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A NOTE ON POINTWISE HEMI-SLANT CONFORMAL SUBMERSIONS

MOHAMMAD SHUAIB *

Abstract. The current study investigates the concept of pointwise hemi-slant conformal submersions from almost contact metric manifolds to Riemannian manifold. We investigate the geometrical implications of the horizontal and vertical vector fields ξ while studying the distribution integrability and total geodesicness of distribution leaves. Finally, we explore ϕ-pluriharmonicity from the almost contact metric manifold.

Keywords: Sasakian manifolds, Riemannian submersions, Pointwise hemi-slant conformal submersions, Conformal submersions.

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1. Introduction

Immersions and submersions are well known to be special tools in both differential and Riemannian geometry. It is important in Riemannian geometry, particularly when the involved manifolds have a differentiable structures. Four decades ago, B. O'Neill [22] and A. Gray [12] separately formulated the cornerstone of the theory of Riemannian submersions, which has experienced major advances over the past two decades. In mathematics and physics, Riemannian submersions have been widely used, particularly in theories like Yang-Mills and Kaluza-Klein (see [8], [35], [20], [17]).

Riemannian submersions from almost Hermitian manifolds to Riemannian manifolds were studied in 1976 by B. Watson [34]. On the foundation of the findings from this study, B. Sahin [26] assessed the geometry and distinctive features of anti-invariant Riemannian submersions onto Riemannian manifolds. Afterwards, authors explored this field further, looking at slant submersions [10], [28], semi-slant submersions [16], [23], anti-invariant submersions [3], [26] and semi-invariant submersions [27] among other topics. Tastan, Sahin, and Yanan [33] defined and studied hemi-slant submersions from almost Hermitian manifolds as a generalized case of semi-invariant and semi-slant submersions.

J. W. Lee and B. Sahin [19] introduced pointwise slant submersions from almost Hermitian manifolds to Riemannian manifolds in this addition, thus expanding the concept of slant submersions even further. They established characterizations for pointwise slant submersions in

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* Corresponding author

Mohammad Shuaib ⋄ shuaibyousuf6@gmail.com.

addition to offering examples of this kind of submersion. As a generalization of Riemannian submersions, B. Fuglede [13] and T. Ishihara [18] presented the idea of conformal submersion and talked about some of its geometric characteristics. As a generalization of holomorphic submersions, conformal holomorphic submersions were studied by Gudmundsson and Wood [15]. They determined the necessary and sufficient conditions for harmonic morphisms of conformal holomorphic submersions. Later, conformal semi-invariant submersions [4], conformal slant submersions [2], conformal anti-invariant submersions [29], [24] and conformal semi-slant submersions [1] were studied and defined by Akyol and Sahin. Conformal hemi-slant submersions [31], [32], conformal bi-slant submersions [5] and quasi bi-slant conformal submersions [6] have all been discussed in the context of geometric studies recently, along with an assortment of decomposition theorems. Moreover, from almost Hermitian manifolds to almost contact metric manifolds, the concept of pluriharmonicity was extended.

We study the geometry of pointwise hemi-slant conformal submersions by considering both the horizontal and vertical aspects of the structural vector field ξ. The paper is organized as follows: Our investigation's goals can be achieved by introducing almost contact manifolds, such as the Sasakian manifold, which we discuss in Section 2. In the third part of this investigation, we define pointwise hemi-slant conformal submersions and describe some interesting results by considering the horizontal structure of the Reeb vector field ξ. A thorough examination of the total geodesic and the integrability of distributions is also given in Section 3. While in section 4, we consider vertical aspect of Reeb vector field ξ and study of pointwise hemi-slant conformal submersions. Additionally, as the nature of the Reeb vector field differs in Sections 3 and 4, we compared the findings of these two sections.

Note: We use the following abbreviations in this article.
Pointwise hemi-slant conformal submersion- $\mathcal{PWHSCS}$
Riemannian submersion- $\mathcal{RS}$
Riemannian Manifold-$\mathcal{RM}$
Horizontally Conformal Submersion-$\mathcal{HCS}$
Sasakian manifold-$\mathcal{SM}$
Almost contact metric manifold-$\mathcal{ACMM}$

2. Preliminaries

Authors find the study of $\mathcal{RS}$s to be an extremely intriguing topic. We now begin with a discussion of a few significant points and some useful results that are highly beneficial to our research.

Definition 2.1. [34] *Let (Ξ_1, g_1) and (Ξ_2, g_2) be two $\mathcal{RM}$s and $\bar{\alpha}$ be a smooth map between (Ξ_1, g_1) and (Ξ_2, g_2) where, m_1 and m_2 are the dimensions of Ξ_1 and Ξ_2 respectively. Then $\bar{\alpha}$ is called horizontally weakly conformal or semi conformal at $x \in \Xi_1$ if either*

(i) $\bar{\alpha}_{*x} = 0$, *or*

(ii) $\bar{\alpha}_{*x}$ *maps horizontal space* $\aleph_x = (\ker(\bar{\alpha}_{*x}))^{\perp}$ *conformally onto* $T_{\bar{\alpha}*}(\Xi_2)$ *i.e.,* $\bar{\alpha}_{*x}$ *is surjective and there exits a number* $\Lambda(x) \neq 0$ *such that*

$$g_2(\bar{\alpha}_{*x}\beta_1, \bar{\alpha}_{*x}\beta_2) = \Lambda(x) g(\beta_1, \beta_2),$$

for any $\beta_1, \beta_2 \in \aleph_x$.

Above equation can be reduce as:

$$(\bar{\alpha}_* g_2)_x \mid_{\aleph_x \times \aleph_x} = \Lambda(x) g(x) \mid_{\aleph_x \times \aleph_x}.$$

A point x is a critical point of $\bar{\alpha}$ if it fulfills (i) in the definition above. A point that satisfies (ii) is known as a regular point. At a critical point, $\bar{\alpha}_{*x}$ has rank 0, whereas at a regular point, it has rank n and defines a submersion. The number $\lambda(x)$ is called the square dilation and its square root $\lambda(x) = \sqrt{\Lambda(x)}$ is called the dilation of $\bar{\alpha}$ at x which is necessarily non-negative. A map $\bar{\alpha}$ is considered horizontally weakly conformal or semi-conformal on Ξ_1 if it is weakly conformal at all points of Ξ_1. If $\bar{\alpha}$ has no critical points, it is considered a (horizontally) conformal submersion.

Definition 2.2. [7] *Let $\bar{\alpha}$ be a $\mathcal{RS}$ between two $\mathcal{RM}s$. Then $\bar{\alpha}$ is called a horizontally conformal submersion, if there is a positive function λ such that*

$$g_1(\omega_1, \hat{\omega_2}) = \frac{1}{\lambda^2} g_2(\bar{\alpha}_* \omega_1, \bar{\alpha}_* \omega_2), \tag{2.1}$$

for any $\omega_1, \omega_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$. It is obvious that every $\mathcal{RS}s$ is particularly a horizontally conformal submersion with $\lambda = 1$.

Let be a $\mathcal{RS}$ $\bar{\alpha} : (\Xi_1, g_1) \to (\Xi_2, g_2)$. If $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^\perp$, then a vector field β_1 on Ξ_1 is referred to as a basic vector field and $\bar{\alpha}$-related to Ξ_2 using a vector field β_1 i.e $\bar{\alpha}_*(\beta_1(q)) = \beta_1 \bar{\alpha}(q)$ for $q \in \Xi_1$.

According to O'Neill, the two equations of the $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ are:

$$\mathcal{A}_{E_1} F_1 = \aleph \nabla_{\aleph E_1} v F_1 + v \nabla_{\aleph E_1} \aleph F_1, \tag{2.2}$$

$$\mathcal{T}_{E_1} F_1 = \aleph \nabla_{v E_1} v F_1 + v \nabla_{v E_1} \aleph F_1, \tag{2.3}$$

for any $E_1, F_1 \in \Gamma(T\Xi_1)$ and ∇ is a Levi-Civita connection of g_1. From above two equations of O'Neill, we can deduce that

$$\nabla_{\omega_1} \omega_2 = \mathcal{T}_{\omega_1} \omega_2 + v \nabla_{\omega_1} \omega_2 \tag{2.4}$$

$$\nabla_{\omega_1} \beta_1 = \mathcal{T}_{\omega_1} \beta_1 + \aleph \nabla_{\omega_1} \beta_1 \tag{2.5}$$

$$\nabla_{\beta_1} \omega_1 = \mathcal{A}_{\beta_1} \omega_1 + v_1 \nabla_{\beta_1} \omega_1 \tag{2.6}$$

$$\nabla_{\beta_1} \beta_2 = \aleph \nabla_{\beta_1} \beta_2 + \mathcal{A}_{\beta_1} \beta_2 \tag{2.7}$$

for any vector fields $\omega_1, \omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$ [11].

We note that

$$g(\mathcal{A}_{\beta_1} E_1, F_1) = -g(E_1, \mathcal{A}_{\beta_1} F_1),\ g(\mathcal{T}_{\omega_2} E_1, F_1) = -g(E_1, \mathcal{T}_{\omega_2} F_1),$$

for any vector fields $E_1, F_1 \in \Gamma(T\Xi_1)$. In the unique scenario where $\bar{\alpha}$ represents a $\mathcal{HCS}$, we possess:

Proposition 2.1. [14] *Let $\bar{\alpha} : (\Xi_1, g_1) \to (\Xi_2, g_2)$ be a horizontally conformal submersion with dilation λ and β_1, β_2 be the horizontal vectors, then*

$$A_{\beta_1} \beta_2 = \frac{1}{2} \{ v[\beta_1, \beta_2] - \lambda^2 g(\beta_1, \beta_2) grad_v (\frac{1}{\lambda^2}) \} \tag{2.8}$$

measures the obstruction integrability of the horizontal distribution.

The second fundamental form of smooth map $\bar{\alpha}$ is provided by the formula

$$(\nabla\bar{\alpha}_*)(\omega_1,\omega_2) = \nabla^{\bar{\alpha}}_{\omega_1}\bar{\alpha}_*\omega_2 - \bar{\alpha}_*\nabla_{\omega_1}\omega_2, \tag{2.9}$$

and the map be totally geodesic if $(\nabla\bar{\alpha}_*)(\omega_1,\omega_2) = 0$ for all $\omega_1,\omega_2 \in \Gamma(T\Xi_1)$ where ∇ and $\nabla^{\bar{\alpha}}$ are Levi-Civita and pullback connections.

Lemma 2.1. *Let $\bar{\alpha} : \Xi_1 \to \Xi_2$ be a $\mathcal{HCS}$. Then, we have*

(i) $(\nabla\bar{\alpha}_*)(\beta_1,\beta_2) = \beta_1(\ln\lambda)\bar{\alpha}_*(\beta_2) + \beta_2(ln\lambda)\bar{\alpha}_*(\beta_1) - g_1(\beta_1,\beta_2)\bar{\alpha}_*(\text{grad } ln\lambda)$,
(ii) $(\nabla\bar{\alpha}_*)(\omega_1,\omega_2) = -\bar{\alpha}_*(\mathcal{T}_{\omega_1}\omega_2)$,
(iii) $(\nabla\bar{\alpha}_*)(\beta_1,\omega_1) = -\bar{\alpha}_*(\nabla_{\beta_1}\omega_1) = -\bar{\alpha}_*(\mathcal{A}_{\beta_1}\omega_1)$

for any $\beta_1,\beta_2 \in \Gamma(ker\bar{\alpha}_*)^{\perp}$ and $\omega_1,\omega_2 \in \Gamma(ker\bar{\alpha}_*)$ [34].

