process modifies the metric of a Riemannian manifold M over time, helping to smooth out its irregularities. The Ricci flow is defined by:

$$\frac{\partial g}{\partial t} = -2Ric \tag{1.1}$$

where g represents the components of the metric tensor, Ric is the Ricci curvature tensor, and t denotes the time parameter. Ricci solitons represent self-similar solutions to the Ricci flow. They have attracted considerable attention in both differential geometry and general relativity due to their strong connection with the Ricci flow and their role as a generalization of Einstein metrics. On a Riemannian manifold (M, g), a Ricci soliton is a particular type of

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solution to the Ricci flow equation. It can be viewed as a natural extension of an Einstein metric and is characterized by a triple (g, V, a), where the following condition holds:

$$\mathfrak{L}_V g + 2Ric + 2ag = 0, \tag{1.2}$$

where Ric is the Ricci curvature tensor of the metric g, $\mathfrak{L}_V g$ denotes the Lie derivative of g along the vector field V on M, and a is a scalar constant. A Ricci soliton (g, V, a) on a manifold M is classified as shrinking, steady, or expanding depending on the sign of the constant a. Specifically, the soliton is: shrinking if $a < 0$: the manifold contracts over time, steady if $a = 0$: the metric evolves trivially under the flow, and expanding if $a > 0$: the manifold expands over time.

Ricci solitons are fundamental in the analysis of geometric flows and general relativity, acting as self-similar solutions to the Ricci flow equation. Their importance has led to extensive investigations across various spacetime geometries, as they offer a deeper understanding of how geometric structures evolve over time. Notably, exploring Ricci solitons within perfect fluid spacetimes offers meaningful contributions to the fields of relativistic hydrodynamics and cosmological modeling. A perfect fluid spacetime is an idealized model in which the energy-momentum tensor represents a fluid with no viscosity or heat conduction, making it a fundamental framework in general relativity. When such a spacetime admits a Lorentzian Concircular structure, it imposes additional geometric constraints that influence the curvature properties and behavior of Ricci solitons. The Lorentzian Concircular structure, characterized by a Concircular vector field, plays a significant role in studying the conformal geometry of spacetime and its interaction with fluid dynamics.

As a natural extension of the Lorentzian para-Sasakian manifold (commonly referred to as the LP-Sasakian manifold, introduced by Matsumoto) A.A. Shaikh [21] developed the concept of Lorentzian Concircular structure manifolds, exploring their existence and significance in both cosmology and the general theory of relativity. These manifolds, denoted as $(LCS)_n$-manifolds, form a notable subclass within the broader category of semi-Riemannian manifolds and play a crucial role in the analysis of spacetime geometry, especially in four-dimensional equipped with a Lorentzian metric g with signature $(-, +, +, +)$. The Lorentzian metric, which stands out among indefinite metrics, introduces a distinct geometric framework where not all directions are equivalent. This leads to a classification of vectors into timelike, lightlike (null), and spacelike, depending on how they interact with the metric. The foundation of Lorentzian geometry lies in understanding the causal character of these vectors. This causal structure is what makes Lorentzian manifolds particularly well-suited for modeling spacetime in the context of Einsteins theory of general relativity [14].

In recent years, Ricci solitons have been extensively explored by several geometers across a wide range of geometric frameworks in [1], [4], [5], [7], [9], [10], [11], [16], [17], [19], [20]. Furthermore, a number of researchers have investigated perfect fluid spacetimes from the perspective of Ricci soliton geometry, highlighting their structural and curvature properties in [3], [6], [8], [18], [24].

Based on the above this research investigates the relationship between Ricci solitons and the geometric framework of a perfect fluid spacetime that possesses a Lorentzian Concircular structure. By examining the essential characteristics of these spacetimes, the study aims to understand the role Ricci solitons play in shaping their geometric progression and stability. Additionally, the work seeks to identify the criteria necessary for the existence of such solitons, thereby offering valuable insights into their significance within the realms of differential geometry and theoretical physics.

Relativistic fluid models play a vital role across various areas of physics, including astrophysics, plasma physics, and nuclear physics. In the context of general relativity, perfect fluids serve as simplified yet powerful models for describing matter distributions, such as those found within stars or in an isotropic universe. Einsteins field equations can be employed to analyze the dynamics of a perfect fluid enclosed in a spherical body, while the FLRW equations are instrumental in modeling the large-scale evolution of the universe. Within general relativity, the energy-momentum tensor acts as the source of spacetime curvature. A perfect fluid is fully described by its mass density in the rest frame and its isotropic pressure. It lacks shear stresses, viscosity, and heat conduction, and its energy-momentum tensor takes the following form:

$$T(U,V) = pg(U,V) + (\sigma + p)\eta(U)\eta(V). \tag{1.3}$$

For any vector fields $U, V \in \chi(M)$, where p denotes the isotropic pressure, σ represents the energy density, and g is the Minkowski metric tensor, the velocity vector of the fluid is given by $\xi := \sharp(\eta)$, satisfying the normalization condition $g(\xi,\xi) = -1$. When the relation $\sigma = -p$ holds, the energy-momentum tensor becomes Lorentz-invariant, expressed as $T = -\sigma g$, corresponding to the vacuum state. Alternatively, when $\sigma = 3p$, the matter content is identified as a radiation fluid.