Let M be a $(2n+1)$-dimensional almost contact manifold with almost contact structures (ϕ,ξ,η), where a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta\otimes\xi,\ \ \phi\xi = 0,\ \ \eta\circ\phi = 0,\ \ \eta(\xi) = 1, \tag{2.10}$$

where I is the identity tensor and there exists a Riemannian metric g in such a way that

$$g(\phi\omega_1,\phi\omega_2) = g(\omega_1,\omega_2) - \eta(\omega_1)\eta(\omega_2), \tag{2.11}$$

which can be noticed as follows:

$$\eta(\omega_1) = g(\omega_1,\xi), \tag{2.12}$$

for any $\omega_1,\ \omega_2 \in \Gamma(TM)$. Then (ϕ,ξ,η,g)-structure is called an almost contact metric structure. A normal contact metric structure is called a Sasakian structure, which satisfies

$$(\nabla_{\omega_1}\phi)\omega_2 = g(\omega_1,\omega_2)\xi - \eta(\omega_2)\omega_1 \tag{2.13}$$

where ∇ is the Levi-Civita connection of g. For a $\mathcal{SM}$, we can deduce that

$$\nabla_{\omega_1}\xi = -\phi\omega_1. \tag{2.14}$$

The covariant derivative of ϕ is defined by

$$(\nabla_{\beta_1}\phi)\beta_2 = \nabla_{\beta_1}\phi\beta_2 - \phi\nabla_{\beta_1}\beta_2, \tag{2.15}$$

for all vector fields β_1,β_2 in M. S. A. Sepet and M. Ergut [30] defined pointwise slant submersion as:

Definition 2.3. *Let $(\bar{\Xi}_1,\phi,\xi,\eta,g_1)$ be an $\mathcal{ACMM}$ and $(\bar{\Xi}_1,\phi,\xi,\eta,g_1)$ be a $\mathcal{RM}$.Let $\bar{\alpha}$ be a $\mathcal{RS}$ from Ξ_1 to Ξ_2. If the wirtinger angle $\theta(\beta_1)$ between $\phi\beta_1$ and the space* $\ker\bar{\alpha}_*$ *is independent of the choice of the non-zero vector field $\beta_1 \in \Gamma(\ker\bar{\alpha}_*) - <\xi>$ at each given point $q \in \Xi_1$, then $\bar{\alpha}$ is a pointwise slant submersion. The angle θ represents a function on $\bar{\Xi}_1$, known as the slant function of the pointwise slant submersion.*

Now, we extended the concept of ϕ-pluriharmonicity from almost contact metric manifolds to $(\Xi_1,\phi,\xi,\eta,g_1)$ which was once studied and defined by Y. Ohnita [21]. Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from almost contact metric manifolds to $(\Xi_1,\phi,\xi,\eta,g_1)$ onto a Riemannian manifold (Ξ_2,g_2). Then $\mathcal{PWHSCS}$ is $\mathfrak{D}^{\perp}$-ϕ-pluriharmonic, $\mathfrak{D}^{\theta}$-ϕ-pluriharmonic, $(\mathfrak{D}^{\perp}-\mathfrak{D}^{\theta})$-$\phi$ pluriharmonic, $\ker\bar{\alpha}_*$-ϕ-pluriharmonic, $(\ker\bar{\alpha}_*)^{\perp}$-$\phi$-pluriharmonic and $((\ker\bar{\alpha}_*)^{\perp}-\ker\bar{\alpha}_*)$-$\phi$-pluriharmonic if

$$(\nabla\bar{\alpha}_*)(\beta_1,\beta_2) + (\nabla\bar{\alpha}_*)(\phi\beta_1,\phi\beta_2) = 0, \tag{2.16}$$

for any $\beta_1, \beta_2 \in \Gamma(\mathfrak{D}^{\perp})$, for any $\beta_1, \beta_2 \in \Gamma(\mathfrak{D}^{\theta})$, for any $\beta_1 \in \Gamma(\mathfrak{D}^{\perp}), \beta_2 \in \Gamma(\mathfrak{D}^{\theta})$, for any $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)$, for any $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$ and for any $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)$.

3. Pointwise hemi-slant $\xi^{\perp}$-conformal submersions

In this section, we will revisit the idea of $\mathcal{PWHSCS}$ with $\xi \in \Gamma(ker\bar{\alpha})^{\perp}$.

Definition 3.1. *Let $\bar{\alpha} : (\Xi_1, \phi, \xi, \eta, g_1) \to (\Xi_2, g_2)$ be a $\mathcal{HCS}$ where $(\Xi_1, \phi, \xi, \eta, g_1)$ is an $\mathcal{ACMM}$ and (Ξ_2, g_2) is a $\mathcal{RM}$. A $\mathcal{HCS}$ $\bar{\alpha}$ is called a pointwise hemi-slant conformal submersion with $\xi \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$ if there exists two distributions $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$ such that $\ker\bar{\alpha}_* = \mathfrak{D}^{\theta} \oplus \mathfrak{D}^{\perp}$, $\phi(\mathfrak{D}^{\perp}) \subseteq \Gamma(\ker\bar{\alpha}_*)^{\perp}$ and for any given point $q \in \Xi_1$ and $\beta_1 \in (\mathfrak{D}^{\theta})_q$, the angle $\theta = \theta(\beta_1)$ between $\phi\beta_1$ and space $(\mathfrak{D}^{\theta})_q$ is independent of choice of non-zero vector $\beta_1 \in (\mathfrak{D}^{\theta})_q$, where $\mathfrak{D}^{\theta}$ is the orthogonal complement of $\mathfrak{D}^{\perp}$ in $\ker\bar{\alpha}_*$. In this case, the angle θ can be regarded as a slant function and called pointwise hemi-slant function of submersion.*

Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2). Then, for any $\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$, we have

$$\omega_2 = \mathfrak{P}\omega_2 + \mathfrak{Q}\omega_2 \tag{3.17}$$

where $\mathfrak{P}$ and $\mathfrak{Q}$ are the projections morphism onto $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$. Now, for any $\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$, we have

$$\phi\omega_2 = \bar{\delta}\omega_2 + \bar{\mathcal{F}}\omega_2 \tag{3.18}$$

where $\bar{\delta}\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\bar{\mathcal{F}}\omega_2 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$. From (3.17) and (3.18), we have

$$\begin{aligned}\phi\zeta_1 =&\phi(\mathfrak{P}\zeta_1) + \phi(\mathfrak{Q}\zeta_1)\\ =&\bar{\delta}(\mathfrak{P}\zeta_1) + \bar{\mathcal{F}}(\mathfrak{P}\zeta_1) + \bar{\delta}(\mathfrak{Q}\zeta_1) + \bar{\mathcal{F}}(\mathfrak{Q}\zeta_1),\end{aligned}$$

for any $\zeta_1 \in \Gamma(\ker\bar{\alpha}_*)$. Since $\phi\mathfrak{D}^{\perp} \subseteq \Gamma(\ker\bar{\alpha}_*)^{\perp}$, we have $\bar{\delta}(\mathfrak{P}\zeta_1) = 0$, we have

$$\phi\zeta_1 = \bar{\mathcal{F}}(\mathfrak{P}\zeta_1) + \bar{\delta}(\mathfrak{Q}\zeta_1) + \bar{\mathcal{F}}(\mathfrak{Q}\zeta_1).$$

Now, we have the following decomposition

$$(\ker\bar{\alpha}_*)^{\perp} = \bar{\mathcal{F}}\mathfrak{D}^{\theta} \oplus \bar{\mathcal{F}}\mathfrak{D}^{\perp} \oplus \nu, \tag{3.19}$$

where ν is the orthogonal complement to $\bar{\mathcal{F}}\mathfrak{D}^{\theta} \oplus \bar{\mathcal{F}}\mathfrak{D}^{\perp}$ in $(\ker\bar{\alpha}_*)^{\perp}$ such that ν is invariant with respect to ϕ. Now, for any $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$, we have

$$\phi\beta_1 = \mathcal{J}\beta_1 + \mathcal{N}\beta_1 \tag{3.20}$$

where $\mathcal{J}\beta_1 \in \Gamma(\ker\bar{\alpha}_*)$ and $\mathcal{N}\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$.

Lemma 3.1. *Let $(\Xi_1, \phi, \xi, \eta, g_1)$ be an $\mathcal{ACMM}$ and (Ξ_2, g_2) be a $\mathcal{RM}$. If $\bar{\alpha} : \Xi \to \Xi_2$ is a $\mathcal{PWHSCS}$, then we have*

$$-\omega_1 = -\bar{\delta}^2\omega_1 + \mathcal{J}\bar{\mathcal{F}}\omega_1,\ \bar{\mathcal{F}}\bar{\delta}\omega_1 + \mathcal{N}\bar{\mathcal{F}}\omega_1 = 0,\ -\beta_1 = \bar{\mathcal{F}}\mathcal{J}\beta_1 + \mathcal{N}^2\beta_1,\ \eta(\beta_1)\xi = \bar{\delta}\mathcal{J}\beta_1 + \mathcal{J}\mathcal{N}\beta_1,$$

for any vector field $\omega_1 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^{\perp}$.

Proof. By considering the (3.18) and (3.20), the proof of Lemma exists. ☐

Let us now provide some helpful outcomes, which will be applied throughout the research paper.

Lemma 3.2. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2), then we have*

$$\bar{\delta}^2\zeta_2 = (-\cos^2\theta)\zeta_2, \tag{3.21}$$

for any vector fields $\zeta_2 \in \Gamma(\mathfrak{D}^\theta)$.

Lemma 3.3. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2), then we have*

(i) $g_1(\bar{\delta}\zeta_1, \bar{\delta}\zeta_2) = \cos^2\theta\, g_1(\zeta_1, \zeta_2)$,
(ii) $g_1(\bar{F}\zeta_1, \bar{F}\zeta_2) = \sin^2\theta g_1(\zeta_1, \zeta_2)$,

for any vector fields $\zeta_1, \zeta_2 \in \Gamma(\mathfrak{D}^\theta)$.

Proof. The proof of the preceding Lemmas is identical to the proof of Theorem (2.2) of [9]. As a result, we omit the proofs. □

Lemma 3.4. *Let $\bar{\alpha} : \Xi_1 \to \Xi_2$ be a $\mathcal{PWHSCS}$ with hemi-slant function θ where, $(\Xi_1, \phi, \xi, \eta, g_1)$ a $\mathcal{SM}$ and (Ξ_2, g_2) a $\mathcal{RM}$, then we have*

(i) $\mathcal{A}_{\beta_1}\mathcal{N}\beta_2 + v\nabla_{\beta_1}\mathcal{J}\beta_2 = \mathcal{J}\aleph\nabla_{\beta_1}\beta_2 + \bar{\delta}\mathcal{A}_{\beta_1}\beta_2$
(ii) $\aleph\nabla_{\beta_1}\mathcal{N}\beta_2 + \mathcal{A}_{\beta_1}\mathcal{J}\beta_2 = \mathcal{N}\aleph\nabla_{\beta_1}\beta_2 + \bar{F}\mathcal{A}_{\beta_1}\beta_2 + g_1(\beta_1, \beta_2)\xi - \eta(\beta_2)\beta_1$
(iii) $v\nabla_{\beta_1}\bar{\delta}\omega_2 + \mathcal{A}_{\beta_1}\bar{F}\omega_2 = \mathcal{J}\mathcal{A}_{\beta_1}\omega_2 + \bar{\delta}v\nabla_{\beta_1}\omega_2$
(iv) $\mathcal{A}_{\beta_1}\bar{\delta}\omega_2 + \aleph\nabla_{\beta_1}\bar{F}\omega_2 = \mathcal{N}\mathcal{A}_{\beta_1}\omega_2 + \bar{F}v\nabla_{\beta_1}\omega_2$
(v) $v\nabla_{\omega_2}\mathcal{J}\beta_1 + \mathcal{T}_{\omega_2}\mathcal{N}\beta_1 = \bar{\delta}\mathcal{T}_{\omega_2}\beta_1 + \mathcal{J}\aleph\nabla_{\omega_2}\beta_1 + \eta(\beta_1)\omega_2$
(vi) $\mathcal{T}_{\omega_2}\mathcal{J}\beta_1 + \aleph\nabla_{\omega_2}\mathcal{N}\beta_1 = \bar{F}\mathcal{T}_{\omega_2}\beta_1 + \mathcal{N}\aleph\nabla_{\omega_2}\beta_1$
(vii) $v\nabla_{\omega_1}\bar{\delta}\omega_2 + \mathcal{T}_{\omega_1}\bar{F}\omega_2 = \bar{\delta}v\nabla_{\omega_1}\omega_2 + \mathcal{J}\mathcal{T}_{\omega_1}\omega_2$
(viii) $\mathcal{T}_{\omega_1}\bar{\delta}\omega_2 + \aleph\nabla_{\omega_1}\bar{F}\omega_2 = \mathcal{N}\mathcal{T}_{\omega_1}\omega_2 + \bar{F}v\nabla_{\omega_1}\omega_2 - g_1(\omega_1, \omega_2)\xi$,

for any vector fields $\omega_1, \omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. Using (2.13), (2.15), and (2.7) (3.20), we obtain the first two relations (i) and (ii). Equations (2.13), (2.15) (2.7), (2.4)-(2.7), and (3.18) (3.20 yield the expected results. □

To investigate the geometry of $\mathcal{PWHSCS}$ $\bar{\alpha} : \Xi_1 \to \Xi_2$, we will now review several key findings. Direct calculations might lead to the following conclusions:

(a) $(\nabla_{\omega_1}\bar{\delta})\omega_2 = v\nabla_{\omega_1}\bar{\delta}\omega_2 - \bar{\delta}v\nabla_{\omega_1}\omega_2$
(b) $(\nabla_{\omega_1}\bar{F})\omega_2 = \aleph\nabla_{\omega_1}\bar{F}\omega_2 - \bar{F}v\nabla_{\omega_1}\omega_2$
(c) $(\nabla_{\beta_1}\mathcal{J})\beta_2 = v\nabla_{\beta_1}\mathcal{J}\beta_2 - \mathcal{J}\aleph\nabla_{\beta_1}\beta_2$
(d) $(\nabla_{\beta_1}\mathcal{N})\beta_2 = \aleph\nabla_{\beta_1}\mathcal{N}\beta_2 - \mathcal{N}\aleph\nabla_{\beta_1}\beta_2$,

for any vector fields $\omega_1, \omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Lemma 3.5. *Let $\bar{\alpha} : \Xi_1 \to \Xi_2$ be a $\mathcal{PWHSCS}$ with hemi-slant function θ from a $\mathcal{SM}$ onto a $\mathcal{RM}$, then we have*

(i) $(\nabla_{\omega_1}\bar{\delta})\omega_2 = \mathcal{J}\mathcal{T}_{\omega_1}\omega_2 - \mathcal{T}_{\omega_1}\bar{F}\omega_2$
(ii) $(\nabla_{\omega_1}\bar{F})\omega_2 = \mathcal{N}\mathcal{T}_{\omega_1}\omega_2 - \mathcal{T}_{\omega_1}\bar{\delta}\omega_2 + g_1(\omega_1, \omega_2)\xi$
(iii) $(\nabla_{\beta_1}\mathcal{J})\beta_2 = \bar{\delta}\mathcal{A}_{\beta_1}\beta_2 - \mathcal{A}_{\beta_1}\mathcal{N}\beta_2$
(iv) $(\nabla_{\beta_1}\mathcal{N})\beta_2 = \bar{F}\mathcal{A}_{\beta_1}\beta_2 - \mathcal{A}_{\beta_1}\mathcal{J}\beta_2 - \eta(\beta_2)\beta_1 + g_1(\beta_1, \beta_2)\xi$,

for all vector fields $\omega_1, \omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. The results may be obtained by using the above-mentioned formulae $(a)-(d)$, as well as (2.15), (2.4)-(2.7). $\square$

The tensor fields $\bar{\delta}$ and $\bar{F}$, if they are parallel with regard to the Levi-Civita connection ∇ of Ξ_1, then we obtain

$$\mathcal{J}\mathcal{T}_{\omega_1}\omega_2 = \mathcal{T}_{\omega_1}\bar{F}\omega_2,\ \mathcal{N}\mathcal{T}_{\omega_1}\omega_2 - g_1(\omega_1,\omega_2)\xi = \mathcal{T}_{\omega_1}\bar{\delta}\omega_2$$

for any vector fields $\omega_1,\omega_2 \in \Gamma(T\Xi_1)$.

We are going to discuss about the anti-invariant distribution $\mathfrak{D}^\perp$ and the slant distribution $\mathfrak{D}^\theta$, as well as their integrability and total geodesic.

Theorem 3.1. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from a $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ onto a $\mathcal{RM}$ (Ξ_2,g_2) such that the structure vector field ξ is horizontal. Then the following are equivalent*

(i) *The anti-invariant distribution $\mathfrak{D}^\perp$ is integrable.*

(ii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_2}\bar{\alpha}_*(\bar{F}\zeta_1) - \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2), \bar{\alpha}_*(\bar{F}\omega_1))$
$= g(\nabla_{\zeta_1}\bar{F}\bar{\delta}\zeta_2 + \nabla_{\zeta_2}\bar{F}\bar{\delta}\zeta_1, \omega_1) - g(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2 + \mathcal{T}_{\zeta_2}\bar{F}\zeta_1, \bar{\delta}\omega_1)$

for any $\zeta_1,\zeta_2 \in \Gamma(D^\perp)$ and $\omega_1 \in \mathfrak{D}^\theta$.

Proof. By using (2.11), (2.13),(3.18), (2.4) and (2.5), we can write

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = -g_1(\phi\nabla_{\zeta_1}\bar{\delta}\zeta_2,\omega_1) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) + g_1(\aleph\nabla_{\zeta_1}\bar{F}\zeta_2,\bar{F}\omega_1),$$

for any $\zeta_1,\zeta_2 \in \Gamma(D)^\perp$ and $\omega_1 \in \Gamma(D)^\theta$. Taking account the fact from (3.17), (2.13), (2.5), and (3.18), we have

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = -g_1(\nabla_{\zeta_1}\bar{\delta}^2\zeta_2,\omega_1) - g_1(\nabla_{\zeta_1}\bar{F}\bar{\delta}\zeta_2,\omega_1) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) + g_1(\aleph\nabla_{\zeta_1}\bar{F}\zeta_2,\bar{F}\omega_1).$$

By using the horizontal conformality of $\bar{\alpha}$ and changing the role of ζ_1,ζ_2 and from (2.9) with Lemma 2.1, Lemma 3.2, we finally get

$$\begin{aligned}\sin^2\theta g_1([\zeta_1,\zeta_2],\omega_1) =& g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) + g_1(\mathcal{T}_{\zeta_2}\bar{F}\zeta_1,\bar{\delta}\omega_1) - g_1(\nabla_{\zeta_1}\bar{F}\bar{\delta}\zeta_2,\omega_1) - g_1(\nabla_{\zeta_2}\bar{F}\bar{\delta}\zeta_1,\omega_1)\\ &+ \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_2}(\bar{F}\zeta_1),\bar{\alpha}_*(\bar{F}\omega_1)) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}(\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)).\end{aligned}$$

This completes the proof. $\square$

Theorem 3.2. *Let $\bar{\alpha}:\Xi_1 \to \Xi_2$ be a $\mathcal{PWHSCS}$ from a $\mathcal{SM}$ Ξ_1 onto a $\mathcal{RM}$ Ξ_2 such that structure vector field ξ is horizontal. Then the following are equivalent.*

(i) *Slant distribution $\mathfrak{D}^\theta$ is integrable.*

(ii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\omega_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1)) + g_1([\omega_1,\zeta_1],\omega_2)$
$= \sin2\theta g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2) - g_1(\mathcal{T}_{\omega_2}\bar{F}\omega_1,\phi\zeta_1),$

for any $\omega_1,\omega_2 \in \Gamma(\mathfrak{D}^\theta)$ and $\zeta_1 \in \Gamma(\mathfrak{D}^\perp)$.

Proof. For any $\omega_1,\omega_2 \in \Gamma(D)^\theta$ and $\zeta_1 \in \Gamma(D)^\perp$ with taking account the fact from (2.11), (2.13) and (2.15), we get

$$g_1([\omega_1,\omega_2],\zeta_1) = -g_1([\omega_1,\zeta_1],\omega_2) - g_1(\nabla_{\zeta_1}\phi\omega_1,\phi\omega_2) - g_1(\nabla_{\omega_2}\phi\omega_1,\phi\zeta_1).$$

By using (2.4), (2.5) and (2.13), we can write

$$\begin{aligned}g_1([\omega_1,\omega_2],\zeta_1) =& -g_1([\omega_1,\zeta_1],\omega_2) + g_1(\nabla_{\zeta_1}\bar{\delta}^2\omega_1,\omega_2) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2)\\ &- g_1(\aleph\nabla_{\zeta_1}\bar{F}\omega_1,\bar{F}\omega_2) - g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\phi\zeta_1) - g_1(\aleph\nabla_{\omega_2}\bar{F}\omega_1,\phi\zeta_1).\end{aligned}$$

In the light of (2.1) with Lemma 3.2, we get

$$\begin{aligned} g_1([\omega_1,\omega_2],\zeta_1) =& - g_1([\omega_1,\zeta_1],\omega_2) + \sin 2\theta\zeta_1(\theta)g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) \\ &- \frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\zeta_1}\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) - \frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\omega_2}\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1)) \\ &- g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\phi\zeta_1) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2). \end{aligned}$$

By using the horizontal conformality of $\bar{\alpha}$ with Lemma 2.1 and (2.5), we finally have

$$\begin{aligned} g_1([\omega_1,\omega_2],\zeta_1) =& - g_1([\omega_1,\zeta_1],\omega_2) + \sin^2\theta\zeta_1(\theta)g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) \\ &- g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) \\ &- \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\omega_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1)) + \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_2,\bar{F}\omega_1),\bar{\alpha}_*(\phi\zeta_1)) \\ &- g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\phi\zeta_1) + \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)). \end{aligned}$$

□

This is the required proof of theorem.

Theorem 3.3. *Let $\bar{\alpha} : (\Xi_1,\phi,\xi,\eta,g_1) \to (\Xi_2,g_2)$ be a $\mathcal{PWHSCS}$ from a $\mathcal{SM}$ Ξ_1 onto a $\mathcal{RM}$ Ξ_2 such that structure vector field ξ is horizontal. Then the following are equivalent.*