The motion of a perfect fluid is governed by Einstein's field equations, which describe the interaction between matter and the curvature of spacetime:

$$Ric(U,V) + (\lambda - \frac{r}{2})g(U,V) = kT(U,V). \tag{1.4}$$

For any vector fields $U, V \in \chi(M)$, where λ denotes the cosmological constant, k represents the gravitational constant (often taken as $8\pi G$ in geometric units with G the universal gravitational constant), Ric is the Ricci curvature tensor, and r is the scalar curvature associated with the metric g. These modified field equations arise from Einsteins original formulation, where the cosmological constant was introduced in an attempt to model a static universe. In contemporary cosmology, however, λ is interpreted as a potential form of dark energy responsible for the observed accelerated expansion of the universe. Substituting the expression for T from equation (1.3) into (1.4) we obtain:

$$Ric(U,V) = -\left[\lambda - \frac{r}{2} - kp\right] g(U,V) + (\sigma k + pk)\eta(U)\eta(V), \tag{1.5}$$

for any $U, V \in \chi(M)$. Recall that a manifold exhibits a particular geometric property when its Ricci tensor Ric can be written as a functional combination of g and $\eta \otimes \eta$, for η the g dual 1-form of a unitary vector field, is called quasi-Einstein.

By contracting equation (1.5) and considering that $g(\xi,\xi) = -1$, we obtain:

$$r = 4\lambda + k\sigma - 3kp. \tag{1.6}$$

Therefore, the resulting expression simplifies to:

$$Ric(U,V)=\left[\frac{2\lambda+k\sigma-pk}{2}\right]g(U,V)+(k\sigma+kp)\eta(U)\eta(V). \tag{1.7}$$

This relation holds for all vector fields $U,V\in\chi(M)$.

$$QU=\left[\frac{2\lambda+k\sigma-pk}{2}\right]U+(k\sigma+kp)\eta(U)\xi. \tag{1.8}$$

Here, Ric denotes the Ricci tensor associated with the Ricci operator Q, defined by the relation $Ric(U,V)=g(QU,V)$.

2. Basic Concepts of Lorentzian Concircular structure manifold

In this section, we explore key concepts of Lorentzian Concircular structure manifolds: A Lorentzian Concircular structure manifold is a smooth manifold M equipped with both a Lorentzian metric g and a Concircular structure.

An n-dimensional smooth, connected, and paracontact Hausdorff manifold M, equipped with a Lorentzian metric g, is referred to as a Lorentzian manifold. This implies that M possesses a smooth, symmetric tensor field g of type $(0,2)$ such that for each point $p\in M$, the bilinear form $g_p:T_pM\times T_pM\longrightarrow R$ defines a non-degenerate inner product with signature $(-,+,\dots,+)$. Here, T_pM denotes the tangent vector space to M at p, and R represents the field of real numbers. A non-zero vector $v\in T_pM$ is classified as timelike if $g_p(v,v)<0$, non-spacelike if $g_p(v,v)\leq 0$, null if $g_p(v,v)=0$, or spacelike $g_p(v,v)>0$, [15]. The causal character of a vector refers to the category it falls into based on this classification.

A Lorentzian manifold M admits a unit timelike concircular vector field ξ, referred to as the characteristic vector field of the manifold. The manifold satisfies the following fundamental conditions:

$$g(\mathcal{U},\xi)=\eta(\mathcal{U}),\quad g(\xi,\xi)=-1,\quad (\nabla_{\mathcal{U}}\eta)(\mathcal{V})=\alpha\{g(\mathcal{U},\mathcal{V})+\eta(\mathcal{U})\eta(\mathcal{V})\}, \tag{2.9}$$

where g denotes the Lorentzian metric, ξ is the unit timelike concircular vector field, η is the associated 1-form, ∇ represents the Levi-Civita connection, and α is a smooth scalar function on M. From equation (2.9), we obtain:

$$\nabla_{\mathcal{U}}\xi=\alpha\{\mathcal{U}+\eta(\mathcal{U})\xi\}. \tag{2.10}$$

For any vector fields $\mathcal{U},\mathcal{V}$ on M and ∇ represents the covariant derivative operator associated with the Lorentzian metric g, and α is a non-zero scalar function that satisfies:

$$\nabla_{\mathcal{U}}\alpha=(\mathcal{U}\alpha)=d\alpha(\mathcal{U})=\rho\eta(\mathcal{U}). \tag{2.11}$$

Let ρ denote a scalar function defined by $\rho=-(\xi\alpha)$. By setting:

$$\nabla_{\mathcal{U}}\xi=\alpha\phi\mathcal{U}. \tag{2.12}$$

Then, using equations (2.10) and (2.12), we get:

$$\phi\mathcal{U}=\mathcal{U}+\eta(\mathcal{U})\xi. \tag{2.13}$$

Here, ϕ is a $(1,1)$-type tensor field, referred to as the structure tensor of M. A Lorentzian manifold M, equipped with a unit timelike concircular vector field ξ, its associated 1-form