(i) *Vertical distribution $(\ker\bar{\alpha}_*)$ is integrable.*

(ii) $$\begin{aligned} &\tfrac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\mathfrak{P}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) + \tfrac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\mathfrak{Q}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) \\ &= \cos^2\theta g_1(\nabla_{\beta_1}\mathfrak{Q}\omega_1,\omega_2) - \sin 2\theta\beta_1(\theta)g_1(\mathfrak{Q}\omega_1,\omega_2) + g_1([\omega_1,\beta_1],\omega_2) - g_1(\mathcal{A}_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{N}\beta_1) \\ &- g_1(v\nabla_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{J}\beta_1) + g_1(\mathcal{A}_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\bar{\delta}\omega_2) - g_1(\nabla_{\beta_1}\bar{F}\bar{\delta}\mathfrak{Q}\omega_1,\omega_2) + g_1(\mathcal{A}_{\beta_1}\bar{F}\mathfrak{Q}\omega_1,\bar{\delta}\omega_2) \\ &+ g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{N}\beta_1) + g_1(v\nabla_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1) + g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1) + g_1(\aleph\nabla_{\omega_2}\bar{F}\omega_1,\mathcal{N}\beta_1) \\ &+ g_1(\beta_1,\bar{F}\omega_2)g_1(\bar{F}\mathfrak{P}\omega_1,\operatorname{grad}\ln\lambda) + g_1(\bar{F}\mathfrak{P}\omega_1,\bar{F}\omega_2)g_1(\beta_1,\operatorname{grad}\ln\lambda) \\ &- g_1(\beta_1,\bar{F}\mathfrak{P}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda) + g_1(\beta_1,\bar{F}\omega_2)g_1(\bar{F}\mathfrak{Q}\omega_1,\operatorname{grad}\ln\lambda) \\ &+ g_1(\bar{F}\mathfrak{Q}\omega_1,\bar{F}\omega_2)g_1(\beta_1,\operatorname{grad}\ln\lambda) - g_1(\beta_1,\bar{F}\mathfrak{Q}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda), \end{aligned}$$

for any $\omega_1,\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. By using (2.11), (2.13), (2.15) and (3.17), we have

$$g_1([\omega_1,\omega_2],\beta_1) = -g_1([\omega_1,\beta_1],\omega_2) - g_1(\nabla_{\beta_1}\phi\omega_1,\phi\omega_2) + g_1(\nabla_{\omega_2}\phi\omega_1,\phi\beta_1),$$

for any $\omega_1,\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^\perp$. In the light of (2.4)-(2.7) and (2.15), we can write

$$\begin{aligned} g_1([\omega_1,\omega_2],\beta_1) =& - g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{N}\beta_1) - g_1(v\nabla_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\omega_2}\bar{F}\omega_1,\mathcal{J}\beta_1) \\ &- g_1([\omega_1,\beta_1],\omega_2) + g_1(\mathcal{A}_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{N}\beta_1) + g_1(v\nabla_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{J}\beta_1) \\ &- g_1(\mathcal{A}_{\beta_1}\bar{F}\mathfrak{P}\omega_1,\bar{\delta}\omega_2) - g_1(\aleph\nabla_{\beta_1}\bar{F}\mathfrak{P}\omega_1,\bar{F}\omega_2) + g_1(\nabla_{\beta_1}\phi\bar{\delta}\mathfrak{Q}\omega_1,\omega_2) \\ &- g_1((\nabla_{\beta_1}\phi)\bar{\delta}\mathfrak{Q}\omega_1,\omega_2) - g_1(\mathcal{A}_{\beta_1}\bar{F}\mathfrak{Q}\omega_1,\bar{\delta}\omega_2) - g_1(\aleph\nabla_{\beta_1}\bar{F}\mathfrak{Q}\omega_1,\bar{F}\omega_2) \\ &- g_1(\aleph\nabla_{\omega_2}\bar{F}\omega_1,\mathcal{N}\beta_1). \end{aligned}$$

By using Lemma 3.2 with (2.1), we get

$$\begin{aligned} g_1([\omega_1,\omega_2],\beta_1) = &- g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{N}\beta_1) - g_1(v\nabla_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1) \\ &+ \sin 2\theta\beta_1(\theta)g_1(\mathfrak{Q}\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\beta_1}\mathfrak{Q}\omega_1,\omega_2) + g_1(\nabla_{\beta_1}\bar{F}\bar{\delta}\mathfrak{Q}\omega_1,\omega_2) \\ &- g_1([\omega_1,\beta_1],\omega_2) + g_1(\mathcal{A}_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{N}\beta_1) + g_1(v\nabla_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{J}\beta_1) \\ &- \frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\beta_1}\bar{F}\mathfrak{P}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) - \frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\beta_1}\bar{F}\mathfrak{Q}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) \\ &- g_1(\mathcal{A}_{\beta_1}\bar{F}\mathfrak{Q}\omega_1,\bar{\delta}\omega_2) - g_1(\aleph\nabla_{\omega_2}\bar{F}\omega_1,\mathcal{N}\beta_1) - g_1(\mathfrak{A}_{\beta_1}\bar{F}\mathfrak{P}\omega_1,\bar{\delta}\omega_2). \end{aligned}$$

□

By using the horizontal conformality of $\bar{\alpha}$ from Lemma 2.1 and with (2.9), we finally deduce that

$$\begin{aligned} g_1([\omega_1,\omega_2],\beta_1) = &- g_1([\omega_1,\beta_1],\omega_2) + g_1(\mathcal{A}_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{N}\beta_1) + g_1(v\nabla_{\beta_1}\bar{\delta}\mathfrak{P}\omega_1,\mathcal{J}\beta_1) \\ &+ \sin 2\theta\beta_1(\theta)g_1(\mathfrak{Q}\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\beta_1}\mathfrak{Q}\omega_1,\omega_2) + g_1(\nabla_{\beta_1}\bar{F}\bar{\delta}\mathfrak{Q}\omega_1,\omega_2) \\ &- g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{N}\beta_1) - g_1(v\nabla_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\omega_2}\bar{\delta}\omega_1,\mathcal{J}\beta_1) \\ &- g_1(\beta_1,\bar{F}\omega_2)g_1(\bar{F}\mathfrak{P}\omega_1,\operatorname{grad}\ln\lambda) - g_1(\bar{F}\mathfrak{P}\omega_1,\bar{F}\omega_2)g_1(\beta_1,\operatorname{grad}\ln\lambda) \\ &- g_1(\beta_1,\bar{F}\mathfrak{P}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\mathfrak{P}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) \\ &- g_1(\beta_1,\bar{F}\omega_2)g_1(\bar{F}\mathfrak{Q}\omega_1,\operatorname{grad}\ln\lambda) - g_1(\bar{F}\mathfrak{Q}\omega_1,\bar{F}\omega_2)g_1(\beta_1,\operatorname{grad}\ln\lambda) \\ &- g_1(\beta_1,\bar{F}\mathfrak{Q}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\mathfrak{Q}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) \\ &- g_1(\mathfrak{A}_{\beta_1}\bar{F}\mathfrak{P}\omega_1,\bar{\delta}\omega_2) - g_1(\mathcal{A}_{\beta_1}\bar{F}\mathfrak{Q}\omega_1,\bar{\delta}\omega_2) - g_1(\aleph\nabla_{\omega_2}\bar{F}\omega_1,\mathcal{N}\beta_1). \end{aligned}$$

This completes the proof of theorem.

Theorem 3.4. *Let $\bar{\alpha} : (\Xi_1,\phi,\xi,\eta,g_1) \to (\Xi_2,g_2)$ be a $\mathcal{PWHSCS}$ where $(\Xi_1,\phi,\xi,\eta,g_1)$ is a $\mathcal{SM}$ and (Ξ_2,g_2) is a $\mathcal{RM}$ such that structure vector field ξ is horizontal. Then the anti-invariant distribution $\mathfrak{D}^\perp$ defines totally geodesic foliation if and only if*

$$\begin{aligned} \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)) = &\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)) - g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{F}) \\ &- g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) = &\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) - g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{N}\beta_1) \\ &- g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\mathcal{J}\beta_1) + g_1(\bar{\delta}\zeta_1,\zeta_2)\eta(\beta_1), \end{aligned} \tag{3.23}$$

for any $\zeta_1,\zeta_2 \in \Gamma(\mathfrak{D}^\perp), \omega_1 \in \Gamma(\mathfrak{D}^\theta)$ and $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. By using (2.11), (2.13) and (2.15), we get

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = g_1(\nabla_{\zeta_1}\bar{\delta}\zeta_2,\phi\omega_1) + g_1(\nabla_{\zeta_1}\bar{F}\zeta_2,\phi\omega_1).$$

In the light of (2.1), (2.4), and (2.5), we can write

$$\begin{aligned} g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = &g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) + \frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\zeta_1}\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)) \\ &+ g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{F}\omega_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1). \end{aligned}$$

By using the horizontal conformality of $\bar{\alpha}$ with Lemma 3.2 and (2.1), (2.5), we finally get

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{F}\omega_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\bar{\delta}\omega_1) - \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\bar{F}\omega_1)).$$

On the other hand, for any $\zeta_1,\zeta_2\in\Gamma(\mathfrak{D}^\perp)$, $\beta_1\in\Gamma(\ker\bar{\alpha}_*)^\perp$ and by using (2.11), (2.13), (2.15), we have

$$g_1(\nabla_{\zeta_1}\zeta_2,\beta_1) = g_1(\nabla_{\zeta_1}\phi\zeta_2,\phi\beta_1) + g_1(\bar{\delta}\zeta_1,\zeta_2)\eta(\beta_1).$$

Finally, in the light of (2.4), (2.5), (2.1) and (2.9), we get

$$g_1(\nabla_{\zeta_1}\zeta_2,\beta_1) = \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) - \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) + g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{N}\beta_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{J}\beta_1) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2,\mathcal{J}\beta_1) + g_1(\bar{\delta}\zeta_1,\zeta_2)\eta(\beta_1).$$

This is the required proof of theorem. □

Theorem 3.5. *Let $\bar{\alpha}:(\Xi_1,\phi,\xi,\eta,g_1)\to(\Xi_2,g_2)$ be a $\mathcal{PWHSCS}$ from a $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ onto a $\mathcal{RM}$ (Ξ_2,g_2) such that the structure vector field ξ is horizontal. Then the slant distribution defines totally geodesic foliation if and only if*

$$\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) - \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2) = g_1([\omega_1,\zeta_1],\omega_2) - \sin 2\theta\zeta_1(\theta)g_1(\omega_1,\omega_2) + \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) - g_1(\mathcal{T}_{\zeta_1}\bar{F}\bar{\delta}W,\omega_2)$$

and

$$\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_2,\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) = g_1([\omega_1,\beta_2],\omega_2) - \sin 2\theta\beta_2(\theta)g_1(\omega_1,\omega_2) + \cos^2\theta g_1(\nabla_{\beta_2}\omega_1,\omega_2) - g_1(\mathfrak{A}_{\beta_2}\bar{F}\bar{\delta}\omega_1,\omega_2) + g_1(\mathfrak{A}_{\beta_2}\bar{F}\omega_1,\bar{\delta}\omega_2) + g_1(\beta_2,\bar{F}\omega_2)g_1(\bar{F}\omega_1,\operatorname{grad}\ln\lambda) + g_1(\bar{F}\omega_1,\bar{F}\omega_2)g_1(\beta_2,\operatorname{grad}\ln\lambda) - g_1(\beta_2,\bar{F}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda),$$

for any $\omega_1,\omega_2\in\Gamma(\mathfrak{D}^\theta),\zeta_1\in\Gamma(\mathfrak{D}^\perp)$ and $\beta_2\in\Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. For any $\omega_1,\omega_2\in\Gamma(\mathfrak{D}^\theta)$ and $\zeta_1\in\Gamma(\mathfrak{D}^\perp)$ with using (2.10) , (2.11), (2.12) and (2.15), we have

$$g_1(\nabla_{\omega_1}\omega_2,\zeta_1) = g_1([\omega_1,\zeta_1],\omega_2) + g_1(\nabla_{\zeta_1}\bar{\delta}^2\omega_1,\omega_2) + g_1(\nabla_{\zeta_1}\bar{F}\bar{\delta}\omega_1,\omega_2) - g_1(\nabla_{\zeta_1}\bar{F}\omega_1,\phi\omega_2).$$

In the light of (2.5), (2.10) and (2.9) with Lemma 3.2, we can write

$$g(\nabla_{\omega_1}\omega_2,\zeta_1) = -g_1([\omega_1,\zeta_1],\omega_2) + \sin 2\theta\zeta_1(\theta)g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) + g_1(\mathcal{T}_{\zeta_1}\bar{F}\omega_1,\bar{\delta}\omega_2) - \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2)).$$

On the other hand, for any $\omega_1,\omega_2\in\Gamma(\mathfrak{D}^\theta)$ and $\beta_2\in\Gamma(\ker\bar{\alpha}_*)^\perp$ with using (2.10), (2.11), (2.12) and (2.15), we get

$$g_1(\nabla_{\omega_1}\omega_2,\beta_2) = -g_1([\omega_1,\beta_2],\omega_2) - g_1(\nabla_{\beta_2}\bar{\delta}^2\omega_1,\omega_2) + g_1(\nabla_{\beta_2}\bar{F}\bar{\delta}\omega_1,\omega_2) - g_1(\nabla_{\beta_2}\bar{F}\omega_1,\phi\omega_2).$$

From (2.7) and with lemma 3.2, we have

$$g_1(\nabla_{\omega_1}\omega_2,\beta_2)=-\,g_1([\omega_1,\beta_2],\omega_2)+\sin 2\theta\beta_2(\theta)g_1(\omega_1,\omega_2)-\cos^2\theta g_1(\nabla_{\beta_2}\omega_1,\omega_2)$$
$$+\,g_1(\mathcal{A}_{\beta_2}\bar{F}\bar{\delta}\omega_1,\omega_2)-g_1(\mathcal{A}_{\beta_2}\bar{F}\omega_1,\bar{\delta}\omega_2)-g_1(\aleph\nabla_{\beta_2}\bar{F}\omega_1,\bar{F}\omega_2).$$

Finally by using the horizontal conformality of $\bar{\alpha}$ with (2.1), (2.9) and Lemma 2.1, we can deduce that

$$\begin{aligned}g_1(\nabla_{\omega_1}\omega_2,\beta_2)=&-\,g_1([\omega_1,\beta_2],\omega_2)+\sin 2\theta\beta_2(\theta)g_1(\omega_1,\omega_2)-\cos^2\theta g_1(\nabla_{\beta_2}\omega_1,\omega_2)\\&+g_1(\mathcal{A}_{\beta_2}\bar{F}\bar{\delta}\omega_1,\omega_2)-g_1(\mathcal{A}_{\beta_2}\bar{F}\omega_1,\bar{\delta}\omega_2)-\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\omega_2))\\&-g_1(\beta_2,\bar{F}\omega_2)g_1(\bar{F}\omega_1,\operatorname{grad}\ln\lambda)-g_1(\bar{F}\omega_1,\bar{F}\omega_2)g_1(\beta_2,\operatorname{grad}\ln\lambda)\\&+g_1(\beta_2,\bar{F}\omega_1)g_1(\bar{F}\omega_2,\operatorname{grad}\ln\lambda).\end{aligned}$$

□

This completes the proof of theorem.

Theorem 3.6. *Let $\bar{\alpha}:\Xi_1\to\Xi_2$ be a $\mathcal{PWHSCS}$ from $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ onto $\mathcal{RM}$ (Ξ_2,g_2) with structure vector field ξ is horizontal. Then $(\ker\bar{\alpha}_*)^\perp$ defines totally geodesic foliation if and only if*

$$\begin{aligned}&\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\mathcal{N}\beta_2),\bar{\alpha}_*(\bar{F}\zeta_1))\\&=g_1(\beta_1,\bar{F}\zeta_1)g_1(\mathcal{N}\beta_2,\operatorname{grad}\ln\lambda)+g_1(\mathcal{N}\beta_2,\bar{F}\zeta_1)g_1(\beta_1,\operatorname{grad}\ln\lambda)\\&\quad-g_1(\beta_1,\mathcal{N}\beta_2)g_1(\bar{F}\zeta_1,\operatorname{grad}\ln\lambda)-g_1(\mathfrak{A}_{\beta_1}\mathcal{J}\beta_2,\bar{F}\zeta_1)\\&\quad-g_1(v\nabla_{\beta_1}\mathcal{J}\beta_2,\bar{\delta}\zeta_1)-g_1(\mathfrak{A}_{\beta_1}\mathcal{N}\beta_2,\bar{\delta}\zeta_1)-\eta(\beta_2)g_1(\beta_1,\bar{F}\zeta_1),\end{aligned}$$

for any $\beta_1,\beta_2\in\Gamma(\ker\bar{\alpha}_*)^\perp$ and $\zeta_1\in\Gamma(\ker\bar{\alpha}_*)$.

Proof. For any $\beta_1,\beta_2\in\Gamma(\ker\bar{\alpha}_*)^\perp$ and $\zeta_1\in\Gamma(\ker\bar{\alpha}_*)$ with using (2.10), (2.11), (2.12), (2.13), (2.10) and (2.7), we can write

$$\begin{aligned}g_1(\nabla_{\beta_1}\beta_2,\zeta_1)=&g_1(\mathcal{A}_{\beta_1}\mathcal{J}\beta_2,\bar{F}\zeta_1)+g_1(v\nabla_{\beta_1}\mathcal{J}\beta_2,\bar{\delta}\zeta_1)+g_1(\mathcal{A}_{\beta_1}\mathcal{N}\beta_2,\bar{\delta}\zeta_1)\\&+g_1(\mathcal{H}\nabla_{\beta_1}\mathcal{N}\beta_2,\bar{F}\zeta_1)+\eta(\beta_2)g_1(\beta_1,\bar{F}\zeta_1).\end{aligned}$$

By using the horizontal conformality of $\bar{\alpha}$ with (2.1), (2.9) and Lemma 2.1, we finally get

$$\begin{aligned}g_1(\nabla_{\beta_1}\beta_2,\zeta_1)=&g_1(\mathcal{A}_{\beta_1}\mathcal{J}\beta_2,\bar{F}\zeta_1)+g_1(v\nabla_{\beta_1}\mathcal{J}\beta_2,\bar{\delta}\zeta_1)+g_1(\mathcal{A}_{\beta_1}\mathcal{N}\beta_2,\bar{\delta}\zeta_1)\\&-g_1(\beta_1,\bar{F}\zeta_1)g_1(\mathcal{N}\beta_2,\operatorname{grad}\ln\lambda)-g_1(\mathcal{N}\beta_2,\bar{F}\zeta_1)g_1(\beta_1,\operatorname{grad}\ln\lambda)\\&+g_1(\beta_1,\mathcal{N}\beta_2)g_1(\bar{F}\zeta_1,\operatorname{grad}\ln\lambda)-\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\mathcal{N}\beta_2),\bar{\alpha}_*(\bar{F}\zeta_1))\\&+\eta(\beta_2)g_1(\beta_1,\bar{F}\zeta_1),\end{aligned}$$

which is the required proof of the theorem. □

Theorem 3.7. *Let $\bar{\alpha}:(\Xi_1,\phi,\xi,\eta,g_1)\to(\Xi_2,g_2)$ be a $\mathcal{PWHSCS}$ where, $(\Xi_1,\phi,\xi,\eta,g_1)$ a $\mathcal{SM}$ and (Ξ_2,g_2) a $\mathcal{RM}$ with structure vector field ξ is horizontal. Then the following are equivalent.*

(i) $(\ker\bar{\alpha}_*)$ *defines a totally geodesic foliation.*

(ii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_2}\bar{\alpha}_*(\bar{F}Q\beta_1),\bar{\alpha}_*(\bar{F}\beta_2))+g_1([\beta_1,\bar{F}],\beta_2)+g_1(\mathcal{A}_{\zeta_2}\bar{\delta}P\beta_1,\bar{F}\beta_2)+g_1(\mathcal{A}_{\zeta_2}\bar{F}P\beta_1,\bar{\delta}\beta_2)$
$=\sin 2\theta\zeta_2(\theta)g_1(Q\beta_1,\beta_2)-\cos^2\theta g_1(\nabla_{\zeta_2}Q\beta_1,\beta_2)-g_1(v\nabla_{\zeta_2}\bar{\delta}P\beta_1,\bar{\delta}\zeta_2)$

$$- g_1(\mathcal{H}\nabla_{\zeta_2}\bar{F}P\beta_1, \bar{F}\beta_2) - g_1(\mathcal{A}_{\zeta_2}\bar{F}Q\beta_1, \bar{\delta}\beta_2) - \eta(\beta_1)g_1(\zeta_2, \bar{F}\beta_2) + \eta(\beta_2)g_1(\bar{\delta}\zeta_2, \beta_1)$$
$$+ g_1(\zeta_2, \bar{F}\beta_2)g_1(\bar{F}Q\beta_1, \mathrm{grad}\ln\lambda) + g_1(\bar{F}Q\beta_1, \bar{F}\beta_2)g_1(\zeta_2, \mathrm{grad}\ln\lambda)$$
$$- g_1(\zeta_2, \bar{F}Q\beta_1)g_1(\bar{F}\beta_2, \mathrm{grad}\ln\lambda),$$

for any $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)$ *and* $\zeta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. Considering the fact

$$g_1(\nabla_{\beta_1}\beta_2, \zeta_2) = -g_1([\beta_1, \zeta_2], \beta_2) - g_1(\nabla_{\zeta_2}\phi\beta_1, \phi\beta_2) - \eta(\beta_1)g_1(\zeta_2, \bar{F}\beta_2) + \eta(\beta_2)g_1(\bar{\delta}\zeta_2, \beta_1),$$

by using (2.11), (2.15), (2.10) and (2.13) for any $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)$ and $\zeta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$. In the light of (3.17), (3.18) and (3.20), we can write

$$\begin{aligned} g_1(\nabla_{\beta_1}\beta_2, \zeta_2) =& - g_1([\beta_1, \zeta_2], \beta_2) - g_1(\nabla_{\zeta_2}\bar{\delta}P\beta_1, \phi\beta_2) - g_1(\nabla_{\zeta_2}\bar{F}P\beta_1, \phi\beta_2) \\ & - g_1(\nabla_{\zeta_2}\bar{F}Q\beta_1, \phi\beta_2) - \eta(\beta_1)g_1(\zeta_2, \bar{F}\beta_2) + \eta(\beta_2)g_1(\bar{\delta}\zeta_2, \beta_1) \\ & - g_1(\nabla_{\zeta_2}\bar{\delta}Q\beta_1, \phi\beta_2). \end{aligned}$$

From (2.10), (2.7), (2.15) and (2.13), we have

$$\begin{aligned} g_1(\nabla_{\beta_1}\beta_2, \zeta_2) =& - g_1([\beta_1, \zeta_2], \beta_2) - g_1(\mathcal{A}_{\zeta_2}\bar{\delta}P\beta_1, \bar{F}\beta_2) - g_1(v\nabla_{\zeta_2}\bar{\delta}P\beta_1, \bar{F}\beta_2) \\ & - g_1(\mathcal{H}\nabla_{\zeta_2}\bar{F}P\beta_1, \bar{F}\beta_2) + g_1(\nabla_{\zeta_2}\bar{\delta}^2Q\beta_1, \beta_2) - g_1(\mathcal{A}_{\zeta_2}\bar{F}Q\beta_1, \bar{\delta}\beta_2) \\ & - g_1(\mathcal{H}\nabla_{\zeta_2}\bar{F}Q\beta_1, \bar{F}\beta_2) - \eta(\beta_1)g_1(\zeta_2, \bar{F}\beta_2) + \eta(\beta_2)g_1(\bar{\delta}\zeta_2, \beta_1) \\ & - g_1(\mathcal{A}_{\zeta_2}\bar{F}P\beta_1, \bar{\delta}\beta_2). \end{aligned}$$

By using (2.1), (2.9) and from the fact that $\bar{\alpha}$ is a $\mathcal{PWHSCS}$, we finally get

$$\begin{aligned} g_1(\nabla_{\beta_1}\beta_2, W) =& - g_1([\beta_1, \zeta_2], \beta_2) - g_1(\mathcal{A}_{\zeta_2}\bar{\delta}P\beta_1, \bar{F}\beta_2) - g_1(v\nabla_{\zeta_2}\bar{\delta}P\beta_1, \bar{F}\beta_2) \\ & - g_1(\mathcal{H}\nabla_{\zeta_2}\bar{F}P\beta_1, \bar{F}\beta_2) + \sin 2\theta\zeta_2(\theta)g_1(Q\beta_1, \beta_2) - \cos^2\theta g_1(\nabla_{\zeta_2}Q\beta_1, \beta_2) \\ & - g_1(\mathcal{A}_{\zeta_2}\bar{F}Q\beta_1, \bar{\delta}\beta_2) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_2}\bar{\alpha}_*(\bar{F}Q\beta_1), \bar{\alpha}_*(\bar{F}\beta_2)) \\ & + g_1(\zeta_2, \bar{F}\beta_2)g_1(\bar{F}Q\beta_1, \mathrm{grad}\ln\lambda) + g_1(\bar{F}Q\beta_1, , \bar{F}\beta_2)g_1(\zeta_2, \mathrm{grad}\ln\lambda) \\ & - g_1(\zeta_2, \bar{F}Q\beta_1)g_1(\bar{F}\beta_2, \mathrm{grad}\ln\lambda) - \eta(\beta_1)g_1(\zeta_2, \bar{F}\beta_2) + \eta(\beta_2)g_1(\bar{\delta}\zeta_2, X) \\ & - g_1(\mathcal{A}_{\zeta_2}\bar{F}P\beta_1, \bar{\delta}\beta_2). \end{aligned}$$