η, and structure tensor field ϕ, is known as a Lorentzian Concircular structure manifold, abbreviated as an $(LCS)_n$-manifold [21]. In such a manifold, the following fundamental relations are satisfied:

$$\phi^2\mathcal{V} = \mathcal{V} + \eta(\mathcal{V})\xi, \quad \eta(\phi\mathcal{V}) = 0, \quad \eta(\xi) = -1, \quad \phi \cdot \xi = 0, \tag{2.14}$$

$$g(\phi\mathcal{U}, \phi\mathcal{V}) = g(\mathcal{U}, \mathcal{V}) + \eta(\mathcal{U})\eta(\mathcal{V}), \tag{2.15}$$

$$\eta(R(\mathcal{U}, \mathcal{V})Z) = (\alpha^2 - \rho)[g(\mathcal{V}, Z)\eta(\mathcal{U}) - g(\mathcal{U}, Z)\eta(\mathcal{V})], \tag{2.16}$$

$$R(\mathcal{U}, \mathcal{V})\xi = (\alpha^2 - \rho)[\eta(\mathcal{V})\mathcal{U} - \eta(\mathcal{U})\mathcal{V}], \tag{2.17}$$

$$Ric(\mathcal{U}, \xi) = (n-1)(\alpha^2 - \rho)\eta(\mathcal{U}). \tag{2.18}$$

For all vector fields $\mathcal{U}, \mathcal{V}, Z$ on M, let R denote the Riemann curvature tensor associated with the Lorentzian metric g, and let Ric represent the Ricci tensor corresponding to the Ricci operator Q, defined by $Ric(\mathcal{U}, \mathcal{V}) = g(Q\mathcal{U}, \mathcal{V})$.

The Concircular curvature tensor Ç [3] is defined by the expression:

$$Ç(\mathcal{U}, \mathcal{V})Z = R(\mathcal{U}, \mathcal{V})Z - \frac{r}{n(n-1)}[g(\mathcal{V}, Z)\mathcal{U} - g(\mathcal{U}, Z)\mathcal{V}], \tag{2.19}$$

where R is the Riemann curvature tensor, r denotes the scalar curvature, and Ric represent the Ricci tensor associated with operator Q, that is, $Ric(\mathcal{U}, \mathcal{V}) = g(Q\mathcal{U}, \mathcal{V})$.

3. Certain Geometric Properties of a Perfect Fluid Spacetime Admitting Lorentzian Concircular structure manifolds

In this portion of the paper, we investigate the geometric characteristics of a perfect fluid spacetime modeled on a Lorentzian Concircular structure manifold (denoted as $(LCS)_n$-manifold). Our aim is to explore how the intrinsic geometry of such a manifold interacts with the energy-momentum distribution of a perfect fluid. We begin by recalling the fundamental definitions and structure tensors associated with a Lorentzian Concircular structure manifold and the energy-momentum tensor for a perfect fluid, followed by derivations of curvature conditions, symmetry properties, and physical interpretations relevant to relativistic fluid dynamics.

The Concircular curvature tensor Ç [3] in perfect fluid spacetime endowed with a 4-dimensional Lorentzian Concircular structure manifold is defined as follows:

$$Ç(\mathcal{U}, \mathcal{V})Z = R(\mathcal{U}, \mathcal{V})Z - \frac{r}{12}[g(\mathcal{V}, Z)\mathcal{U} - g(\mathcal{U}, Z)\mathcal{V}]. \tag{3.20}$$

After Covariantly differentiating equation (3.20) and contracting, we derive

$$(divÇ)(\mathcal{U}, \mathcal{V})Z = (divR)(\mathcal{U}, \mathcal{V})Z - \frac{1}{12}[g(\mathcal{V}, Z)(\mathcal{U}r) - g(\mathcal{U}, Z)(\mathcal{V}r)]. \tag{3.21}$$

Assuming $divÇ = \nabla \cdot Ç = 0$, where div represents the divergence, we derive the following from equation (3.21):

$$(\nabla_{\mathcal{U}} Ric)(\mathcal{V}, Z) - (\nabla_{\mathcal{V}} Ric)(\mathcal{U}, Z) = \frac{1}{12}[g(\mathcal{V}, Z)(\mathcal{U}r) - g(\mathcal{U}, Z)(\mathcal{V}r)]. \tag{3.22}$$

Since r is a constant scalar curvature, the preceding relation indicates that

$$g((\nabla_{\mathcal{U}} Q)\mathcal{V} - (\nabla_{\mathcal{V}} Q)\mathcal{U}, Z) = 0 \Longrightarrow (\nabla_{\mathcal{U}} Q)\mathcal{V} - (\nabla_{\mathcal{V}} Q)\mathcal{U} = 0. \tag{3.23}$$

By substituting equation (1.8) into (3.23), we obtain:

$$k(\sigma + p)[g(\mathcal{V}, \nabla_{\mathcal{U}}\xi)\xi + \eta(\mathcal{V})\nabla_{\mathcal{U}}\xi - g(\mathcal{U}, \nabla_{\mathcal{V}}\xi)\xi - \eta(\mathcal{U})\nabla_{\mathcal{V}}\xi] = 0. \tag{3.24}$$

Inserting (2.12), (2.13) in (3.24) and on simplification, we get:

$$k(\sigma + p)\alpha[\eta(\mathcal{V})\mathcal{U} - \eta(\mathcal{U})\mathcal{V}] = 0. \tag{3.25}$$

Here, k denotes the gravitational constant defined as $k = 8\pi G$ in geometrized units, where G is the universal gravitational constant. The preceding equation leads to the condition $p = -\sigma$, under the assumption that $\alpha \neq 0$. This leads us to the following conclusion:

Theorem 3.1. *Let (M, g) be a general relativistic perfect fluid spacetime endowed with a Lorentzian Concircular structure manifold satisfying (1.7). If the divergence of the Concircular curvature tensor vanishes, that is, $div Ç = \nabla \cdot Ç = 0$, then the pressure and energy density satisfy $p = -\sigma$ provided that $\alpha \neq 0$.*

If the Concircular curvature tensor is Concircularly flat, that is, $Ç(\mathcal{U}, \mathcal{V})Z = 0$, then by equation (1.6), it follows that:

$$R(\mathcal{U}, \mathcal{V})Z = \frac{4\lambda + k(\sigma - 3p)}{12}[g(\mathcal{V}, Z)\mathcal{U} - g(\mathcal{U}, Z)\mathcal{V}]. \tag{3.26}$$

If the condition $p = -\sigma$ is imposed in equation (3.26), then it results in the following expression:

$$R(\mathcal{U}, \mathcal{V})Z = \frac{\lambda + k\sigma}{3}[g(\mathcal{V}, Z)\mathcal{U} - g(\mathcal{U}, Z)\mathcal{V}]. \tag{3.27}$$

Based on this, we propose the following statement:

Theorem 3.2. *Let (M, g) be a general relativistic perfect fluid spacetime that satisfies equation (1.6) and possesses a Lorentzian Concircular structure. If the Concircular curvature tensor $Ç = 0$ vanishes, implying that the manifold is Concircularly flat, then the spacetime has constant curvature given by $\frac{\lambda + k\sigma}{3}$.*

In this case, we investigate the curvature properties under the assumption that the spacetime is 4-dimensional ξ-Concircularly flat that is:
Using the definition of the Concircular curvature tensor, we have:

$$Ç(\mathcal{U}, \mathcal{V})\xi = R(\mathcal{U}, \mathcal{V})\xi - \frac{r}{12}[g(\mathcal{V}, \xi)\mathcal{U} - g(\mathcal{U}, \xi)\mathcal{V}]. \tag{3.28}$$

If the manifold is ξ-Concircularly flat, then $Ç(\mathcal{U}, \mathcal{V})\xi = 0$, which implies:

$$R(\mathcal{U}, \mathcal{V})\xi = \frac{r}{12}[g(\mathcal{V}, \xi)\mathcal{U} - g(\mathcal{U}, \xi)\mathcal{V}]. \tag{3.29}$$

By applying equations (1.6) and (2.17) in (3.29), the following result is derived:

$$\left[\frac{-[4\lambda + k(\sigma - 3p)] + 12(\alpha^2 - \rho)}{12}\right][\eta(\mathcal{V})\mathcal{U} - \eta(\mathcal{U})\mathcal{V}] = 0. \tag{3.30}$$

From (3.30), we get:

$$p = \frac{4\lambda + k\sigma - 12(\alpha^2 - \rho)}{3k}. \tag{3.31}$$

As a result, we conclude the following:

Theorem 3.3. *Let (M, g) be a general relativistic perfect fluid spacetime satisfying equation (1.7) and admitting a Lorentzian Concircular structure. If the Concircular curvature tensor satisfies $\c{C}(\mathcal{U}, \mathcal{V})\xi = 0$, then the pressure p is equal to $\frac{4\lambda+k\sigma-12(\alpha^2-\rho)}{3k}$.*

Definition 3.1. *A second-order tensor ℓ is referred to as a parallel tensor if its covariant derivative vanishes, i.e., $\nabla\ell = 0$, where ∇ represents the covariant derivative operator associated with the metric tensor g.*

Consider ℓ as a symmetric second-order tensor. It is said to be parallel if it satisfies $\nabla\ell = 0$. Employing the Ricci commutation identity, we proceed as follows:

$$\nabla^2_{\mathcal{U},\mathcal{V}}\ell(\mathfrak{Z}, \mathfrak{W}) - \nabla^2_{\mathcal{U},\mathcal{V}}\ell(\mathfrak{W}, \mathfrak{Z}) = 0. \tag{3.32}$$

Accordingly, the resulting expression is:

$$\ell(R(\mathcal{U}, \mathcal{V})\mathfrak{Z}, \mathfrak{W}) + \ell(\mathfrak{Z}, R(\mathcal{U}, \mathcal{V})\mathfrak{W}) = 0. \tag{3.33}$$

For arbitrary vector fields $\mathcal{U}, \mathcal{V}, \mathfrak{Z}, \mathfrak{W}$ on M, by assigning $\mathfrak{Z} = \mathfrak{W} = \xi$ in equation (3.33), and applying equation (3.27) along with the symmetry property of ℓ, we obtain:

$$\frac{2(\lambda + \sigma k)}{3}[\eta(\mathcal{V})\ell(\mathcal{U}, \xi) - \eta(\mathcal{U})\ell(\mathcal{V}, \xi)] = 0. \tag{3.34}$$

From equation(3.34), it follows that either $\lambda = -\sigma k$, which is equivalent to $\sigma = \frac{-\lambda}{k}$ or

$$\eta(\mathcal{V})\ell(\mathcal{U}, \xi) - \eta(\mathcal{U})\ell(\mathcal{V}, \xi) = 0. \tag{3.35}$$