This is complete proof of the theorem. □

Theorem 3.8. *Let* $\bar{\alpha} : (\Xi_1, \phi, \xi, \eta, g_1) \to (\Xi_1, g_2)$ *be a* $\mathcal{PWHSCS}$ *from a* $\mathcal{SM}$ Ξ_1 *onto a* $\mathcal{RM}$ Ξ_2 *such that the structure vector field* ξ *is horizontal. Then* $\bar{\alpha}$ *is totally geodesic map if and only if*

(i) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_2}\bar{\alpha}_*(\bar{F}\omega_1), \bar{\alpha}_*(\bar{F}\zeta_1)) = \sin 2\theta\beta_2(\theta)g_1(\omega_1, \zeta_1) - \cos^2\theta g_1(\nabla_{\beta_2}\omega_1, \zeta_1)$
$- g_1(\mathcal{A}_{\beta_2}\bar{F}\bar{\delta}\omega_1, \zeta_1) - g_1(\mathcal{A}_{\beta_2}\bar{F}\omega_1, \bar{\delta}\zeta_1) + g_1(\beta_2, \bar{F}\zeta_1)g_1(\bar{F}\omega_1, \mathrm{grad}\ln\lambda)$
$+ g_1(\bar{F}\omega_1, \bar{F}\zeta_1)g_1(\beta_2, \mathrm{grad}\ln\lambda) - g_1(\beta_2, \bar{F}\omega_1)g_1(\bar{F}\zeta_1, \mathrm{grad}\ln\lambda) - g_1([\beta_2, \omega_1], \zeta_1)$

(ii) $\mathcal{T}_{\zeta_2}\mathcal{J}\bar{F}\omega_1 + \nabla_{\zeta_2}\mathcal{N}\bar{F}\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$

(iii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\bar{\delta}\omega_1), \bar{\alpha}_*(\beta_2)) = \cos^2\theta g_1(\nabla_{\beta_1}\omega_1, \beta_2) - g_1(\mathfrak{A}_{\beta_1}\bar{F}\omega_1, \mathcal{J}\beta_2)$
$+ \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_1, \bar{F}\bar{\delta}\omega_1), \bar{\alpha}_*(\beta_2)) + \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_1, \bar{F}\omega_1), \bar{\alpha}_*(\mathcal{N}\beta_2))$
$- \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\omega_1), \bar{\alpha}_*(\mathcal{N}\beta_2)) - g_1(\mathcal{J}\beta_1, \omega_1)\eta(\beta_2),$

for any $\omega_1, \zeta_1 \in \Gamma(\mathfrak{D}^\theta), \zeta_2, \omega_2 \in \Gamma(\mathfrak{D}^\perp)$ *and* $\beta_1, \beta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Proof. By using (2.1), (2.15), (2.9) and (2.13), we can write

$$\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2)) = -g([\beta_2,\omega_1],\zeta_1) - g_1(\nabla_{\beta_2}\phi\omega_1,\phi\zeta_1),$$

for any $\omega_1,\zeta_1 \in \Gamma(\mathfrak{D}^\theta)$ and $\beta_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp$. In the light of (3.18), (2.13) and Lemma 3.2, we get

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2)) =& - g([\beta_2,\omega_1],\zeta_1) + \sin 2\theta\beta_2(\theta)g_1(\omega_1,\zeta_1) - \cos^2\theta g_1(\nabla_{\beta_2}\omega_1,\zeta_1)\\ &- g_1(\mathcal{A}_{\beta_2}\bar{F}\omega_1,\bar{\delta}\zeta_1) - g_1(\mathcal{H}\nabla_{\beta_2}\bar{F}\omega_1,\bar{F}\zeta_1).\end{aligned}$$

From (2.9) and by using the horizontal conformality from Lemma 2.1, we have

$$\begin{aligned}&\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2))\\ &= -g([\beta_2,\omega_1],\zeta_1) + \sin 2\theta\beta_2(\theta)g_1(\omega_1,\zeta_1) - \cos^2\theta g_1(\nabla_{\beta_2}\omega_1,\zeta_1)\\ &\quad - g_1(\mathcal{A}_{\beta_2}\bar{F}\omega_1,\bar{\delta}\zeta_1) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_2}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\bar{F}\zeta_1))\\ &\quad + g_1(\beta_2,\bar{F}\zeta_1)g_1(\bar{F}\omega_1,\operatorname{grad}\ln\lambda) + g_1(\bar{F}\omega_1,\bar{F}\zeta_1)g_1(\beta_2,\operatorname{grad}\ln\lambda)\\ &\quad - g_1(\beta_2,\bar{F}\omega_1)g_1(\bar{F}\zeta_1,\operatorname{grad}\ln\lambda),\end{aligned}$$

which is the proof of (i). On the other hand, from (2.1), (2.9), (2.13) and (2.15), we get

$$\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2)) = -g_1(\nabla_{\zeta_2}\phi\omega_2,\phi\beta_1),$$

for $\zeta_2,\omega_2 \in \Gamma(\mathfrak{D}^\perp)$ and $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^\perp$. By using the fact $\phi\omega_2 \in \Gamma(\ker\bar{\alpha}_*)^\perp, \omega_2 \in \Gamma(\mathfrak{D}^\perp)$ and from (2.4), (2.5), we have

$$\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2)) = g_1(\mathcal{A}_{\zeta_2}\mathcal{J}\bar{F}\omega_2,\beta_1) + g_1(\mathcal{H}\nabla_{\zeta_2}\mathcal{N}\bar{F}\omega_2,\beta_1),$$

which proves the (ii) part. For part (iii), by using (2.1), (2.9), (2.11) and (2.13), we can write

$$\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_1,\omega_1),\bar{\alpha}_*(\beta_2)) = -g_1(\nabla_{\beta_1}\phi\omega_1,\phi\beta_2) - g_1(\mathcal{J}\beta_1,\omega_1)\eta(\beta_2),$$

for any $\omega_1 \in \Gamma(\mathfrak{D}^\theta)$ and $\beta_1,\beta_2 \in \Gamma(\bar{\alpha}_*)^\perp$. In the light of (3.18), (2.11), (2.13), (2.7) and Lemma 3.2, we get

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2)) =& - \cos^2\theta g_1(\nabla_{\beta_1}\omega_1,\beta_2) - g_1(\mathcal{H}\nabla_{\beta_1}\bar{F}\bar{\delta}\omega_1,\beta_2) - g_1(\mathcal{A}_{\beta_1}\bar{F}\omega_1,\mathcal{J}\beta_2)\\ &- g_1(\mathcal{H}\nabla_{\beta_1}\bar{F}\omega_1,\mathcal{N}\beta_2) - g_1(\mathcal{J}\beta_1,\omega_1)\eta(\beta_2).\end{aligned}$$

By using the horizontal conformality of $\bar{\alpha}$ with (2.9) and Lemma 2.1, we finally have

$$\begin{aligned}&\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\omega_1,\zeta_1),\bar{\alpha}_*(\beta_2))\\ &= -\cos^2\theta g_1(\nabla_{\beta_1}\omega_1,\beta_2) - g_1(\mathcal{A}_{\beta_1}\bar{F}\omega_1,\mathcal{J}\beta_2) - g_1(\mathcal{J}\beta_1,\omega_1)\eta(\beta_2)\\ &\quad + \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_1,\bar{F}\bar{\delta}\omega_1),\bar{\alpha}_*(\beta_2)) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\bar{\delta}\omega_1),\bar{\alpha}_*(\beta_2))\\ &\quad + \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\beta_1,\bar{F}\omega_1),\bar{\alpha}_*(\mathcal{N}\beta_2)) - \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\beta_1}\bar{\alpha}_*(\bar{F}\omega_1),\bar{\alpha}_*(\mathcal{N}\beta_2)).\end{aligned}$$

This completes the proof of theorem. ∎

Theorem 3.9. *Let $\bar{\alpha} : (\Xi_1, \phi, \xi, \eta, g_1) \to (\Xi_2, g_2)$ be a $\mathcal{PWHSCS}$ where, $(\Xi_1, \phi, \xi, \eta, g_1)$ a $\mathcal{SM}$ and (Ξ_2, g_2) a $\mathcal{RM}$ with structure vector field ξ is horizontal. Suppose $\bar{\alpha}$ is $\mathfrak{D}^\theta$-ϕ-pluriharmonic. Then the following are equivalent.*

(i) *$\mathfrak{D}^\theta$ defines a totally geodesic foliation.*

(ii) $\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\zeta_2) + \nabla^{\bar{\alpha}}_{\phi\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_1) - \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2) = \cos^2\theta\bar{\alpha}_*(\mathcal{NH}\nabla_{\bar{\delta}\zeta_1}\zeta_2 + \bar{F}\mathcal{T}_{\bar{\delta}\zeta_1}\zeta_2$
$+ \bar{F}v\nabla_{\bar{F}\zeta_1}\zeta_2) - \sin 2\theta(\bar{\delta}\zeta_1(\theta)\bar{\alpha}_*(\bar{F}\zeta_2) + \bar{F}\zeta_1(\theta)\bar{\alpha}_*(\bar{F}\zeta_2)) - \eta(\nabla_{\bar{F}\zeta_1}\bar{\delta}\zeta_2)\bar{\alpha}_*\xi$
$- \bar{\alpha}_*(\mathcal{H}\nabla_{\bar{\delta}\zeta_1}\zeta_2 + \bar{F}\mathcal{A}_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2 + \mathcal{NH}\nabla_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2) - \bar{F}\zeta_1(\ln\lambda)\bar{\alpha}_*(\bar{F}\zeta_2)$
$- \bar{F}\zeta_2(\ln\lambda)\bar{\alpha}_*(\bar{F}\zeta_1) + g_1(\bar{F}\zeta_1, \bar{F}\zeta_2)\bar{\alpha}_*(\mathrm{grad}\ln\lambda) - \bar{\alpha}_*(\mathcal{H}_{\bar{\delta}\zeta_1}\bar{F}\zeta_2) + \mathcal{NA}_{\bar{F}\zeta_1}\zeta_2,$

for any $\zeta_1, \zeta_2 \in \Gamma(\mathfrak{D}^\theta)$.

Proof. By using the concept of ϕ-pluriharmonicity with (3.18) and (2.9), we have

$$\bar{\alpha}_*\nabla_{\zeta_1}\zeta_2 = \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\zeta_2) + \nabla^{\bar{\alpha}}_{\bar{\delta}\zeta_1}\bar{\alpha}_*(\phi\zeta_2) - \bar{\alpha}_*\nabla_{\bar{\delta}\zeta_1}\bar{\delta}\zeta_2 - \bar{\alpha}_*\nabla_{\bar{\delta}\zeta_1}\bar{F}\zeta_2 - \bar{\alpha}_*\nabla_{\bar{F}\zeta_1}\bar{\delta}\zeta_2 - \bar{\alpha}_*\nabla_{\bar{F}\zeta_1}\bar{F}\zeta_2,$$

for any $\zeta_1, \zeta_2 \in \Gamma(\mathfrak{D}^\theta)$. In the light of (2.5), (2.13) and (2.12), we can write

$$\begin{aligned}
&\bar{\alpha}_*\nabla_{\zeta_1}\zeta_2 \\
&= \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\zeta_2) + \nabla^{\bar{\alpha}}_{\phi\zeta_1}\bar{\alpha}_*(\phi\zeta_2) + \bar{\alpha}_*(\phi\nabla_{\bar{\delta}\zeta_1}\phi\bar{\delta}\zeta_2 + \eta(\nabla_{\bar{\delta}\zeta_1}\bar{\delta}\zeta_2)\xi) \\
&\quad + \bar{\alpha}_*(\phi\nabla_{\bar{F}\zeta_1}\phi\bar{\delta}\zeta_2 + \eta(\nabla_{\bar{F}\zeta_1}\bar{\delta}\zeta_2)\xi) - \bar{\alpha}_*\nabla_{\bar{F}\zeta_1}\bar{F}\zeta_2 - \bar{\alpha}_*\nabla_{\bar{\delta}\zeta_1}\bar{F}\zeta_2.
\end{aligned}$$

By using (2.4), (2.7) and Lemma 3.2, we get

$$\begin{aligned}
&\bar{\alpha}_*\nabla_{\zeta_1}\zeta_2 \\
&= \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\zeta_2) + \nabla^{\bar{\alpha}}_{\phi\zeta_1}\bar{\alpha}_*(\phi\zeta_2) + \sin 2\theta\bar{\delta}\zeta_1(\theta)\bar{\alpha}_*(\phi\zeta_2) - \cos^2\theta\bar{\alpha}_*(\phi\nabla_{\bar{\delta}\zeta_1}\zeta_2) \\
&\quad + \sin 2\theta\bar{F}\zeta_1(\theta)\bar{\alpha}_*(\phi\zeta_2) - \cos^2\theta\bar{\alpha}_*(\phi\nabla_{\bar{F}\zeta_1}\zeta_2) + \bar{\alpha}_*\{\phi(\mathcal{T}_{\bar{\delta}\zeta_1}\zeta_2 + \mathcal{H}\nabla_{\bar{\delta}\zeta_1}\zeta_2)\} \\
&\quad + \eta(\nabla_{\bar{\delta}\zeta_1}\bar{\delta}\zeta_2)\bar{\alpha}_*\xi + \bar{\alpha}_*\{\phi(\mathcal{A}_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2 + \mathcal{H}\nabla_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2)\} + \eta(\nabla_{\bar{F}\zeta_1}\bar{\delta}\zeta_2)\bar{\alpha}_*\xi \\
&\quad - \nabla^{\bar{\alpha}}_{\bar{F}\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2) + (\nabla\bar{\alpha}_*)(\bar{F}\zeta_1, \bar{F}\zeta_2) - \bar{\alpha}_*(\mathcal{T}_{\bar{\delta}\zeta_1}\bar{F}\zeta_2 + \mathcal{H}_{\bar{\delta}\zeta_1}\bar{F}\zeta_2).
\end{aligned}$$

Finally, by using the horizontal conformality of $\bar{\alpha}$ from Lemma 2.1 and with (2.7), (2.1), we have

$$\begin{aligned}
\bar{\alpha}_*\nabla_{\zeta_1}\zeta_2 =& \nabla^{\bar{\alpha}}_{\phi\zeta_1}\bar{\alpha}_*(\phi\zeta_2) + \sin 2\theta\bar{\delta}\zeta_1(\theta)\bar{\alpha}_*(\bar{F}\zeta_2) - \cos^2\theta\bar{\alpha}_*(\mathcal{NH}\nabla_{\bar{\delta}\zeta_1}\zeta_2 + \bar{F}\mathcal{T}_{\bar{\delta}\zeta_1}\zeta_2) \\
&+ \sin 2\theta\bar{F}\zeta_1(\theta)\bar{\alpha}_*(\bar{F}\zeta_2) - \cos^2\theta\bar{\alpha}_*(\mathcal{NA}_{\bar{F}\zeta_1}\zeta_2 + \bar{F}v\nabla_{\bar{F}\zeta_1}\zeta_2) + \bar{\alpha}_*(\mathcal{H}\nabla_{\bar{\delta}\zeta_1}\zeta_2) \\
&+ \eta(\nabla_{\bar{\delta}\zeta_1}\bar{\delta}\zeta_2)\bar{\alpha}_*\xi + \bar{\alpha}_*\{\bar{F}(\mathcal{A}_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2 + \mathcal{NH}\nabla_{\bar{F}\zeta_1}\bar{F}\bar{\delta}\zeta_2)\} + \eta(\nabla_{\bar{F}\zeta_1}\bar{\delta}\zeta_2)\bar{\alpha}_*\xi \\
&+ \bar{F}\zeta_1(\ln\lambda)\bar{\alpha}_*(\bar{F}\zeta_2) + \bar{F}\zeta_2(\ln\lambda)\bar{\alpha}_*(\bar{F}\zeta_1) - g_1(\bar{F}\zeta_1, \bar{F}\zeta_2)\bar{\alpha}_*(\mathrm{grad}\ln\lambda) \\
&- \bar{\alpha}_*(\mathcal{H}_{\bar{\delta}\zeta_1}\bar{F}\zeta_2) + \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\zeta_2) - \nabla^{\bar{\alpha}}_{\bar{F}\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2).
\end{aligned}$$

□

This completes the proof of the theorem.

4. Pointwise hemi-slant conformal submersions with vertical Reeb vector field-ξ

This section will go over the definitions and results that will help us understand and investigate the concept of pointwise hemi-slant conformal submersions from $\mathcal{ACMM}s$ by considering the Reeb vector filed ξ vertical.

Definition 4.1. *Let $\bar{\alpha} : (\Xi_1, \phi, \xi, \eta, g_1) \to (\Xi_2, g_2)$ be a $\mathcal{HCS}$ where $(\Xi_1, \phi, \xi, \eta, g_1)$ is an $\mathcal{ACMM}$ and (Ξ_2, g_2) is a $\mathcal{RM}$. A $\mathcal{HCS}$ $\bar{\alpha}$ is called a pointwise hemi-slant conformal submersion if there exists distributions $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$ such that $\ker\bar{\alpha}_* = \mathfrak{D}^{\theta} \oplus \mathfrak{D}^{\perp} \oplus < \xi >$, $\phi(\mathfrak{D}^{\perp}) \subseteq \Gamma(\ker\bar{\alpha})^{\perp}$ and for any given point $q \in \Xi_1$ and $\beta_1 \in (\mathfrak{D}^{\theta})_q$, the angle $\theta = \theta(\beta_1)$ between $\phi\beta_1$ and space $(\mathfrak{D}^{\theta})_q$ is independent of choice of non-zero vector $\beta_1 \in (\mathfrak{D}^{\theta})_q$, where $\mathfrak{D}^{\theta}$ is the orthogonal complement of $\mathfrak{D}^{\perp}$ in $\ker\bar{\alpha}_*$ and $< \xi >$ is 1-dimensional distribution. The angle θ is a slant function, often known as the pointwise hemi-slant function of submersion.*

Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2) with vertical Reeb vector field ξ. Then, for any $Y \in (\ker\bar{\alpha}_*)$, we have

$$Y = \mathfrak{P}\beta_2 + \mathfrak{Q}\beta_2 + \eta(\beta_2)\xi \tag{4.24}$$

where $\mathfrak{P}$ and $\mathfrak{Q}$ are the projections morphism onto $\mathfrak{D}^{\perp}$ and $\mathfrak{D}^{\theta}$.

Lemma 4.1. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2), then we have*

$$\bar{\delta}^2\omega_2 = -\cos^2\theta(I - \eta \otimes \xi)\omega_2, \tag{4.25}$$

for any vector field $\omega_2 \in \Gamma(\ker\bar{\alpha}_*)$.

Lemma 4.2. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ with vertical ξ, from an $\mathcal{ACMM}$ $(\Xi_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ (Ξ_2, g_2), then we have*

(i) $g_1(\bar{\delta}\zeta_1, \bar{\delta}\zeta_2) = \cos^2\theta\{g_1(\zeta_1, \zeta_2) - \eta(\zeta_1)\eta(\zeta_2)\}$,
(ii) $g_1(\bar{F}\zeta_1, \bar{F}\zeta_2) = \sin^2\theta\{g_1(\zeta_1, \zeta_2) - \eta(\zeta_1)\eta(\zeta_2)\}$,

for any vector fields $\zeta_1, \zeta_2 \in \Gamma(\ker\bar{\alpha}_*)$.

Moving further, we shall talk about the integrability of slant and anti-invariant distributions $\mathfrak{D}^{\theta}$ and $\mathfrak{D}^{\perp}$ respectively.

Theorem 4.1. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from $\mathcal{SM}$ onto a $\mathcal{RM}$ with vertical ξ and θ is a hemi-slant function, tfae*

(i) *The anti-invariant distribution $\mathfrak{D}^{\perp}$ is integrable.*
(ii) $\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_2}\bar{\alpha}_*(\bar{F}\zeta_1) - \nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{F}\zeta_2), \bar{\alpha}_*(\bar{F}\omega_1))$
$= g(\nabla_{\zeta_1}\bar{F}\bar{\delta}\zeta_2 + \nabla_{\zeta_2}\bar{F}\bar{\delta}\zeta_1, \omega_1) - g(\mathcal{T}_{\zeta_1}\bar{F}\zeta_2 + \mathcal{T}_{\zeta_2}\bar{F}\zeta_1, \bar{\delta}\omega_1)$,

for any $\zeta_1, \zeta_2 \in \Gamma(D^{\perp})$ and $\omega_1 \in \mathfrak{D}^{\theta}$.

By comparing the preceding conclusion with Theorem 3.1, it is inescapable that there is no influence of the Reeb vector field ξ, whether horizontal or vertical. For slant distribution, we have

Lemma 4.3. *Let $\bar{\alpha} : (\Xi_1, \phi, \xi, \eta, g_1) \to (\Xi_2, g_2)$ be a $\mathcal{PWHSCS}$ with $\xi \in \Gamma(ker\bar{\alpha}_*)$ where, $(\Xi_1, \phi, \xi, \eta, g_1)$ a $\mathcal{SM}$ and (Ξ_2, g_2) a $\mathcal{RM}$. Then the slant distribution is not integrable.*

Since the slant distribution is not integrable, now we will discuss about distribution $\mathfrak{D}^{\theta} \oplus < \xi >$.

Theorem 4.2. *Let $\bar{\alpha} : \Xi_1 \to \Xi_2$ be a $\mathcal{PWHSCS}$ from a $\mathcal{SM}$ Ξ_1 onto a $\mathcal{RM}$ Ξ_2 such that ξ is vertical. Then the following are equivalent.*

(i) *Slant distribution* $\mathfrak{D}^\theta \oplus <\xi>$ *is integrable.*

(ii) $\frac{1}{\lambda^2} g_2(\nabla^{\bar\alpha}_{\zeta_1} \bar\alpha_*(\bar F\omega_1), \bar\alpha_*(\bar F\omega_2)) + \frac{1}{\lambda^2} g_2(\nabla^{\bar\alpha}_{\omega_2} \bar\alpha_*(\bar F\omega_1), \bar\alpha_*(\phi\zeta_1)) + g_1([\omega_1,\zeta_1],\omega_2)$
$= \sin 2\theta g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) - g_1(\mathcal{T}_{\zeta_1}\bar F\omega_1, \bar\delta\omega_2) - g_1(\mathcal{T}_{\omega_2}\bar F\omega_1, \phi\zeta_1)$,

for any $\omega_1,\omega_2 \in \Gamma(\mathfrak{D}^\theta \oplus <\xi>)$ *and* $\zeta_1 \in \Gamma(\mathfrak{D}^\perp)$.