By inserting $\mathcal{U} = \xi$ in (3.35) and on simplification, we obtain:

$$\ell(\mathcal{V}, \xi) = -\eta(\mathcal{V})\ell(\xi, \xi). \tag{3.36}$$

The fact that ℓ is parallel, combined with equation (3.36), leads to the conclusion that $\ell(\xi, \xi)$ is constant:

$$\begin{aligned}&(\nabla_{\mathfrak{U}}\ell)(V, \xi) + \ell(\nabla_{\mathfrak{U}}V, \xi) + \ell(V, \nabla_{\mathfrak{U}}\xi)\\ &= -\{[g(\nabla_{\mathfrak{U}}V, \xi) + g(V, \nabla_{\mathfrak{U}}\xi)]\ell(\xi, \xi) + \eta(V)[(\nabla_{\mathfrak{U}}\ell)(\xi, \xi) + 2\ell(\nabla_{\mathfrak{U}}\xi, \xi)]\}.\end{aligned} \tag{3.37}$$

By considering $\nabla\ell = 0$ and by virtue of (3.36) in (3.37), we obtain:

$$\ell(\nabla_{\mathfrak{U}}V, \xi) + \alpha\ell(V, \phi\mathfrak{U}) = -\{[\eta(\nabla_{\mathfrak{U}}V) + \alpha g(V, \phi\mathfrak{U})]\ell(\xi, \xi) + 2\alpha\eta(V)\ell(\phi\mathfrak{U}, \xi)\}. \tag{3.38}$$

By using (3.36) in (3.38) and by virtue of (2.13) and on simplification, we have:

$$\ell(\mathfrak{U}, V) = -g(\mathfrak{U}, V)\ell(\xi, \xi). \tag{3.39}$$

For any arbitrary vector fields $\mathfrak{U}, V$ on M and assuming ℓ is parallel, it follows that $\ell(\xi, \xi)$ remains constant. Therefore, we conclude the following:

Theorem 3.4. *In a perfect fluid spacetime that is Concircularly flat and equipped with a Lorentzian Concircular structure manifolds, the presence of a symmetric parallel tensor of second order implies that either the condition $\lambda = -\sigma k$ is satisfied, or the tensor must be a constant scalar multiple of the metric tensor g.*

If $\lambda + \sigma k \neq 0$ in equation (3.34), then Concircularly flat perfect fluid spacetime endowed with a Lorentzian Concircular structure manifolds that admits a second-order symmetric parallel tensor is a regular spacetime. Accordingly, we present the following corollary:

Corollary 3.1. *In a Concircularly flat, regular perfect fluid spacetime endowed with a Lorentzian Concircular structure, a second-order symmetric parallel tensor possesses the same directional and symmetry characteristics as the metric tensor and is uniformly scaled by a constant factor across the manifold. Hence, it is a constant multiple of the metric tensor* g.

4. Perfect fluid spacetime satisfying $(\xi, \cdot)_R \cdot Ric = 0$ and Lorentzian Concircular structure manifold

This section focuses on the analysis of a perfect fluid spacetime in Lorentzian Concircular structure manifolds that fulfills the curvature condition $(\xi, \cdot)_R \cdot Ric = 0$, which is equivalent to the following [3]:

$$\begin{aligned} ((\xi, U)_R \cdot Ric)(V, Z) &= ((\xi \wedge_R U) \cdot Ric)(V, Z) \\ &= Ric((\xi \wedge_R U)V, Z) + Ric(V, (\xi \wedge_R U)Z), \end{aligned} \tag{4.40}$$

where the curvature operator $(U \wedge_R V)$ is defined by its action on a vector field Z as $(U \wedge_R V)Z = R(U, V)Z$. Utilizing this definition, equation (4.40) leads to:

$$Ric(R(\xi, U)V, Z) + Ric(V, R(\xi, U)Z) = 0. \tag{4.41}$$

Inserting (1.7) in (4.41), we get

$$\begin{aligned} &\left(\frac{2\lambda + k(\sigma - p)}{2}\right)[g(R(\xi, U)V, Z) + g(V, R(\xi, U)Z)] \\ &+(k\sigma + kp)[\eta(R(\xi, U)V)\eta(Z) + \eta(V)\eta(R(\xi, U)Z)] = 0. \end{aligned} \tag{4.42}$$

By substituting equations (2.16) and (2.17) into (4.42), we obtain the following expression:

$$(k\sigma + kp)(\alpha^2 - \rho)[g(U, V)\eta(Z) + g(U, Z)\eta(V) + 2\eta(U)\eta(V)\eta(Z)] = 0. \tag{4.43}$$

By replacing Z with ξ in equation (4.43), we obtain:

$$(\alpha^2 - \rho)(k\sigma + kp)[g(U, V) + \eta(U)\eta(V)] = 0, \tag{4.44}$$

we obtain $\sigma = -p$ and $(\alpha^2 - \rho) \neq 0$. Therefore, we can conclude the following:

Theorem 4.1. *Let (M, g) be a perfect fluid spacetime in general relativity that admits a Lorentzian Concircular structure and satisfies equation (1.7). If the curvature condition $(\xi, \cdot)_R \cdot Ric = 0$ holds, then the pressure p and energy density σ satisfy $\sigma = -p$.*

5. Perfect fluid spacetime satisfying $(\xi, \cdot)_{Ric} \cdot R = 0$ and Lorentzian Concircular structure manifold

The current section is concerned with the perfect fluid spacetimes in Lorentzian Concircular structure manifolds that satisfy the curvature condition $(\xi, \cdot)_{Ric} \cdot R = 0$. This condition is equivalent to the following expression:

$$((\xi,U)_{Ric}\cdot R)(V,Z)W = (\xi\wedge_{Ric}U)R(V,Z)W + R((\xi\wedge_{Ric}U)V,Z)W + R(V,(\xi\wedge_{Ric}U)Z)W + R(V,Z)(\xi\wedge_{Ric}U)W, \tag{5.45}$$

where $(U\wedge_{Ric}V)Z = Ric(V,Z)U - Ric(U,Z)V$. Rewriting the preceding relation, we obtain:

$$\begin{aligned} &Ric(U,R(V,Z)W)\xi - Ric(\xi,R(V,Z)W)U + Ric(U,V)R(\xi,Z)W \\ &-Ric(\xi,V)R(U,Z)W + Ric(U,Z)R(V,\xi)W - Ric(\xi,Z)R(V,U)W \\ &+Ric(U,W)R(V,Z)\xi - Ric(\xi,W)R(V,Z)U = 0. \end{aligned} \tag{5.46}$$

Applying the inner product with the vector field ξ in equation (5.46), we obtain:

$$\begin{aligned} &-Ric(U,R(V,Z)W) - Ric(\xi,R(V,Z)W)\eta(U) + Ric(U,V)\eta(R(\xi,Z)W) \\ &-Ric(\xi,V)\eta(R(U,Z)W) + Ric(U,Z)\eta(R(V,\xi)W) - Ric(\xi,Z)\eta(R(V,U)W) \\ &+Ric(U,W)\eta(R(V,Z)\xi) - Ric(\xi,W)\eta(R(V,Z)U) = 0. \end{aligned} \tag{5.47}$$

Inserting (1.7) in (5.47), we get

$$\begin{aligned} &\left(\frac{2\lambda + k(\sigma-p)}{2}\right)[-g(U,R(V,Z)W) - \eta(R(V,Z)W)\eta(U) + g(U,V)\eta(R(\xi,Z)W) \\ &-\eta(V)\eta(R(U,Z)W) + g(U,Z)\eta(R(V,\xi)W) - \eta(Z)\eta(R(V,U)W) + g(U,W)\eta(R(V,Z)\xi) \\ &-\eta(W)\eta(R(V,Z)U)] + k(\sigma+p)[\eta(U)\eta(V)\eta(R(\xi,Z)W) + \eta(V)\eta(R(U,Z)W) \\ &+\eta(U)\eta(Z)\eta(R(V,\xi)W) + \eta(Z)\eta(R(V,U)W) + \eta(U)\eta(W)\eta(R(V,Z)\xi) \\ &+\eta(W)\eta(R(V,Z)U)] = 0. \end{aligned} \tag{5.48}$$

By using (2.16), (2.17) in (5.48), we arrive at

$$\begin{aligned} &\left(\frac{2\lambda + k(\sigma-p)}{2}\right)[-g(U,R(V,Z)W) + (\alpha^2-\rho)\{2g(V,W)\eta(U)\eta(Z) \\ &-2g(Z,W)\eta(U)\eta(V) - g(U,V)g(Z,W) + g(U,Z)g(V,W)\}] \\ &+k(\sigma+p)(\alpha^2-\rho)[g(U,Z)\eta(W)\eta(V) - g(U,V)\eta(W)\eta(Z)] = 0. \end{aligned} \tag{5.49}$$

By inserting $Z = W = \xi$ in (5.49) and on simplification, we have

$$(\alpha^2-\rho)[2\lambda - 2kp][g(U,V) + \eta(U)\eta(V)] = 0. \tag{5.50}$$

From the above equation, it follows that $p = \frac{\lambda}{k}$ and $(\alpha^2-\rho) \neq 0$. Consequently, we present the following result:

Theorem 5.1. *Let (M,g) be a general relativistic perfect fluid spacetime satisfying equation (1.7) and admitting a Lorentzian Concircular structure manifold. If the curvature condition $(\xi,\cdot)_{Ric}\cdot R = 0$ holds, then the pressure satisfies $p = \frac{\lambda}{k}$, provided that $(\alpha^2-\rho) \neq 0$.*

6. Perfect fluid spacetime satisfying $(\xi,\cdot)_{Ç}\cdot Ric = 0$ and Lorentzian Concircular structure manifold

In this section, we investigate perfect fluid spacetimes admitting a Lorentzian Concircular structure manifold that satisfy the condition $(\xi,\cdot)_{Ç}\cdot Ric = 0$. This condition is equivalent to the following relation:

$$\begin{aligned}((\xi,U)_{Ç}\cdot Ric)(V,Z) &= ((\xi\wedge_{Ç}U)\cdot Ric)(V,Z)\\ &= Ric((\xi\wedge_{Ç}U)V,Z)+Ric(V,(\xi\wedge_{Ç}U)Z),\end{aligned}\tag{6.51}$$

where the operator $(U\wedge_{Ç}V)$ acts on a vector field Z as $(U\wedge_{Ç}V)Z=Ç(U,V)Z$. Reformulating the preceding expression using this definition, we get:

$$Ric(Ç(\xi,U)V,Z)+Ric(V,Ç(\xi,U)Z)=0,\tag{6.52}$$

Inserting (1.7) in (6.52), we obtain:

$$\begin{aligned}&\left(\frac{2\lambda+k\sigma-kp)}{2}\right)[g(Ç(\xi,U)V,Z)+g(V,Ç(\xi,U)Z)]\\ &+(k\sigma+kp)[\eta(Ç(\xi,U)V)\eta(Z)+\eta(V)\eta(Ç(\xi,U)Z)]=0.\end{aligned}\tag{6.53}$$