Proof. For any $\omega_1,\omega_2 \in \Gamma(\mathfrak{D}^\theta \oplus <\xi>)$ and $\zeta_1 \in \Gamma(D)^\perp$ with taking account the fact from (2.11), (2.13), (2.15), (2.4), (2.5), (2.1) with Lemma 3.2, we get

$$\begin{aligned}
&g_1([\omega_1,\omega_2],\zeta_1)\\
&= -g_1([\omega_1,\zeta_1],\omega_2) + \sin 2\theta\zeta_1(\theta) g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) - g_1(\mathcal{T}_{\omega_2}\bar\delta\omega_1,\phi\zeta_1)\\
&- g_1(\mathcal{T}_{\zeta_1}\bar F\omega_1,\bar\delta\omega_2) - \frac{1}{\lambda^2} g_2(\bar\alpha_*(\aleph\nabla_{\zeta_1}\bar F\omega_1),\bar\alpha_*(\bar F\omega_2)) - \frac{1}{\lambda^2} g_2(\bar\alpha_*(\aleph\nabla_{\omega_2}\bar F\omega_1),\bar\alpha_*(\phi\zeta_1)).
\end{aligned}$$

By using the horizontal conformality of $\bar\alpha$ with Lemma 2.1 and (2.5), we finally have

$$\begin{aligned}
&g_1([\omega_1,\omega_2],\zeta_1)\\
&= -g_1([\omega_1,\zeta_1],\omega_2) + \sin^2\theta\zeta_1(\theta) g_1(\omega_1,\omega_2) - \cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2) - g_1(\mathcal{T}_{\omega_2}\bar\delta\omega_1,\phi\zeta_1)\\
&- g_1(\mathcal{T}_{\zeta_1}\bar F\omega_1,\bar\delta\omega_2) - \frac{1}{\lambda^2} g_2(\nabla^{\bar\alpha}_{\zeta_1}\bar\alpha_*(\bar F\omega_1),\bar\alpha_*(\bar F\omega_2)) + \frac{1}{\lambda^2} g_2((\nabla\bar\alpha_*)(\zeta_1,\bar F\omega_1),\bar\alpha_*(\bar F\omega_2))\\
&- \frac{1}{\lambda^2} g_2(\nabla^{\bar\alpha}_{\omega_2}\bar\alpha_*(\bar F\omega_1),\bar\alpha_*(\phi\zeta_1)) + \frac{1}{\lambda^2} g_2((\nabla\bar\alpha_*)(\omega_2,\bar F\omega_1),\bar\alpha_*(\phi\zeta_1)).
\end{aligned}$$

□

Although the nature of ξ differs, the proofs of Theorem 3.2 and the previous result are identical as well.

Corollary 4.1. *Let* $\bar\alpha : (\Xi_1,\phi,\xi,\eta,g_1) \to (\Xi_2,g_2)$ *be a* $\mathcal{PWHSCS}$ *from* $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ *onto a* $\mathcal{RM}$ (Ξ_2,g_2) *with hemi-slant function* θ*. The following conditions holds.*

Let $\bar\alpha : (\Xi_1,\phi,\xi,\eta,g_1) \to (\Xi_2,g_2)$ *be a* $\mathcal{PWHSCS}$ *from* $\mathcal{SM}$ *onto a* $\mathcal{RM}$ *with hemi slant function* θ*. Then*	*(i)* $\mathfrak{D}^\theta \oplus <\xi>$ *is integrable with* $\xi \in \Gamma(ker\bar\alpha_*)$ *if and only if*	*(ii)* $\mathfrak{D}^\theta$ *is integrable with* $\xi \in \Gamma(ker\bar\alpha_*)^\perp$ *if and only if*
	$\frac{1}{\lambda^2} g_2(\nabla^{\bar\alpha}_{\zeta_1}\bar\alpha_*(\bar F\omega_1),\bar\alpha_*(\bar F\omega_2))$ $+\frac{1}{\lambda^2} g_2(\nabla^{\bar\alpha}_{\omega_2}\bar\alpha_*(\bar F\omega_1),\bar\alpha_*(\phi\zeta_1))$ $+g_1([\omega_1,\zeta_1],\omega_2) = \sin 2\theta$ $g_1(\omega_1,\omega_2) - g_1(\mathcal{T}_{\zeta_1}\bar F\omega_1,\bar\delta\omega_2)$ $-\cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2)$ $-g_1(\mathcal{T}_{\omega_2}\bar F\omega_1,\phi\zeta_1)$	$\frac{1}{\lambda^2} g_2(\nabla^{\bar\alpha}_{\zeta_1}\bar\alpha_*(\bar F\omega_1),\bar\alpha_*(\bar F\omega_2))$ $+\frac{1}{\lambda^2} g_2(\nabla^{\bar\alpha}_{\omega_2}\bar\alpha_*(\bar F\omega_1),\bar\alpha_*(\phi\zeta_1))$ $+g_1([\omega_1,\zeta_1],\omega_2) = \sin 2\theta$ $g_1(\omega_1,\omega_2) - g_1(\mathcal{T}_{\omega_2}\bar F\omega_1,\phi\zeta_1)$ $-\cos^2\theta g_1(\nabla_{\zeta_1}\omega_1,\omega_2)$ $-g_1(\mathcal{T}_{\zeta_1}\bar F\omega_1,\bar\delta\omega_2)$

Then, for (i), $\omega_1,\omega_2 \in \Gamma(\mathfrak{D}^\theta \oplus <\xi>)$ and $\zeta_1 \in \Gamma(\mathfrak{D}^\perp)$, for (ii), $\omega_1,\omega_2 \in \Gamma(\mathfrak{D}^\theta)$ and $\zeta_1 \in \Gamma(\mathfrak{D}^\perp)$.

For totally geodesicness of anti-invariant distribution $\mathfrak{D}^\perp$ considering ξ to be vertical specially $\mathfrak{D}^\perp \oplus <\xi>$, we have

Theorem 4.3. *Let* $\bar\alpha$ *be a* $\mathcal{PWHSCS}$ *from* $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ *onto a* $\mathcal{RM}$ (Ξ_2,g_2) *with hemi-slant function* θ *and vertical Reeb vector field* ξ*. Then the anti-invariant distribution* $\mathfrak{D}^\perp \oplus <\xi>$ *defines a totally geodesic foliation if and only if*

$$\begin{aligned}
\frac{1}{\lambda^2} g_2(\nabla^{\bar\alpha}_{\zeta_1}\bar\alpha_*(\bar F\zeta_2),\bar\alpha_*(\bar F\omega_1)) =& \frac{1}{\lambda^2} g_2((\nabla\bar\alpha_*)(\zeta_1,\bar F\zeta_2),\bar\alpha_*(\bar F\omega_1)) - g_1(\mathcal{T}_{\zeta_1}\bar\delta\zeta_2,\bar F)\\
&- g_1(v\nabla_{\zeta_1}\bar\delta\zeta_2,\bar\delta\omega_1) - g_1(\mathcal{T}_{\zeta_1}\bar F\zeta_2,\bar\delta\omega_1)
\end{aligned}$$

and

$$\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) = \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) - g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{N}\beta_1)$$
$$- g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\zeta_1}\bar{\mathcal{F}}\zeta_2,\mathcal{J}\beta_1) + g_1(\zeta_1,\mathfrak{B}\beta_1)\eta(\zeta_2),$$

for any $\zeta_1,\zeta_2 \in \Gamma(\mathfrak{D}^\perp \oplus <\xi>), \omega_1 \in \Gamma(\mathfrak{D}^\theta)$ *and* $\beta_1 \in \Gamma(\ker \bar{\alpha}_*)^\perp$.

Proof. From (2.11), (2.13, (2.1), (2.4), (2.5) and (2.15), we obtain

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = g_1(\mathcal{T}_{\zeta_1}\bar{\mathcal{F}}\zeta_2,\bar{\delta}\omega_1) + \frac{1}{\lambda^2}g_2(\bar{\alpha}_*(\aleph\nabla_{\zeta_1}\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\bar{\mathcal{F}}\omega_1))$$
$$+ g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{\mathcal{F}}\omega_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1).$$

Lemma 3.2, (2.1), and the horizontal conformality of $\bar{\alpha}$ allow us to ultimately obtain

$$g_1(\nabla_{\zeta_1}\zeta_2,\omega_1) = g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{\mathcal{F}}\omega_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1) + g_1(\mathcal{T}_{\zeta_1}\bar{\mathcal{F}}\zeta_2,\bar{\delta}\omega_1)$$
$$- \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\bar{\mathcal{F}}\omega_1)) + \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\bar{\mathcal{F}}\omega_1)).$$

However, using (2.11), (2.13), (2.4), (2.5), (2.1), (2.9) and (2.15), for any $\zeta_1,\zeta_2 \in \Gamma(\mathfrak{D}^\perp)$ and $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^\perp$, we have

$$g_1(\nabla_{\zeta_1}\zeta_2,\beta_1) = \frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) - \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1))$$
$$+ g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{N}\beta_1) + g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{J}\beta_1) + g_1(\mathcal{T}_{\zeta_1}\bar{\mathcal{F}}\zeta_2,\mathcal{J}\beta_1)$$
$$+ g_1(\zeta_1,\mathfrak{B}\beta_1)\eta(\zeta_2).$$

This is the required proof of theorem. □

Theorem 4.4. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ onto a $\mathcal{RM}$ (Ξ_2,g_2) with hemi-slant function θ and vertical Reeb vector field ξ. Then the slant distribution $\mathfrak{D}^\perp$ not defines totally geodesic foliation.*

Since the slant distribution is not defines totally geodesic foliation, we can discuss the total geodesicness of $\mathfrak{D}^\theta \oplus <\xi>$ as follows :

Corollary 4.2. *Let $\bar{\alpha}$ be a $\mathcal{PWHSCS}$ from $\mathcal{SM}$ $(\Xi_1,\phi,\xi,\eta,g_1)$ onto a $\mathcal{RM}$ (Ξ_2,g_2) with hemi-slant function θ and vertical Reeb vector field ξ. Then the slant distribution $\mathfrak{D}^\perp \oplus <\xi>$ defines a totally geodesic foliation if and only if*

$$\begin{aligned}\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\bar{\mathcal{F}}\omega_1)) = &\frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\bar{\mathcal{F}}\omega_1)) - g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\bar{\mathcal{F}})\\ &- g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\bar{\delta}\omega_1) - g_1(\mathcal{T}_{\zeta_1}\bar{\mathcal{F}}\zeta_2,\bar{\delta}\omega_1)\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}&\frac{1}{\lambda^2}g_2(\nabla^{\bar{\alpha}}_{\zeta_1}\bar{\alpha}_*(\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1))\\ &= \frac{1}{\lambda^2}g_2((\nabla\bar{\alpha}_*)(\zeta_1,\bar{\mathcal{F}}\zeta_2),\bar{\alpha}_*(\mathcal{N}\beta_1)) - g_1(\mathcal{T}_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{N}\beta_1)\\ &- g_1(v\nabla_{\zeta_1}\bar{\delta}\zeta_2,\mathcal{J}\beta_1) - g_1(\mathcal{T}_{\zeta_1}\bar{\mathcal{F}}\zeta_2,\mathcal{J}\beta_1) - g_1(\bar{\delta}\bar{\mathcal{F}}\zeta_1,\beta_1)\eta(\zeta_2),\end{aligned} \tag{4.27}$$

for any $\zeta_1,\zeta_2 \in \Gamma(\mathfrak{D}^\perp), \omega_1 \in \Gamma(\mathfrak{D}^\theta)$ *and* $\beta_1 \in \Gamma(\ker\bar{\alpha}_*)^\perp$.

Theorem 3.4 provides an easy way to prove the above conclusion by taking the vertical character of ξ-. When we compare the proof of both results, there is no change in equations

(3.22) and (4.26), but in comparison (3.23) and (4.27), single term $g_1(\bar{\delta}\zeta_1, \zeta_2)\eta(\beta_1)$ is substituted by $-g_1(\bar{\delta}\bar{F}\zeta_1, \beta_1)\eta(\zeta_2)$.

5. Conclusion

This research article examined the effect of a vector field ξ- with dual nature (vertical and horizontal) on pointwise hemi-slant conformal submersions from Sasakian manifolds. The conditions of distribution integrabiities and their leaves' total geodesicness are also examined.

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(M. Shuaib) Department of Mathematics, Lovely Professional University, Punjab, India.