By using (2.16), (2.17), and (2.19) in (6.53), we get:

$$\begin{aligned}&(k\sigma+kp)\left\{\frac{12(\alpha^2-\rho)-[4\lambda+k(\sigma-3p)]}{12}\right\}[g(U,V)\eta(Z)+g(U,Z)\eta(V)\\ &+2\eta(U)\eta(V)\eta(Z)]=0.\end{aligned}\tag{6.54}$$

Replacing Z by ξ in equation (6.54), gives the following result:

$$(k\sigma+kp)\left\{\frac{12(\alpha^2-\rho)-[4\lambda+k(\sigma-3p)]}{12}\right\}[g(U,V)+\eta(U)\eta(V)]=0.\tag{6.55}$$

Therefore, we arrive at two possibilities: either $\sigma=-p$ or $p=\frac{4\lambda+k\sigma-12(\alpha^2-\rho)}{3k}$. As a consequence, we establish the following result:

Theorem 6.1. *Let (M,g) be a general relativistic perfect fluid spacetime endowed with a Lorentzian Concircular structure manifolds satisfying equation (1.7). If the condition $(\xi,\cdot)_{Ç}\cdot Ric=0$ holds, then either $\sigma=-p$ or $p=\frac{4\lambda+k\sigma-12(\alpha^2-\rho)}{3k}$.*

7. Perfect fluid spacetime satisfying $(\xi,\cdot)_{Ric}\cdot Ç=0$ and Lorentzian Concircular structure manifolds

In this section, we examine perfect fluid spacetimes in Lorentzian Concircular structure manifolds satisfying the condition $(\xi,\cdot)_{Ric}\cdot Ç=0$. This condition is equivalent to the following:

$$\begin{aligned}((\xi,U)_{Ric}\cdot Ç)(V,Z)W &= (\xi\wedge_{Ric}U)Ç(V,Z)W+Ç((\xi\wedge_{Ric}U)V,Z)W\\ &\quad+Ç(V,(\xi\wedge_{Ric}U)Z)W+Ç(V,Z)(\xi\wedge_{Ric}U)W\end{aligned}\tag{7.56}$$

where $(X\wedge_{Ric}Y)Z=Ric(Y,Z)X-Ric(X,Z)Y$.

The preceding equation can be expressed as:

$$\begin{aligned}&Ric(U,Ç(V,Z)W)\xi-Ric(\xi,Ç(V,Z)W)U+Ric(U,V)Ç(\xi,Z)W\\ &-Ric(\xi,V)Ç(U,Z)W+Ric(U,Z)Ç(V,\xi)W-Ric(\xi,Z)Ç(V,U)W\\ &+Ric(U,W)Ç(V,Z)\xi-Ric(\xi,W)Ç(V,Z)U=0.\end{aligned}\tag{7.57}$$

Taking the inner product of equation (7.57) with ξ, we obtain:

$$-Ric(U,Ç(V,Z)W)-Ric(\xi,Ç(V,Z)W)\eta(U)+Ric(U,V)\eta(Ç(\xi,Z)W)$$
$$-Ric(\xi,V)\eta(Ç(U,Z)W)+Ric(U,Z)\eta(Ç(V,\xi)W)-Ric(\xi,Z)\eta(Ç(V,U)W)$$
$$+Ric(U,W)\eta(Ç(V,Z)\xi)-Ric(\xi,W)\eta(Ç(V,Z)U)=0. \tag{7.58}$$

Inserting equation (1.7) into (7.58) gives:

$$\left(\frac{2\lambda+k(\sigma-p)}{2}\right)[-g(U,Ç(V,Z)W)-\eta(Ç(V,Z)W)\eta(U)+g(U,V)\eta(Ç(\xi,Z)W)$$
$$-\eta(V)\eta(Ç(U,Z)W)+g(U,Z)\eta(Ç(V,\xi)W)-\eta(Z)\eta(Ç(V,U)W)+g(U,W)\eta(Ç(V,Z)\xi)$$
$$-\eta(W)\eta(Ç(V,Z)U)]+k(\sigma+p)[\eta(U)\eta(V)\eta(Ç(\xi,Z)W)+\eta(V)\eta(Ç(U,Z)W)$$
$$+\eta(U)\eta(Z)\eta(Ç(V,\xi)W)+\eta(Z)\eta(Ç(V,U)W)+\eta(U)\eta(W)\eta(Ç(V,Z)\xi)$$
$$+\eta(W)\eta(Ç(V,Z)U)]=0. \tag{7.59}$$

By substituting equations (2.16), (2.17) and (2.19) into (7.59), we obtain:

$$-\left(\frac{2\lambda+k(\sigma-p)}{2}\right)g(U,Ç(V,Z)W)$$
$$+\left(\frac{2\lambda+k(\sigma-p)}{2}\right)\left\{\frac{12(\alpha^2-\rho)-[4\lambda+k(\sigma-3p)]}{12}\right\}[2g(V,W)\eta(U)\eta(Z)$$
$$-2g(Z,W)\eta(U)\eta(V)-g(U,V)g(Z,W)+g(U,Z)g(V,W)]$$
$$+(k\sigma+kp)\left\{\frac{12(\alpha^2-\rho)-[4\lambda+k(\sigma-3p)]}{12}\right\}[g(U,Z)\eta(W)\eta(V)$$
$$-g(U,V)\eta(W)\eta(Z)]=0. \tag{7.60}$$

By substituting $Z=W=\xi$ in (7.60) and on simplification, we have

$$[2\lambda-2kp]\left\{\frac{12(\alpha^2-\rho)-[4\lambda+k(\sigma-3p)]}{12}\right\}[g(U,V)+\eta(U)\eta(V)]=0. \tag{7.61}$$

Therefore, either $p=\frac{\lambda}{k}$ or $p=\frac{4\lambda+k\sigma-12(\alpha^2-\rho)}{3k}$. Hence, we present the following:

Theorem 7.1. *Let (M,g) represent a general relativistic perfect fluid spacetime satisfying (1.7), endowed with a Lorentzian Concircular structure manifold. If the condition $(\xi,\cdot)_{Ric}\cdot Ç=0$ holds, then either $p=\frac{\lambda}{k}$ or $p=\frac{4\lambda+k\sigma-12(\alpha^2-\rho)}{3k}$.*

8. Ricci solitons within perfect fluid spacetimes possessing a Lorentzian Concircular structure manifold

This section explores Ricci solitons within the framework of a perfect fluid spacetime endowed with a Lorentzian Concircular structure.

Consider the Ricci solitons equation:

$$\mathfrak{L}_\xi g+2Ric+2ag=0. \tag{8.62}$$

Let g be a pseudo-Riemannian metric, Ric the Ricci tensor, ξ a vector field, and a a real constant. A triple (g,ξ,a) that satisfies equation (8.62) is called a Ricci soliton on M. The soliton is said to be shrinking, steady, or expanding depending on whether a is negative, zero, or positive, respectively.

From the Lie derivative, it follows that:

$$(\mathfrak{L}_\xi g)(U,V) = g(\nabla_U \xi, V) + g(U, \nabla_V \xi). \tag{8.63}$$

By using (8.63) in (8.62), we obtain

$$Ric(U,V) = -ag(U,V) - \frac{1}{2}[g(\nabla_U \xi, V) + g(U, \nabla_V \xi)]. \tag{8.64}$$

By contracting both sides of (8.64), we arrive at:

$$r = -a \cdot dim(M) - div(\xi), \tag{8.65}$$

where $dim(M)$ denotes the dimension of the manifold M, and $div(\xi)$ is the divergence of the vector field ξ.

Let (M, g) be a spacetime representing a general relativistic perfect fluid with a Lorentzian Concircular structure, and let (g, ξ, a) define a Ricci soliton on M. From equations (1.7) and (8.64), we obtain:

$$\begin{aligned}&\left[a + \lambda + \frac{k(\sigma - p)}{2}\right] g(U,V) + k(\sigma + p)\eta(U)\eta(V) \\ &+ \frac{1}{2}[g(\nabla_U \xi, V) + g(U, \nabla_V \xi)] = 0.\end{aligned} \tag{8.66}$$

By an orthonormal frame field e_i substituting $U = V = e_i$ in (8.66) and by incorporating the fluid parameters and the divergence of the vector field ξ, the soliton constant a takes the form:

$$a = -\lambda - \frac{k(\sigma - 3p)}{4} - \frac{div(\xi)}{4}. \tag{8.67}$$

Therefore, we present the following statement:

Theorem 8.1. *A Ricci soliton (g, ξ, a), where $a = -\lambda - \frac{k(\sigma-3p)}{4} - \frac{div(\xi)}{4}$, is classified as steady when $p = \frac{4\lambda}{3k} + \frac{\sigma}{3} + \frac{div\xi}{3k}$; it is expanding if $p > \frac{4\lambda}{3k} + \frac{\sigma}{3} + \frac{div\xi}{3k}$, and shrinking if $p < \frac{4\lambda}{3k} + \frac{\sigma}{3} + \frac{div\xi}{3k}$.*

9. Conclusion

This investigation highlights the fundamental role of Ricci solitons in perfect fluid spacetimes characterized by a Lorentzian Concircular structure and a Concircular curvature tensor. Through a detailed analysis of the relationship between the soliton potential, energy-momentum tensor, and the geometric properties of the manifold, we have identified essential conditions for the existence and nature of Ricci solitons. The results impose meaningful constraints on the curvature and topology of the spacetime, offering new perspectives on the geometric behavior of relativistic fluids. These contributions not only strengthen the theoretical understanding of Ricci solitons but also expand their relevance within the framework of general relativity.

Looking ahead, this study can be extended by examining Ricci solitons in more generalized geometric environments, such as manifolds with torsion or non-metric connections. Further research could also address the stability, evolution, and physical significance of Ricci solitons in dynamic spacetimes. Incorporating numerical approaches and analyzing specific solutions to Einsteins field equations may offer practical insights, enhancing the applicability of the theoretical findings to problems in cosmology and gravitational physics.

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